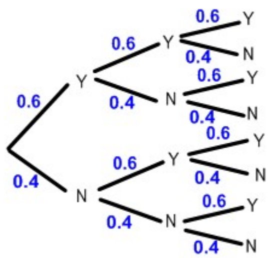
Bernoulli Trials and the Binomial Probability Model

At a college, 60% of the students voted in the latest election. If you select 3 students at random, what is the probability that exactly 2 voted in the election?



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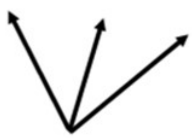

$$P(exactly\ 2\ voted) = {}_3C_2\left(P(voted)\right)^2\left(P(didn't\ vote)\right)^1$$

number of ways the 2 students who voted can be chosen

multiple trials, all independent
probability is constant
each trial only two options: "success" (voted), "failure" (did not vote)

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If we are running a **fixed number of trials** (here, 3 students) the probability answer is always in the form...

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...which looks like a term from a binomial theorem expansion, so this model is called the **Binomial probability model**. More generally:

$$P(exactly\ k\ successes) = {}_nC_k\,(p)^k\,(q)^{n-k}$$

where n = *number of trials*
k = *number of successes*
p = *probability of success for a single trial*
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$$binompdf(n,\ p,\ k)$$

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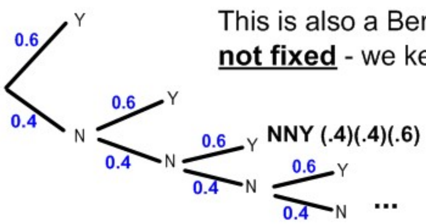

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$geometpdf(p, k)$ probability that 1st success is on the k^{th} trial

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The shape, mean, standard deviation of the Geometric Distribution

The Geometric distribution is a distribution so it has a shape which has a mean and standard deviation.

$$G(\mu, \sigma)$$

$p = 0.2$

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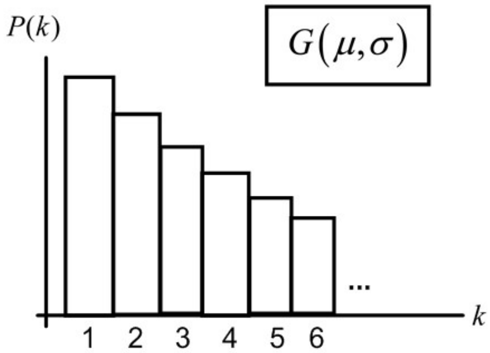

(Math box (p.388) has an interesting derivation for the mean.)

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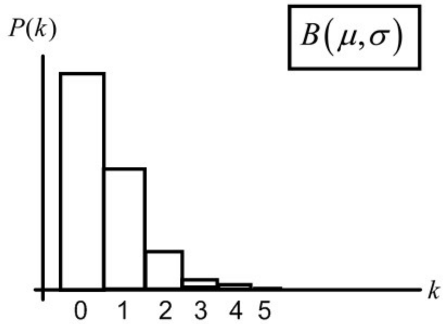

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$$\mu = np \qquad \sigma = \sqrt{npq}$$

The shape of the Binomial Distribution - varying p

Changing the probability of success, p, changes the shape...

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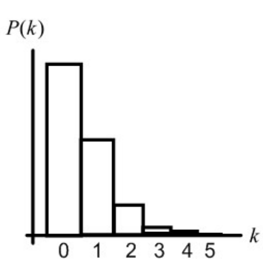

p close to 0
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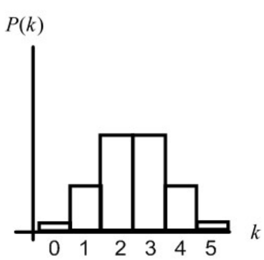
$p = 0.5, n = 5$



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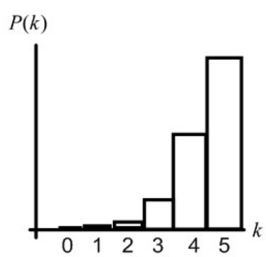

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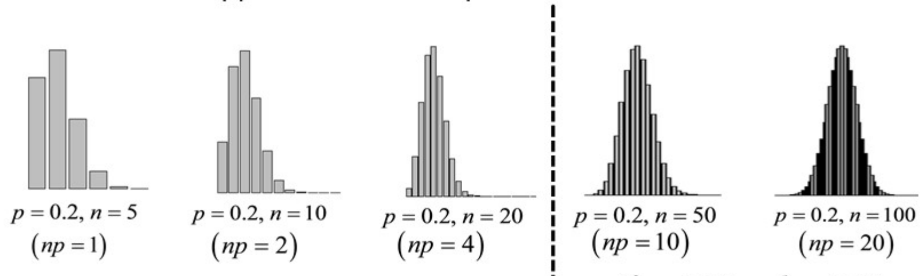
As the number of trials, n, increases, the shape of the Binomial distribution approaches the shape of a Normal distribution:



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If we expect at least 10 successes and 10 failures, we can use a Normal model to approximate the Binomial model. → *If $np \geq 10$ and $nq \geq 10$ the Binomial model is approximately Normal.*

(Math box (p.395) explains this condition in more detail.)

(see practice packet for examples of approximating a Binomial distribution with a Normal model)

Summary of probability models

<u>Binomial Model (Bernoulli trials) Setting</u> $\mu = np$ $\sigma = \sqrt{npq}$

1) Only 2 outcomes (success/failure)
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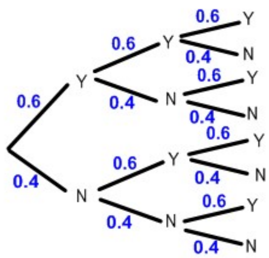

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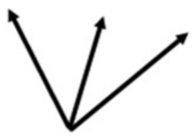

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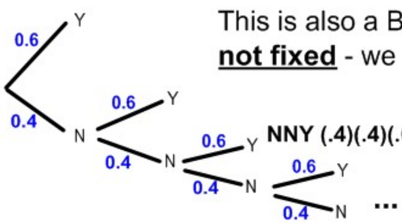

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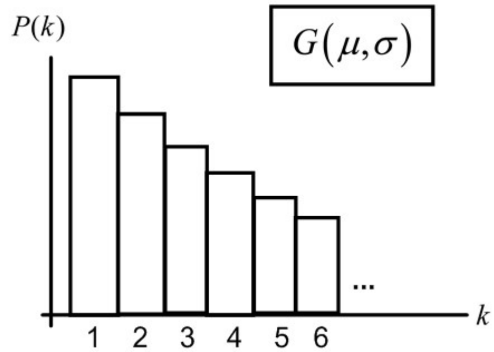

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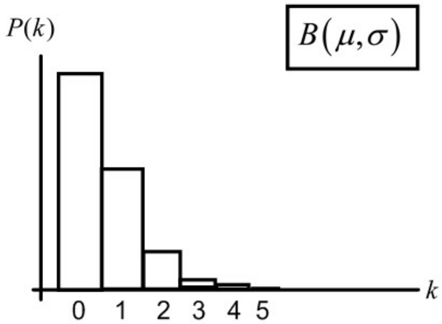

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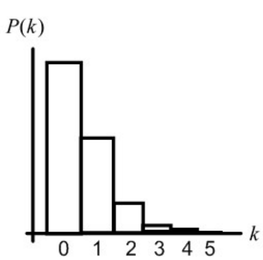

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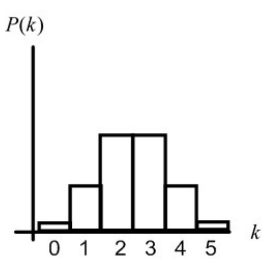

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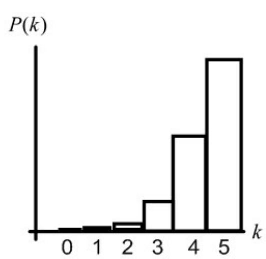

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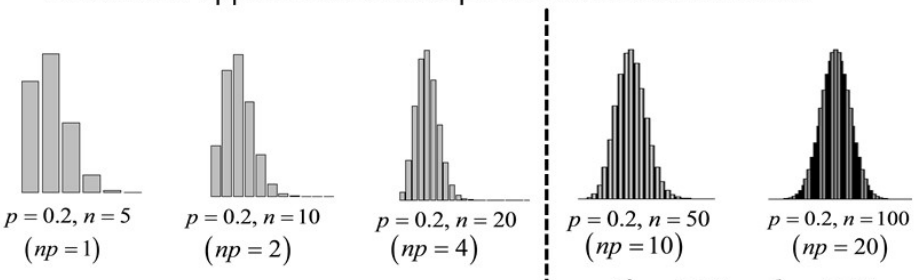

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