2.4: Derivation of the solution method for first-order exact equations

If $z = f(x,y)$ is a function with continuous partial derivatives, then the differential of this function is:

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

For the special case of $f(x,y) = c \ (a\ constant)$ *dz* would be zero:

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Many differential equations can be written in this differential form, for example:

$\frac{dy}{dx} = -\frac{y}{x}$ could be rearranged as: $ydx + xdy = 0$

If a differential equation can be written in this form, and the constant on the RHS is zero, it is called an exact equation. We therefore define an exact equation as a differential equation which can be written in the following form, and introduce *M* and *N* as follows:

$$\begin{aligned} \frac{\partial f}{\partial x}dx \quad &+ \quad \frac{\partial f}{\partial y}dy \quad = 0 \\ M(x,y)dx &+ N(x,y)dy = 0 \end{aligned}$$

Many differential equations can be written in differential form, but to be an exact equation, the RHS constant must be zero. How can we determine if a differential equation is exact?

If a differential equation is exact, there exists some function *f* such that for all *x* the differential is exact:

$$M(x,y)dx + N(x,y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$M(x,y) = \frac{\partial f}{\partial x} \qquad N(x,y) = \frac{\partial f}{\partial y}$$

We know that the two mixed 2nd partial derivatives of a function must be equal (by Clairault's Theorem):

$$M(x,y) = \frac{\partial f}{\partial x} \qquad N(x,y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Therefore:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So if the partial derivative of the M function of the dx term with respect to y is the same as the partial derivative of the N function of the dy term with respect to x, that means there must exist a function $f(x,y)=c \ \ (a\ constant)$ whose derivatives form the original differential equation in differential form, and therefore the equation is an exact equation.

To solve the differential equation, we then just need to find this function $f(x,y)=c$ which is the solution.

We start with the relation: $\frac{\partial f}{\partial x}=M(x,y)$ and we can find the solution *f* by integrating with respect to x:

$$f(x,y)=\int M(x,y)\,dx+g(y)$$

The result is the antiderivative of the M function with an integration constant, but here the integration constant can actually be a function of the other variable y (which is being treated as a constant while working with the partial derivative with respect to x). We don't know this function, so we simply call it *g(y)* for now.

This first step makes our solution function *f(x,y)* match the M term of the original differential equation. To make it match the N term, we now take the partial derivative of the result from the first step but with respect to y:

$$f(x,y)=\int M(x,y)\,dx+g(y)$$
$$\frac{\partial}{\partial y}\left[f(x,y)\right]=\frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right]+\frac{\partial}{\partial y}\left[g(y)\right]$$
$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right]+g'(y)=N(x,y)$$

The result is the partial derivative of *f* with respect to y which is the N function, so we set the result equal to the N function from the dy term in the original differential equation.

We can then solve algebraically for $g'(y)$:

$$g'(y)=N(x,y)-\frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right]$$

Next, we take the integral of *g'* with respect to y to find g(y):

$$g(y)=\int g'(y)\,dy$$
$$g(y)=\int\left(N(x,y)-\frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right]\right)dy$$

We then fill this into the spot for *g(y)* in the partially completed solution earlier:

$$f(x,y)=\int M(x,y)\,dx+g(y)$$
$$f(x,y)=\int M(x,y)\,dx+\int\left(N(x,y)-\frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right]\right)dy$$

The solution is then this $f(x,y)=c$.

This suggests the following procedure for solving first-order, exact differential equations:

1) Put the equation in differential form and verify that the RHS is zero (an exact equation):

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

2) Identify M and N, and take partial derivatives to confirm this is an exact equation:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

3) Postulate a solution of the form $f(x,y) = c$ and initially set $\frac{\partial f}{\partial x} = M(x,y)$, then integrate with respect to x to find the first, partially complete, form of the solution function. The integration constant is an unknown function of y:

$$f(x,y) = \int M(x,y)\,dx + g(y)$$

4) Now differentiate both sides with respect to y to find an expression for $\frac{\partial f}{\partial y}$

5) Set this equal to N: $\frac{\partial f}{\partial y} = N(x,y)$. The form will be (an expression) + *g'(y)* = (an expression). Solve this expression algebraically for *g'(y)*.

6) Integrate the resulting *g'(y)* expression with respect to y to find *g(y)*.

7) Fill *g(y)* in to the partially complete form of the solution from step 3 to obtain *f(x,y)*. The solution is then $f(x,y) = c$ where *c* is a currently unknown constant which is fixed by initial conditions for a particular solution.