Proof: Why variances of independently varying distributions add

Expected values are summations: $E(X \pm Y) = E(X) + E(Y)$

First, we'll prove a 'Lemma' (a side results that we need to prove the main result):

Lemma: $Var(X) = E(x^2) - \mu^2$

Proof:

$$
\begin{aligned}
Var(X) &= E\left((x-\mu)^2\right) \qquad \textit{where } E(X) \textit{ means } \sum (x_i - \mu)^2 P(x_i) \\
&= E\left(x^2 - 2x\mu + \mu^2\right) \\
&= E\left(x^2\right) + E(-2x\mu) + E\left(\mu^2\right) \\
&= E\left(x^2\right) - 2\mu E(x) + E\left(\mu^2\right) \qquad \textit{but } E(X) = \mu \\
&= E\left(x^2\right) - 2\mu\mu + E\left(\mu^2\right) \qquad \mu \textit{ is a constant, so } E\left(\mu^2\right) = \mu^2 \\
&= E\left(x^2\right) - 2\mu\mu + \mu^2 \\
&= E\left(x^2\right) - \mu^2
\end{aligned}
$$

Now we can prove the theorem: If X and Y are independent: $Var(X \pm Y) = Var(X) + Var(Y)$

Proof:

$$
\begin{aligned}
Var(X \pm Y) &= E\left((x+y)^2\right) - \left(\mu_{x\pm y}\right)^2 \qquad \textit{using the Lemma} \\
&= E\left((x+y)^2\right) - \left(\mu_x \pm \mu_y\right)^2 \qquad \textit{because } \mu \textit{ is a constant} \\
&= E\left(x^2 \pm 2xy + y^2\right) - \left(\mu_x^{\ 2} \pm 2\mu_x\mu_y + \mu_y^{\ 2}\right) \\
&= E\left(x^2\right) \pm 2E(xy) + E\left(y^2\right) - \mu_x^{\ 2} \mp 2\mu_x\mu_y - \mu_y^{\ 2} \\
&= E\left(x^2\right) - \mu_x^{\ 2}\left[\pm 2E(xy) \mp 2\mu_x\mu_y\right] + E\left(y^2\right) - \mu_y^{\ 2} \\
&= E\left(x^2\right) - \mu_x^{\ 2} \pm 2\underbrace{\left[E(xy) - \mu_x\mu_y\right]} + E\left(y^2\right) - \mu_y^{\ 2}
\end{aligned}
$$

$E(xy)$ is sum of terms $x_i y_j \bullet P(x_i \cap y_j)$

$\mu_x\mu_y$ is sum of terms $x_i P(x_i) \bullet y_j P(y_j)$

if X and Y are independent, probabilities multiply

and $P(x_i \cap y_j) = P(x_i) \bullet P(y_j)$

so middle term is zero if independent

$$
\begin{aligned}
Var(X \pm Y) &= E\left(x^2\right) - \mu_x^{\ 2} + E\left(y^2\right) - \mu_y^{\ 2} \\
Var(X \pm Y) &= Var(X) + Var(Y)
\end{aligned}
$$

This proof is from the website below, which provides even more background...

http://apcentral.collegeboard.com/apc/members/courses/teachers_corner/50250.html