2.3: Derivation of the solution method for first-order, linear differential equations

Start with the first-order, linear differential equation to solve: $\frac{dy}{dx}+P(x)y=f(x)$

We use a property (not proved here) that the solution can be written as the sum of two solutions…

$$y=y_c+y_p$$

…where y_c is the solution to the associated homogenous DE: $\frac{dy_c}{y_c}+P(x)dx=0$

…and y_p is a particular solution of the original nonhomogenous DE.

First, we plug this postulated solution into the original DE:

$$\frac{d}{dx}\left[y_c+y_p\right]+P(x)\left(y_c+y_p\right)=f(x)$$

Separating the *c* and *p* parts, we get…

$$\left(\frac{d}{dx}[y_c]+P(x)(y_c)\right)+\left(\frac{d}{dx}\left[y_p\right]+P(x)\left(y_p\right)\right)=f(x)$$

Because we defined y_c is the solution of the associated homogenous equation, the first bracketed term is zero, and therefore, the second bracketed term equals the original *f(x)*:

$$\left(\frac{d}{dx}[y_c]+P(x)(y_c)\right)+\left(\frac{d}{dx}\left[y_p\right]+P(x)\left(y_p\right)\right)=f(x)$$
$$0 \qquad + \qquad f(x) \qquad = f(x)$$

The purpose of this is the 'split off' the homogenous part, which turns out to be a separable DE:

$$\frac{dy_c}{dx}+P(x)y_c=0$$
$$\frac{dy_c}{y_c}+P(x)dx=0$$

…we can solve using the method of separable variables:

$$\int \frac{1}{y_c}\, dy_c = -\int P(x)\, dx$$

$$\ln|y_c| = -\int P(x)\, dx + C_1$$

$$y_c = e^{-\int P(x)dx + C_1}$$

$$y_c = e^{C_1} e^{-\int P(x)dx}$$

$$y_c = Ce^{-\int P(x)dx}$$

The resulting solution contains this important exponential term (which we will refer to later):

$$an\ important\ \ factor: \quad e^{-\int P(x)dx}$$

We now turn to solving the original DE by finding the particular solution y_p. If we allowed the integrating constant to instead be a function of *x*, we can define a particular solution:

$$y_p = u(x) e^{-\int P(x)dx}$$

Substituting this solution into the original DE…

$$\frac{dy}{dx} + P(x) y = f(x)$$

$$\frac{d}{dx}\left[u(x) e^{-\int P(x)dx} \right] + P(x)\left(u(x) e^{-\int P(x)dx} \right) = f(x)$$

We need the product rule to evaluate the derivative of the first bracketed term:

$$u(x) \frac{d}{dx}\left[e^{-\int P(x)dx} \right] + e^{-\int P(x)dx} \frac{d}{dx}\left[u(x) \right] + P(x)\left(u(x) e^{-\int P(x)dx} \right) = f(x)$$

Regrouping and factoring out the *u(x)* term:

$$u(x)\left(\frac{d}{dx}\left[e^{-\int P(x)dx} \right] + P(x) e^{-\int P(x)dx} \right) + e^{-\int P(x)dx} \frac{d}{dx}\left[u(x) \right] = f(x)$$

In the first term, replacing with the symbol y_c for the important factor…

$$u(x)\left(\frac{dy_c}{dx}+P(x)y_c\right)+e^{-\int P(x)dx}\frac{d}{dx}\left[u(x)\right]=f(x)$$

…and remembering that : $\frac{dy_c}{dx}+P(x)y_c=0$ means the first term goes to zero:

$$u(x)(0)+e^{-\int P(x)dx}\frac{d}{dx}\left[u(x)\right]=f(x)$$
$$e^{-\int P(x)dx}\frac{d}{dx}\left[u(x)\right]=f(x)$$

Now, we can separate variables and integrate:

$$du=e^{\int P(x)dx}f(x)\,dx$$
$$\int du=\int e^{\int P(x)dx}f(x)\,dx$$
$$u=\int e^{\int P(x)dx}f(x)\,dx$$

…and this means our particular solution is:

$$y_p=ue^{-\int P(x)dx}=\int e^{\int P(x)dx}f(x)\,dx\bullet e^{-\int P(x)dx}=e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx$$

Which means the final solution we originally postulated is:

$$y=y_c+y_p$$
$$y=Ce^{-\int P(x)dx}+e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx$$

If the original DE has a solution, it must be in this form. If we wish to check this, we can plug this into the original DE to verify that it is, indeed, a solution:

$$\frac{dy}{dx}+Py=f(x)$$
$$\frac{d}{dx}\left[Ce^{-\int P(x)dx}+e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx\right]+P\left(Ce^{-\int P(x)dx}+e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx\right)=f(x)$$
$$\frac{d}{dx}\left[Ce^{-\int P(x)dx}\right]+\frac{d}{dx}\left[e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx\right]+P\left(Ce^{-\int P(x)dx}+e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx\right)=f(x)$$

$$(product\ rule)$$

$$Ce^{-\int P(x)dx}\frac{d}{dx}\left[-\int P(x)\,dx\right]+e^{-\int P(x)dx}\frac{d}{dx}\left[\int e^{\int P(x)dx}f(x)\,dx\right]+\int e^{\int P(x)dx}f(x)\,dx\frac{d}{dx}\left[e^{-\int P(x)dx}\right]$$
$$+P(x)\left(Ce^{-\int P(x)dx}+e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx\right)=f(x)$$

$$-Ce^{-\int P(x)dx}P(x)+e^{-\int P(x)dx}e^{\int P(x)dx}f(x)+\int e^{\int P(x)dx}f(x)\,dx\left(e^{-\int P(x)dx}\right)(-P(x))+P(x)Ce^{-\int P(x)dx}$$
$$+P(x)e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx=f(x)$$

$$\left(-Ce^{-\int P(x)dx}P(x)+P(x)Ce^{-\int P(x)dx}\right)+\left(P(x)e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx-P(x)e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx\right)$$
$$+e^{-\int P(x)dx}e^{\int P(x)dx}f(x)=f(x)$$

$$(0)+(0)+e^{-\int P(x)dx}e^{\int P(x)dx}f(x)=f(x)$$

$$e^{\int P(x)dx-\int P(x)dx}f(x)=f(x)$$
$$e^{0}f(x)=f(x)$$
$$f(x)=f(x)\quad verified$$

This is extremely convoluted, but is the procedure used to derive the solution form. Now that we know the solution form, we can use the results to define an easier-to-remember process to quickly find this solution, our 'solution procedure for first-order linear equations'.

During the above procedure, we encountered this important factor:

$$an\ important\ \ factor:\quad e^{-\int P(x)dx}$$

And we now define 1 divided by this factor, the **<u>integrating factor</u>**:

$$integrating\ \ factor\ (I.F.):\quad \frac{1}{e^{-\int P(x)dx}}=e^{\int P(x)dx}$$

If we multiply what we now know is the solution form by the integrating factor we get:

$$y = Ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)\,dx$$

$$e^{\int P(x)dx} y = Ce^{\int P(x)dx} e^{-\int P(x)dx} + e^{\int P(x)dx} e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)\,dx$$

$$e^{\int P(x)dx} y = Ce^{0} + e^{0} \int e^{\int P(x)dx} f(x)\,dx$$

$$e^{\int P(x)dx} y = C + \int e^{\int P(x)dx} f(x)\,dx$$

We can then differentiate both sides of this equation:

$$\frac{d}{dx}\left[e^{\int P(x)dx} y\right] = \frac{d}{dx}[C] + \frac{d}{dx}\left[\int e^{\int P(x)dx} f(x)\,dx\right]$$

$$(\textit{product rule})$$

$$e^{\int P(x)dx} \frac{d}{dx}[y] + y\frac{d}{dx}\left[e^{\int P(x)dx}\right] = 0 + e^{\int P(x)dx} f(x)$$

$$e^{\int P(x)dx} \frac{dy}{dx} + ye^{\int P(x)dx} \frac{d}{dx}\left[\int P(x)\,dx\right] = e^{\int P(x)dx} f(x)$$

$$e^{\int P(x)dx} \frac{dy}{dx} + yP(x)e^{\int P(x)dx} = e^{\int P(x)dx} f(x)$$

Dividing everything by the integrating factor, gives the original DE: $\frac{dy}{dx} + P(x)y = f(x)$

…and this suggests a simple procedure for quickly finding the solution:

1) Put a linear equation in the form: $\frac{dy}{dx} + P(x)y = f(x)$

2) Identify *P(x)* and use it to find the integrating factor: $I.F. = e^{\int P(x)dx}$

3) Multiply the standard form by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the integrating factor and y:

$$\frac{d}{dx}\left[e^{\int P(x)dx} y\right] = e^{\int P(x)dx} f(x)$$

4) Then just integrate both sides of this last equation to obtain the solution.