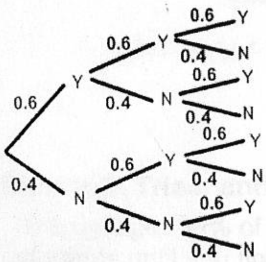
# AP Statistics – Lesson Notes - Chapter 17: Probability Models

## Bernoulli Trials and the Binomial Probability Model

At a college, 60% of the students voted in the latest election. If you select 3 students at random, what is the probability that <u>exactly 2</u> voted in the election?



AND (multiply)

YYN (.6)(.6)(.4)
YNY (.6)(.4)(.6)
NYY (.4)(.6)(.6)

OR (add)

$$P(exactly\ 2\ voted) = P(YYN) + P(YNY) + P(NYY)$$
$$P(exactly\ 2\ voted) = (.6)(.6)(.4) + (.6)(.4)(.6) + (.4)(.6)(.6)$$
$$P(exactly\ 2\ voted) = (.6)^2(.4)^1 + (.6)^2(.4)^1 + (.6)^2(.4)^1$$
$$P(exactly\ 2\ voted) = 3(.6)^2(.4)^1$$
$$P(exactly\ 2\ voted) = {}_3C_2\left(P(voted)\right)^2\left(P(didn't\ vote)\right)^1$$

*number of ways the 2 students who voted can be chosen*

**Situations like this are called Bernoulli Trials**

multiple trials, all independent
probability is constant
each trial only two options: "success" (voted), "failure" (did not vote)

If we are running a **<u>fixed number of trials</u>** (here, 3 students)
the probability answer is always in the form...

$$P(exactly\ 2\ voted) = {}_3C_2\left(P(voted)\right)^2\left(P(didn't\ vote)\right)^1$$

...which looks like a term from a binomial theorem expansion, so this model is called the **<u>Binomial probability model</u>**. More generally:

$$P(exactly\ k\ successes) = {}_nC_k\,(p)^k\,(q)^{n-k}$$

where $n$ = number of trials
$k$ = number of successes
$p$ = probability of success for a single trial
$q$ = probability of failure for a single trial

At a college, 60% of the students voted in the latest election. If you select 3 students at random, what is the probability that <u>exactly 2</u> voted in the election?

We use this model so often that there is a calculator function dedicated to it:

$$binompdf(n,\ p,\ k)$$

#voted: 0 1 2 3

binompdf(3, .6, 2) = $\boxed{.432}$

Slightly different problem:
At a college, 60% of the students voted in the latest election. If you select 3 students at random, what is the probability that <u>up to 2</u> voted in the election?

What are all the possibilities of number of students of 3 who voted?

0 voted 1 voted 2 voted 3 voted

$$\underbrace{binompdf(3, .6, 0) + binompdf(3, .6, 1) + binompdf(3, .6, 2)}_{binomcdf(3,\ .6,\ 2)}$$

*c for cumulative*

# voted: 0 1 Σ 3

binomcdf(3, .6, 2) = $\boxed{.784}$

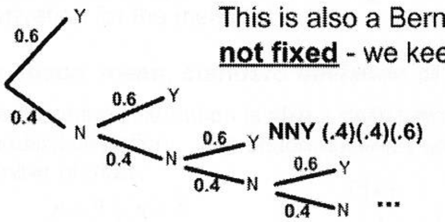
Another variation:
At a college, 60% of the students voted in the latest election. If you select 3 students at random, what is the probability that at least 2 voted in the election?

$\#\text{voted}: \underbrace{0 \quad 1}, \underbrace{2 \quad 3}$

binomcdf(3,.6,1)    1-binomcdf(3,.6,1) = $\boxed{.648}$

**Bernoulli Trials and the Geometric Probability Model**

At a college, 60% of the students voted in the latest election. If you keep asking students until you find one that voted, what is the probability that you have to ask exactly 3 students in order to find one that voted?




This is also a Bernoulli trial case, but **the number of trials is not fixed** - we keep asking *until* we have a 'success' case.

The results is in the form $P(\text{exactly } k \text{ successes}) = (q)^{k-1}(p)$ and this is called the **Geometric probability model**.

Like the binomial model, there are calculator functions for computation:

$geometpdf(p,k)$ probability that 1st success is on the k$^{th}$ trial

$geometcdf(p,k)$ probability that 1st success is on or before the k$^{th}$ trial

At a college, 60% of the students voted in the latest election. If you keep asking students until you find one that voted, what is the probability that you have to ask more than 3 students in order to find one that voted?

$\#\text{voted}: \underbrace{0 \quad 1 \quad 2 \quad 3} \; [4 \quad 5 \quad 6 \quad 7 \ldots$

geometcdf(.6,3)    1-geometcdf(.6,3) = $\boxed{.064}$

**Bernoulli Trials**

Compare these three scenarios...

A) If you draw two cards from a deck, what is the probability they are kings?

$\frac{4}{52}\cdot\frac{3}{51}$ not independent

B) If you flip a fair coin twice, what is the probability you get tails both times?

$\frac{1}{2}\cdot\frac{1}{2}$ independent (Bernoulli trial, Binomial)

C) If the ball pit in a MacDonald's play area contains 10,000 balls, and 10% of them are green, what is the probability that if you randomly choose 2 balls from the pit they will both be green?

$\frac{1}{10}\cdot\frac{1}{10}$ ← approximately independent

rule of thumb: if sample is ≤10% of the population, can consider probability constant

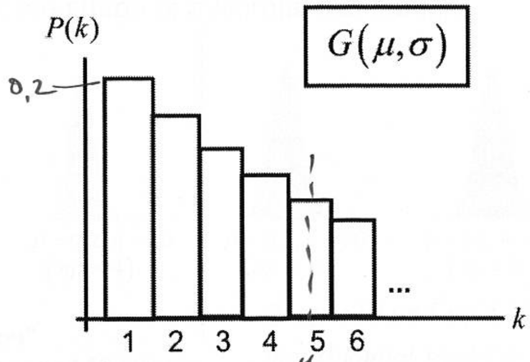
### The shape, mean, standard deviation of the Geometric Distribution

The Geometric distribution is a distribution so it has a shape which has a mean and standard deviation.

$p = 0.2$

$geometpdf(0.2,1) = .200$
$geometpdf(0.2,2) = .160$
$geometpdf(0.2,3) = .128$
$geometpdf(0.2,4) = .102$
$geometpdf(0.2,5) = .082$
$geometpdf(0.2,6) = .066$

$\boxed{G(\mu,\sigma)}$




starts with probability p of success on 1st trial then gets progressively less likely to take longer to find success.

(Math box (p.388) has an interesting derivation for the mean.)

$\mu = \frac{1}{p}$ $\qquad \sigma = \sqrt{\frac{q}{p^2}}$

If $p = .2$

$\mu = \frac{1}{.2} = 5 \qquad \sigma = \sqrt{\frac{.8}{(.2)^2}} = 4.47$

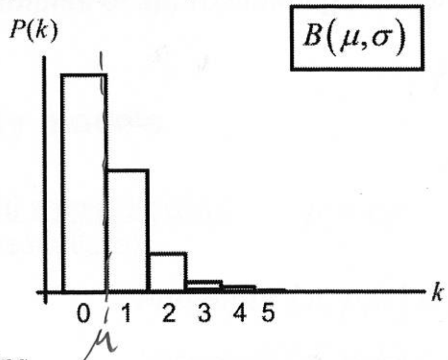
### The shape, mean, standard deviation of the Binomial Distribution

The Binomial distribution is also a distribution and has a shape, a mean, and standard deviation. The shape depends upon the probability of success and the number of trials.

$p = 0.1, n = 5$

$binompdf(5,0.1,0) = .590$
$binompdf(5,0.1,1) = .320$
$binompdf(5,0.1,2) = .073$
$binompdf(5,0.1,3) = .008$
$binompdf(5,0.1,4) = .00045$
$binompdf(5,0.1,5) = .00001$

$\boxed{B(\mu,\sigma)}$




if $p = .1, n = 5$:

$\mu = (5)(.1) = .5$

$\sigma = \sqrt{(5)(.1)(.9)} = .6708$

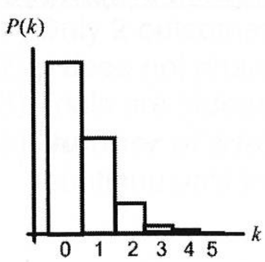
(Math box (p.392) has an interesting derivation for the mean and standard deviation.)

$\mu = np \qquad \sigma = \sqrt{npq}$

### The shape of the Binomial Distribution - varying p

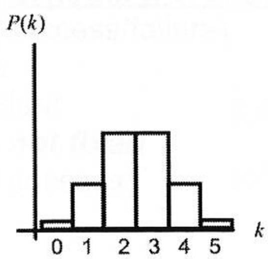
Changing the probability of success, p, changes the shape...

$p = 0.1, n = 5$



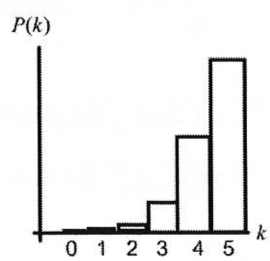

p close to 0
skewed right

$p = 0.5, n = 5$




p close to 0.5
symmetric

$p = 0.9, n = 5$




p close to 1
skewed left

(imagine flipping a coin 5 times with different probability of 'heads')

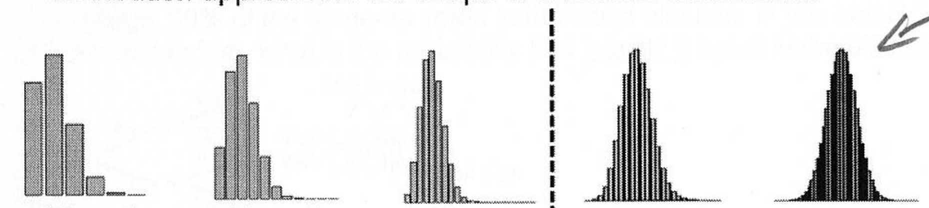
to see distribution:

| L1 | L2 = binompdf(5,.1,L1) |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |

**The shape of the Binomial Distribution - varying n**

As the number of trials, n, increases, the shape of the Binomial distribution approaches the shape of a Normal distribution:

$\mu = np$
$\sigma = \sqrt{npq}$



| $p = 0.2, n = 5$ $(np = 1)$ | $p = 0.2, n = 10$ $(np = 2)$ | $p = 0.2, n = 20$ $(np = 4)$ | $p = 0.2, n = 50$ $(np = 10)$ | $p = 0.2, n = 100$ $(np = 20)$ |
|---|---|---|---|---|

**The "Success/Failure Condition"**
If we expect at least 10 successes and 10 failures, we can use a Normal model to approximate the Binomial model.

*If $np \geq 10$ and $nq \geq 10$ the Binomial model is approximately Normal.*

(Math box (p.395) explains this condition in more detail.)

***(see practice packet for examples of approximating a Binomial distribution with a Normal model)***

## Summary of probability models

**Binomial Model (Bernoulli trials) Setting** $\mu = np$ $\sigma = \sqrt{npq}$

1) Only 2 outcomes (success/failure)
2) p does not change
3) Trials are independent
4) ***Fixed*** number of trials

$$binompdf(n, p, k) = \text{exactly } k \text{ successes out of } n \text{ trials}$$
$$binomcdf(n, p, k) = \text{at most } k \text{ successes out of } n \text{ trials}$$

Can approximate Binomial $B(\mu, \sigma)$ with Normal $N(\mu, \sigma)$ if $np \geq 10$ *and* $nq \geq 10$
$(np = \#successes, \; nq = \#failures)$

**Geometric Model (Bernoulli trials) Setting** $\mu = \dfrac{1}{p}$ $\sigma = \sqrt{\dfrac{q}{p^2}}$

1) Only 2 outcomes (success/failure)
2) p does not change
3) Trials are independent
4) ***Number of trials not fixed*** (continue until first success)

$$geometpdf(p, k) = 1^{st} \text{ success is on } k^{th} \text{ trial}$$
$$geometcdf(p, k) = 1^{st} \text{ success is on or before } k^{th} \text{ trial}$$

*where* $n = $ *number of trials*
$k = $ *number of successes*
$p = $ *probability of success for a single trial*
$q = $ *probability of failure for a single trial*
$q = 1 - p$

***Bernoulli Trials*** *can be considered independent if sample is <10% of the entire population.*