AP Calculus BC Name: answers/solutions

8.1 Worksheet Period: ____________

Verify the solution of the differential equation.

1. Differential Equation: $y' = 5y$ Solution: $y = Ce^{5x}$

(show steps)

2. Differential Equation: $y' = \frac{2xy}{x^2-y^2}$ Solution: $x^2 + y^2 = Cy$

$\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = \frac{d}{dx}[Cy]$

$2x + 2yy' = Cy'$ (also $C = \frac{x^2+y^2}{y}$)

$2yy' - Cy' = -2x$

$(2y - C)y' = -2x$

$y' = \frac{-2x}{2y-C} = \frac{-2x}{2y - \left(\frac{x^2+y^2}{y}\right)}$

plug in DE: $\left[\frac{-2x}{2y - \left(\frac{x^2+y^2}{y}\right)}\right] \stackrel{?}{=} \frac{2xy}{x^2-y^2}$

$\frac{-2xy}{2y^2 - x^2 - y^2}$

$\frac{-2xy}{y^2-x^2} = \frac{-(2xy)}{-(x^2-y^2)} = \frac{2xy}{x^2-y^2}$ $= \checkmark$

Verify the particular solution of the differential equation.

3. Differential Equ: $2y + y' = 2\sin(2x) - 1$

Initial Condition: $y\left(\frac{\pi}{4}\right) = 0$

Particular Solution: $y = \sin(x)\cos(x) - \cos^2(x)$

(show steps)

4. Differential Equ: $y' = -12xy$

Initial Condition: $y(0) = 4$

Particular Solution: $y = 4e^{-6x^2}$

$y' = 4(e^{(-6x^2)} \cdot (-12x))$

$y' = -48xe^{(-6x^2)}$ plug into DE

$y' \stackrel{?}{=} -12xy$

$[-48xe^{(-6x^2)}] \stackrel{?}{=} -12x[4e^{(-6x^2)}]$

$-48xe^{(-6x^2)} = -48xe^{(-6x^2)}$ $\checkmark$

Determine whether the function is a solution of the differential equation $xy' - 2y = x^3e^x$.

5. $y = x^2$

$\boxed{\text{No, not a Solution}}$

6. $y = \ln(x)$ → into DE;

$y' = \frac{1}{x}$

$xy' - 2y \stackrel{?}{=} x^3e^x$

$x[\frac{1}{x}] - 2[\ln x] \stackrel{?}{=} x^3e^x$

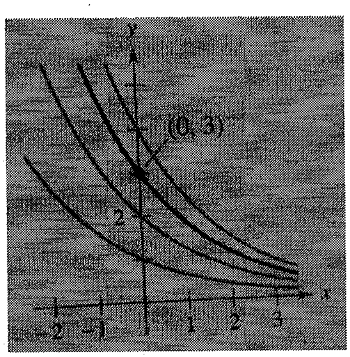
$1 - 2\ln x \stackrel{?}{=} x^3e^x$

$\boxed{\text{No, not a solution}}$

The graph shows some curves from the general solution to the differential equation. Find the particular solution of the differential with the indicated coordinate as an initial condition.

7. $y = Ce^{-x/2}$

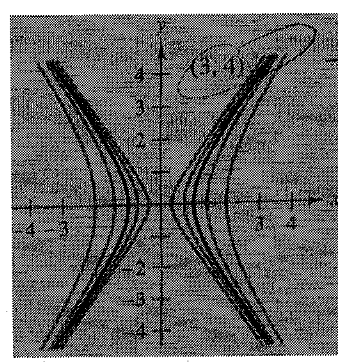
$2y' + y = 0$



$\boxed{y = 3e^{-\frac{x}{2}}}$

8. $2x^2 - y^2 = C$ ← Solution

$yy' - 2x = 0$



(3, 4)
x, y

$2x^2 - y^2 = c$

$2(3)^2 - (4)^2 = c$

$2 = c$

$\boxed{2x^2 - y^2 = 2}$

Verify that the general soln. satisfies the diff. equation. Then find the particular soln. with the initial condition.

9. $y = Ce^{-6x}$

$y' + 6y = 0$

$y = 3 \text{ when } x = 0$

$\boxed{y = 3e^{-6x}}$

10. $3x^2 + 2y^2 = C$

$3x + 2yy' = 0$

$y = 3 \text{ when } x = 1$

verifying: $3x^2 + 2y^2 = c$

$\frac{d}{dx}[3x^2] + \frac{d}{dx}[2y^2] = \frac{d}{dx}[c]$

$6x + 4y\,y' = 0$

$3x + 2y\,y' = 0$ is DE ✓

particular solution:

$3x^2 + 2y^2 = c$ make (1,3) work

$3(1)^2 + 2(3)^2 = c$

$3 + 18 = c$

$21 = c$

so $\boxed{3x^2 + 2y^2 = 21}$

Use integration to find the general equation of the differential equation.

11. $\frac{dy}{dx} = 12x^2$

$\boxed{y = 4x^3 + C}$

12. $\frac{dy}{dx} = 10x^4 - 2x^3$

$y = \int (10x^4 - 2x^3)\,dx$

$y = 10\frac{x^5}{5} - 2\frac{x^4}{4} + C$

$\boxed{y = 2x^5 - \frac{1}{2}x^4 + C}$

13. $\frac{dy}{dx} = \sin(2x)$

$\boxed{y = \frac{1}{2}\cos(2x) + C}$

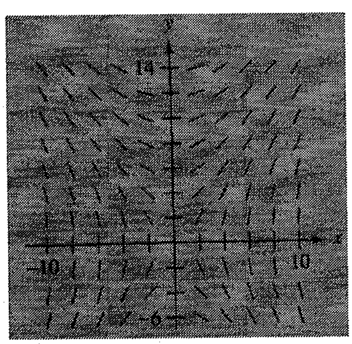
14. $\frac{dy}{dx} = 4\sec^2(x)$

$y = \int 4\sec^2 x\,dx$

$\boxed{y = 4\tan x + C}$

A differential equation and its slope field are given. Complete the table by determining the slopes (if possible) in the slope field at the given points.

15. $\frac{dy}{dx} = \frac{2x}{y}$

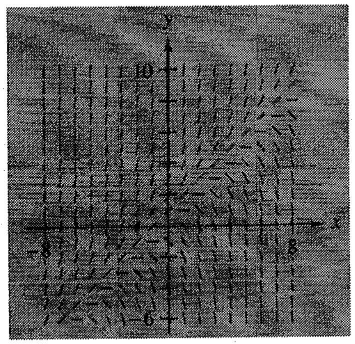


x	−4	−2	0	2	4	8
y	2	0	4	4	6	8
dy/dx						

-2 $\quad \infty$ (undef) $\quad 0 \quad 2 \quad \frac{3}{2} \quad 2$

← these are all estimates of the slopes

16. $\frac{dy}{dx} = y - x$



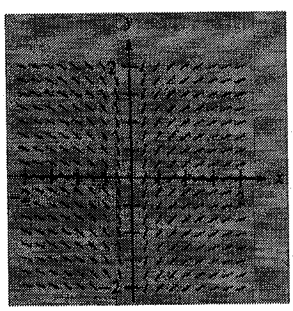
x	−4	−2	0	2	4	8
y	2	0	4	4	6	8
dy/dx						

3 2 3 2 2 0

(these are estimates of the slopes at the given points)

17. Match the slope field with the differential equation.

A



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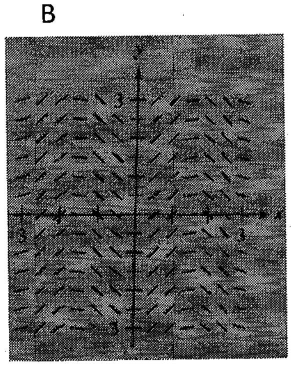

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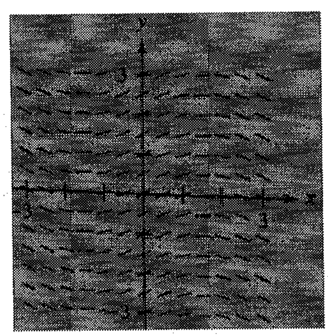

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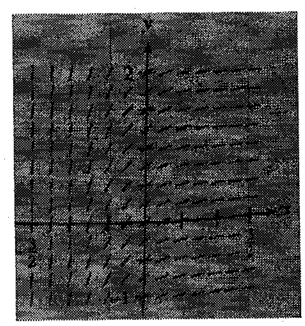


$y' = \sin(2x)$ B $y' = \frac{1}{2}\cos(x)$ C $y' = e^{-2x}$ D $y' = x^{-1}$ A

(hint: find y by integrating, then graph and match the solution curve shapes with the slope fields)

Use Euler's Method to make a table of values for the approximate solution of the differential equation with the specified initial value. Estimate y(0.4).

18. $y' = x + y$ $\quad$ $y(0) = 2$ $\quad$ $n = 10, h = 0.1$

$$\boxed{y(0.4) \approx 2.992}$$

19. $y' = e^{xy}$ $\quad$ $y(0) = 1$ $\quad$ $n = 10, h = 0.1$

(x, y)	$y_{n+1} = y_n + n[y'] = y_n + (0.1)[e^{(x)(y)}]$
$(0, 1)$	$y = 1 + 0.1\,[e^{(0)(1)}] = 1.1$
$(0.1, 1.1)$	$y = 1.1 + 0.1\,(e^{(0.1)(1.1)}] = 1.211627807$
$(0.2, 1.211627807)$	$y = 1.211627807 + 0.1\,(e^{(0.2)(1.211627807)}] = 1.339048703$
$(0.3, 1.339048703)$	$y = 1.339048703 + 0.1\,[e^{(0.3)(1.339048703)}] = 1.488487182$
$(0.4, 1.488487182)$	

so $y(0.4) \approx 1.488487182$

round to 3 decimal places at end of the procedure!

$$\boxed{y(0.4) \approx 1.488}$$

8.2 Worksheet

Find the general solution of the differential equation.

1. $\frac{dr}{ds} = .75r$

$$\boxed{r = Ce^{.75s}}$$

2. $\frac{dr}{ds} = .75s$

$dr = .75s\,ds$

$\int dr = \int .75s\,ds$

$$\boxed{r = \frac{.75}{2}s^2 + c}$$

3. $yy' = 4\sin x$

$$\boxed{y^2 = -8\cos x + C}$$

$\left(\text{or } y = \pm\sqrt{-8\cos x + C}\right)$

4. $xy' = y$

$x\frac{dy}{dx} = y$

$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$

$\ln|y| = (\ln|x| + C_1)$

$|y| = e^{\ln|x|} \cdot \underbrace{e^{C_1}}_{C} = Ce^{\ln|x|} = C|x|$

$$\boxed{y = Cx}$$

5. $\sqrt{1-4x^2}\,y' = x$

$$\boxed{y = -\frac{1}{4}\sqrt{1-4x^2} + C}$$

6. $12yy' - 7e^x = 0$

$12y\,dy = 7e^x\,dx$

$\int 12y\,dy = \int 7e^x\,dx$

$6y^2 = 7e^x + C_1$

$y^2 = \frac{7}{6}e^x + \frac{C_1}{6}$

$$\boxed{y^2 = \frac{7}{6}e^x + C}$$

$\left(\text{or } y = \pm\sqrt{\frac{7}{6}e^x + C}\right)$

Find the particular solution that satisfies the initial condition.

7. $yy' - 2e^x = 0, \quad y(0) = 6$

$\boxed{y^2 = 4e^x + 32}$

(or $y = \pm\sqrt{4e^x + 32}$)

8. $\sqrt{x} + \sqrt{y}\,y' = 0, \quad y(1) = 9$

$\sqrt{y}\,\frac{dy}{dx} = -\sqrt{x}, \quad \int y^{1/2}dy = \int -x^{1/2}dx$

$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C_1$

$y^{3/2} = -x^{3/2} + \frac{3}{2}C_1$

$(\sqrt{y})^3 = -(\sqrt{x})^3 + C$ now, $\underset{x}{y}(1) = \underset{y}{9}$

$(\sqrt{(9)})^3 = -(\sqrt{(1)})^3 + C$

$27 = -1 + C \rightarrow C = 28$

$\boxed{y^{3/2} = -x^{3/2} + 28}$

(or $(\sqrt{y})^3 = -(\sqrt{x})^3 + 28$

or $y = (-x^{3/2} + 28)^{2/3}$)

9. $\frac{du}{dv} = uv\sin(v^2), \quad u(0) = 1$

$\boxed{\ln|u| = -\frac{1}{2}\cos(v^2) + \frac{1}{2}}$

(or $u = \pm e^{\left(-\frac{1}{2}\cos(v^2) + \frac{1}{2}\right)}$)

10. $2xy' - \ln(x^2) = 0, \quad y(1) = 2$

$2x\frac{dy}{dx} = \ln(x^2) = 2\ln x$

$\int dy = \int \frac{\ln x}{x}dx$ u sub: $u = \ln x$

$y = \int u\,du$ $\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}du$

$y = \frac{1}{2}u^2 + C$

$y = \frac{1}{2}(\ln x)^2 + C$ now $\underset{x}{y}(1) = \underset{y}{2}$

$2 = \frac{1}{2}\ln(1)^2 + C$

$2 = 0 + C \rightarrow C = 2$

$\boxed{y = \frac{1}{2}(\ln x)^2 + 2}$

11. $\frac{dr}{ds} = e^{r-2s}$, $r(0) = 0$

$\boxed{r = -\ln\left|\frac{1}{2}e^{-2s} - c\right|}$

12. $dP - kP\,dt = 0$, $P(0) = P_0$

$\int \frac{1}{P}\,dP = \int k\,dt$

$\ln P = kt + C_1$

$P = e^{kt+C_1} = e^{kt}e^{C_1} = Ce^{kt}$

now, $P(0) = P_0$
t P

$P_0 = e^{k(0)} + C \Rightarrow C = P_0$

$\boxed{P = P_0e^{kt}}$

13. A calf that weighs 60 pounds at birth gains weight at the rate

$\frac{dw}{dt} = k(1200 - w)$

Where w is the weight in pounds and t is the time in years.

a. Solve the differential.

b. Use a graphing calculator to graph the particular solutions for k = 0.8, 0.9, and 1.

c. The animal is sold when its weight is 800 pounds. Find the time of sale for each of the models in part b.

d. What is the maximum weight of the animal for each of the models in part b?

a) $\boxed{W = 1200 - 1140e^{-kt}}$

b) (graph should show as k increases animal's weight rises more rapidly)

c) $k = .8$, $t = 1.309$ years
$k = .9$, $t = 1.164$ years
$k = 1$, $t = 1.047$ years

d) $\boxed{1200 \text{ lbs}}$

8.3 Worksheet

Write and solve the differential equation that models the verbal statement.

1. The rate of change of Q with respect to t is inversely proportional to the square of t.

$\boxed{\frac{dQ}{dt} = \frac{K}{t^2}}$ $\boxed{Q = \frac{-K}{t} + C}$

2. The rate of change of P with respect to t is proportional to 25 – t.

$\boxed{\frac{dP}{dt} = K(25-t)}$

$\int dP = \int (25K - Kt)dt$

$\boxed{P = 25Kt - \frac{1}{2}Kt^2 + C}$

Write and solve the differential equation that models the verbal statement. Evaluate the solution at the specified value of the independent variable.

3. The rate of change of N is proportional to N. When t = 0, N = 250, and when t = 1, N = 400. What is the value of N when t = 4?

$\boxed{N(4) = 1638.4}$

4. The rate of change of P is proportional to P. When t = 0, P = 5000, and when t = 1, P = 4750. What is the value of P when t = 5?

$\frac{dP}{dt} = KP$

$\int \frac{1}{P} dP = \int K dt$

$\ln P = Kt + C_1$

$P = e^{(Kt + C_1)} = e^{Kt} e^{C_1}$

$P = Ce^{Kt}$

t	P
0	5000
1	4750

$5000 = Ce^{K(0)} = C$

$P = 5000\, e^{Kt}$

$4750 = 5000\, e^{K(1)}$

$\frac{4750}{5000} = e^K$

$\ln\left(\frac{4750}{5000}\right) = K$

$P = 5000\, e^{\ln\left(\frac{4750}{5000}\right)t}$

$P(5) = 5000\, e^{\ln\left(\frac{4750}{5000}\right) \cdot 5}$

$= \boxed{3868.905}$

Complete the table for the radioactive isotope.

	Isotope	Half-Life (yrs)	Initial Quantity	Amount after 100 yrs	Amount after 10000 yrs
5.	^{226}Ra	1599	20g	19.152g	0,262g
6.	^{239}Pu	24100	0,533g	0,532g	0.4g

(6)

t	Q
0	Q_0
24100	$\frac{1}{2}Q_0$
10000	0,4g

$Q = Q_0 e^{kt}$

$\frac{1}{2}Q_0 = Q_0 e^{k(24100)}$

$\frac{1}{2} = e^{24100k}$

$\ln(\frac{1}{2}) = 24100k$

$k = \frac{\ln(\frac{1}{2})}{24100}$

$Q = Q_0 e^{\left[\frac{\ln(\frac{1}{2})}{24100}\right]t}$

$0,4 = Q_0 e^{\left(\frac{\ln(\frac{1}{2})}{24100}\right)(10000)}$

$0,4 = Q_0\ (0,750051852 6)$

$Q_0 = 0,5332964 2\ g$

$\boxed{Q_0 = 0,533g}$

$Q(100) = (0,533296..)e^{\left[\frac{\ln(\frac{1}{2})}{24100}\right](100)}$

$\boxed{Q(100) = 0,532g}$

7. Radioactive radium has a half-life of approximately 1599 years. What percent of a given amount remains after 100 years?

$\boxed{95,758\%}$

Complete the table for a savings account in which interest is compounded continuously.

	Initial Investment	Annual Rate	Time to Double	Amount after 10 yrs
8.	$1000	12%	5,776yrs	$ 3320, 117
9.	$750	8,94%	7.75 yrs	$1834,3705

(8) $A = Pe^{rt}$

$A = 1000e^{.12t}$

$2000 = 1000e^{.12t}$

$2 = e^{.12t}$

$\ln(2) = .12t$

$t = \frac{\ln(2)}{.12} = \boxed{5,776 \text{ yrs}}$

$A(10) = 1000e^{.12(10)} = \boxed{\$3320,117}$

10. One hundred bacteria are started in a culture and the number N of bacteria is counted each hour for 5 hours. The results are shown in the table below, where t is time in hours.

t	0	1	2	3	4	5
N	100	126	151	198	243	297

a) Use the regression capabilities in your graphing calculator to find an exponential model for the data.

t into L1

N into L2

ExpReg L1, L2

$$\boxed{y = 100.1596(1.2455)^t}$$

b) Use the model to estimate the time required for the population to quadruple in size.

$$y(0) = 100.1596$$

$$y(t) = 4(100.1596) = 100.1596(1.2455)^t$$

$$4 = (1.2455)^t$$

$$\log_{(1.2455)}(4) = t$$

$$\frac{\ln(4)}{\ln(1.2455)} = t$$

$$t = 6.3146$$

$$\boxed{t = 6.315 \text{ hrs}}$$

11. The management at a certain factory has found that a worker can produce at most 30 units in a day. The learning curve for the number of units N produced per day after a new employee has worked t days is

$N = 30(1 - e^{kt})$

After 20 days on the job, a particular worker produces 19 units.

a) Find the learning curve for this worker.

$$\boxed{N = 30\left(1 - e^{\left[\frac{\ln\left(\frac{11}{30}\right)}{20}\right]t}\right)}$$

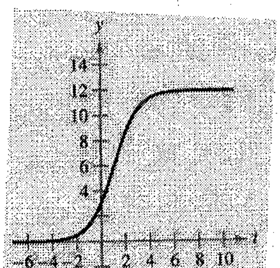
b) How many days should pass before this worker is producing 25 units per day?

$\boxed{\text{36 days}}$

8.4 Worksheet

Match the logistic equation with its graph.

A



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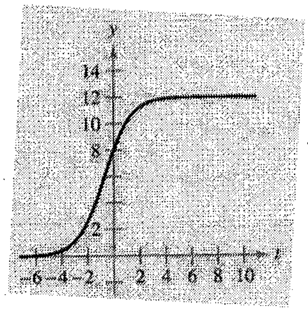

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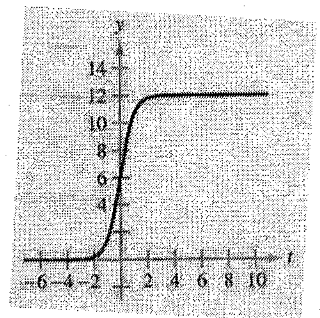

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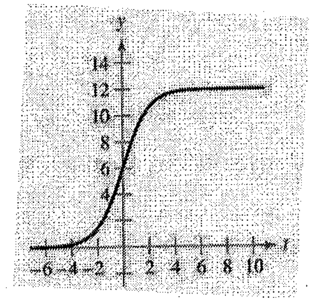


1. $y = \frac{12}{1+e^{-t}}$ $\boxed{D}$

2. $y = \frac{12}{1+3e^{-t}}$ (A)

3. $y = \frac{12}{1+\frac{1}{2}e^{-t}}$ $\boxed{B}$

4. $y = \frac{12}{1+e^{-2t}}$ $\boxed{C}$

(hint: 1st check y intercepts. If same consider how e^{-t} vs e^{-2t} affect the speed of the transition)

The logistic equation models the growth of a population. A) find the value of k, B) find the carrying capacity, C) find the initial population, D) determine when the population will reach 50% of its carrying capacity, and E) write a logistic differential equation that has the solution P(t).

5. $P(t) = \frac{2100}{1+29e^{-0.75t}}$ $\quad P = \frac{L}{1+Ce^{-kt}}$

(a) $\boxed{k = 0.75}$

(b) $\boxed{L = 2100}$

(c) $P(0) = \frac{2100}{1+29e^{0}} = \frac{2100}{30} = \boxed{70}$

(d) $.5(2100) = \frac{2100}{1+29e^{-.75t}}$

$.5(1+29e^{-.75t}) = 1$

$1+29e^{-.75t} = 2$

$29e^{-.75t} = 1$

$e^{-.75t} = \frac{1}{29}$

$-.75t = \ln(\frac{1}{29})$

$t = \frac{\ln(\frac{1}{29})}{-.75} = \boxed{4.4897 \text{ years}}$

(e) form: $\frac{dP}{dt} = kP(1-\frac{P}{L})$

$\boxed{\frac{dP}{dt} = 0.75P\left(1-\frac{P}{2100}\right)}$

6. $P(t) = \frac{6000}{1+4999e^{-0.8t}}$ $P = \frac{L}{1+Ce^{-kt}}$

(a) $\boxed{k=0.8}$

(b) $\boxed{L=6000}$

(c) $P(0) = \frac{6000}{1+4999e^{0}} = \frac{6000}{5000} = \boxed{1.2}$

(d) $.5(6000) = \frac{6000}{1+4999e^{-.8t}}$

$.5(1+4999e^{-.8t}) = 1$

$1+4999e^{-.8t} = 2$

$4999e^{-.8t} = 1$

$e^{-.8t} = \frac{1}{4999}$

$-.8t = \ln\left(\frac{1}{4999}\right)$

$t = \frac{\ln\left(\frac{1}{4999}\right)}{-.8} = \boxed{10.646 \text{ yrs}}$

(e) $\frac{dP}{dt} = kP\left(1-\frac{P}{L}\right)$

$\boxed{\frac{dP}{dt} = 0.8P\left(1-\frac{P}{6000}\right)}$

The logistic differential models the growth rate of a population. A) find the value of k, B) find the carrying capacity, C) ~~use a computer algebra system to graph a slope field~~, Solve if $P(0) = 10$ D) determine the value of P at which the population growth rate is the greatest.

7. $\frac{dP}{dt} = 3P\left(1-\frac{P}{100}\right)$

a) $\boxed{k=3}$

b) $\boxed{L=100}$

c) $\boxed{P = \frac{100}{1+9e^{-3t}}}$

d) $\boxed{50}$

8. $\frac{dP}{dt} = 0.1P - 0.0004P^2$ $\frac{dP}{dt} = kP(1 - \frac{P}{L}) \rightarrow$ solution: $P = \frac{L}{1 + Ce^{-kt}}$

a) $\frac{dP}{dt} = 0.1P(1 - .004P)$

$\frac{dP}{dt} = 0.1P(1 - \frac{P}{250})$ $\boxed{k = 0.1}$

b) $\boxed{L = 250}$

c) $P(0) = 10$

$P = \frac{250}{1 + Ce^{-0.1t}}$

$10 = \frac{250}{1 + Ce^{0}} = \frac{250}{1 + C}$

$1 + C = \frac{250}{10} = 25$

$C = 24$

$\boxed{P = \frac{250}{1 + 24e^{-0.1t}}}$

d) max growth rate

when $P = \frac{1}{2}L$

$P = \frac{1}{2}(250) = \boxed{125}$

Find the logistic equation that satisfies the initial condition. Then use the logistic equation to find y when t = 5 and t = 100.

9. $\frac{dy}{dt} = y\left(1 - \frac{y}{36}\right)$ initial condition (0, 4)

$\boxed{y = \frac{36}{1 + 8e^{-t}}}$

$\boxed{y(5) = 34.1587}$

$\boxed{y(100) = 36}$

10. $\frac{dy}{dt} = \frac{4y}{5} - \frac{y^2}{150}$ initial condition (0, 8)

$$\frac{dy}{dt} = \frac{4}{5}y\left(1 - \frac{\left(\frac{1}{150}y^2\right)}{\left(\frac{4}{5}y\right)}\right)$$

$$\frac{dy}{dt} = \frac{4}{5}y\left(1 - \frac{y}{120}\right) \quad k = \frac{4}{5}, \ L = 120$$

general solution: $y = \frac{L}{1+Ce^{-kt}} = \frac{120}{1+Ce^{-\frac{4}{5}t}}$

$y(0) = 8$

$$8 = \frac{120}{1+Ce^{-\frac{4}{5}(0)}} = \frac{120}{1+C}$$

$$1+C = \frac{120}{8} = 15$$

$$C = 14$$

particular solution: $\boxed{y = \frac{120}{1+14e^{-\frac{4}{5}t}}}$

$\boxed{y(5) = 95.5095 \quad y(100) = 120}$

11. Describe what the value L represents in the logistic differential equation

$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$

(hint: look up the definition of carrying capacity)

12. It is known that $y = \frac{L}{1+be^{-kt}}$ is a solution of the logistic equation $\frac{dy}{dt} = 0.75y\left(1 - \frac{y}{2500}\right)$.

Is it possible to determine L, k, and b from the given information? If not, which value(s) cannot be determined and what information do you need to determine the value(s)?

$\boxed{L = 2500}$

$\boxed{k = 0{,}75}$

... But to get b, we would need to be given an initial condition,

13. A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

a) Write a logistic equation that models the population of panthers in the preserve.

b) Find the population after 5 years.

c) When will the population reach 100?

d) Write a logistic differential equation that models the growth rate of the panther population. Then repeat part b using Euler's method of approximation with a step size of h = ~~1~~ 0,5. Compare the approximation with the exact number.

e) After how many years is the panther population growing most rapidly? Explain.

a) $\boxed{P = \frac{200}{1+7e^{-,264033t}}}$

b) $\boxed{69{,}695 \approx 70 \text{ panthers}}$

c) $\boxed{7{,}37 \text{ years}}$

d) $\boxed{P(5) \approx 68.6025}$ (close to exact value in b) (I used h = 0,5 to reduce number of iterations)

e) $\boxed{7{,}37 \text{ years}}$

8.4 #13, part d:

solution is $P = \frac{200}{1+7e^{-.2640337t}}$, $k = .2640337$, $L = 200$

corresponding DE is $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$

$$\boxed{\frac{dP}{dt} = (.2640337)P\left(1 - \frac{P}{200}\right)} \approx .2640P\left(1 - \frac{P}{200}\right)$$

last known population is 39 at $t = 2$ yrs

if we use $h = .1$ to go from $t = 2$ to $t = 5$ that will take $\frac{5-2}{.1} = 30$ iterations!

(Let's use $h = .5$ to be more reasonable)

(t, P)	$P_{n+1} = P_n + h\left[\frac{dP}{dt}\right] = P_n + (0.5)\left[(.2640)P\left(1 - \frac{P}{200}\right)\right]$
$(2, 39)$	$P = 39 + (0.5)[8.28828] = 43.14414$
$(2.5, 43.14414)$	$P = 43.14414 + (0.5)[8.93298276] = 47.61063$
$(3, 47.61063)$	$P = 47.61063 + (0.5)[9.577067] = 52.399165$
$(3.5, 52.399165)$	$P = 52.399165 + (0.5)[10.206578] = 57.502454$
$(4, 57.502454)$	$P = 57.502454 + (0.5)[10.8160253] = 62.9104667$
$(4.5, 62.9104667)$	$P = 62.9104667 + (0.5)[11.3841638] = 68.6025$
$(5, 68.60254)$	

$\boxed{P(5) \approx 68.6025}$ (actual was 69.695)

still $\approx$ 70 panthers

14. For any logistic growth curve, show that the point of inflection occurs at y = L/2 when the solution starts below the carrying capacity L.

$$\frac{dP}{dt} = kP\left(1-\frac{P}{L}\right) = kP - \frac{k}{L}P^2$$

$$\frac{d^2P}{dt^2} = \frac{d}{dt}\left[kP - \frac{k}{L}P^2\right]$$

$$\frac{d^2P}{dt^2} = k\frac{dP}{dt} - \frac{2k}{L}P\frac{dP}{dt}$$

$$= \left(k - \frac{2k}{L}P\right)\frac{dP}{dt}$$

$$= \left(k - \frac{2k}{L}P\right)\left[kP - \frac{k}{L}P^2\right]$$

$$= k^2P - \frac{k^2}{L}P^2 - \frac{2k^2}{L}P^2 + \frac{2k^2}{L^2}P^3$$

$$= k^2P - \frac{3k^2}{L}P^2 + \frac{2k^2}{L^2}P^3$$

inflection point when 2nd deriv = 0

$$\frac{2k^2}{L^2}P^3 - \frac{3k^2}{L}P^2 + k^2P = 0$$

$$k^2P\underbrace{\left(\frac{2}{L^2}P^2 - \frac{3}{L}P + 1\right)}_{\text{quadratic}} = 0$$

$P = 0$

(trivial solution)

disregard

$$P = \frac{-\left(-\frac{3}{L}\right) \pm \sqrt{\left(\frac{3}{L}\right)^2 - 4\left(\frac{2}{L^2}\right)(1)}}{2\left(\frac{2}{L^2}\right)}$$

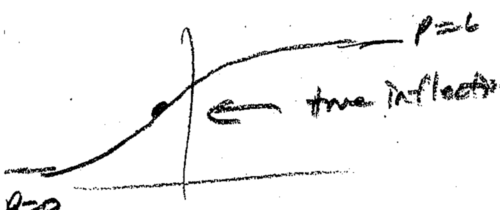
$$= \frac{\frac{3}{L} \pm \sqrt{\frac{1}{L^2}}}{\frac{4}{L^2}} = \frac{\frac{3}{L} \pm \frac{1}{L}}{\frac{4}{L^2}}\left(\frac{L^2}{L^2}\right) = \frac{L(3\pm1)}{4}$$

$$= L\,\frac{3+1}{4} = L \quad \text{or} \quad = L\,\frac{3-1}{4} = \frac{1}{2}L$$

2nd deriv = 0 at $P = 0$,

$P = \frac{1}{2}L$,

$P = L$



← true inflection pt at $\boxed{P = \frac{1}{2}L}$

Calculus 2 - Unit 8 Test REVIEW

Solve the differential equation by separation of variables:

#1. $\frac{dy}{dx} = xy^2$ if $y(1) = 3$.

#2. $\frac{dy}{dx} = (2+x)y^2$ if $y(2) = 4$.

#3. $xyy' = 3 + x^2$ if $y(1) = 2$.

#4. $\frac{dP}{dt} = kP$ if $P(0) = 20$.

#5. $\frac{dy}{dt} = 0.5y - 0.004y^2$ if $y(0) = 30$.

Exponential Growth/Decay problems:

#6. When a child was born, her grandparents place $1000 in a savings account which earns 12% annual interest compounded continuously.

(a) How much money is in the account when the child is 20 years old?
(b) At what time had the $1000 doubled to $2000?

#7. A radioactive sample which contained 50 g of mass at time $t = 0$ decays exponentially. After 75 days, the mass of the sample has decreased to 30 g.

(a) Write a differential equation which models this scenario.
(b) Solve the differential equation to write remaining mass as a function of time.
(c) What mass remains at $t = 100$ days?
(d) At what time was the initial mass reduced by 20%?

#8. A rabbit population with an initial size of 500 rabbits grows at a rate proportional to its size.

(a) Write a differential equation which models this scenario.
(b) Solve the differential equation to write the population of rabbits as a function of time, if there were 1200 rabbits at $t = 10$ days.
(c) How many rabbits will there be at $t = 50$ days ?
(d) When was the rabbit population 900 rabbits?

#9. A large manufacturing firm's training department has found that the number of minutes it takes assembly line workers to build a product decreases over time as the employees become more experienced, and that the time to build a product (in minutes) is inversely proportional to the number of days they have been building this product.

(a) Write a differential equation which models this scenario.
(b) Solve the differential equation to write the number of minutes to build a product as a function of days of experience, if it initially (in this case, the first day when t=1) takes employees 45 minutes to build the product, but they can build the product in 10 minutes when they have 30 days of experience.
(c) How many minutes does it take an employee to build the product after 10 days of experience?
(d) How many days of experience will it take for an employee to build the product in only 6 minutes?

Logistic equation problems:

#10. A virus spreads throughout a population according to the logistic differential equation $\frac{dy}{dt} = 0.5y - 0.004y^2$ where y is the number of people in the community who have been infected with the virus and t is the time in days.

(a) What is the size of this population?
(b) If 1 person is initially infected at $t = 0$, find the solution for the differential equation.
(c) At the time when the virus is spreading most quickly, how many people have caught the virus?
(d) To the nearest day, how many days will it take for half of the population to be infected?
(e) Draw and label a sketch of the graph of y as a function of time. (Label any significant parts)

#11. The rate of growth in a population of rabbits in a forest is proportional to current number of rabbits, but there is a limit in the food supply, so it cannot grow without bound. The carrying capacity of the forest is 40000 rabbits.

(a) What is the logistic form differential equation which models this situation?
(b) If there are 500 rabbits at $t = 0$, and 2000 rabbits at $t = 10$ days, find the solution for the differential equation.
(c) To the nearest day, how many days will it take for the population of rabbits to increase to 20000?
(d) At what time is the population of rabbits growing most rapidly?
(e) Draw and label a sketch of the graph of rabbit population as a function of time.

Euler's Method approximations:

#12. Given $\frac{dy}{dx} = 2x + 3y$ and $y(1) = 3$

use Euler's method to approximate $y(2)$ using increments of 0.2

Sketch slope fields:

#13. Sketch a slope field in quadrants 1 and 2 for the differential equation $\frac{dy}{dx} = -x + 2y$

(include lineal elements for at least 15 points in your sketch)

Unit 8 Test Review

(1) $\frac{dy}{dx} = xy^2 \quad y(1)=3$

$\frac{1}{y^2}dy = x\,dx$

$\int y^{-2}dy = \int x\,dx$

$\frac{y^{-1}}{-1} = \frac{1}{2}x^2 + c$

$-\frac{1}{y} = \frac{1}{2}x^2 + c$

$-\frac{1}{(3)} = \frac{1}{2}(1)^2 + c$

$\left(-\frac{1}{3} = \frac{1}{2} + c\right)6$

$-2 = 3 + 6c$

$6c = -5, \quad c = -5/6$

$-\frac{1}{y} = \frac{1}{2}x^2 - \frac{5}{6}$

$\boxed{y(x) = \frac{-1}{\left(\frac{1}{2}x^2 - \frac{5}{6}\right)}}$

---

(2) $\frac{dy}{dx} = (2+x)y^2 \quad y(2)=4$

$\frac{1}{y^2}dy = (2+x)dx$

$\int y^{-2}dy = \int (2+x)dx$

$\frac{y^{-1}}{-1} = 2x + \frac{1}{2}x^2 + C$

$-\frac{1}{y} = 2x + \frac{1}{2}x^2 + C$

$-\frac{1}{(4)} = 2(2) + \frac{1}{2}(2)^2 + C$

$-\frac{1}{4} = 4 + 2 + C$

$-\frac{1}{4} = 6 + C$

$-1 = 24 + 4C, \quad 4C = -25, \quad C = -\frac{25}{4}$

$-\frac{1}{y} = 2x + \frac{1}{2}x^2 - \frac{25}{4}$

$\boxed{y(x) = \frac{-1}{\left(2x + \frac{1}{2}x^2 - \frac{25}{4}\right)}}$

---

(3) $xy\overset{dy/dx}{y'} = 3 + x^2 \quad y(1) = 2$

$y\frac{dy}{dx} = \frac{3+x^2}{x}$

$y\,dy = \left(\frac{3}{x} + x\right)dx$

$\int y\,dy = \int\left(\frac{3}{x} + x\right)dx$

$\frac{y^2}{2} = 3\ln|x| + \frac{1}{2}x^2 + C$

$\frac{(2)^2}{2} = 3\cancel{\ln|1|}^{0} + \frac{1}{2}(1)^2 + c$

$2 = \frac{1}{2} + c$

$4 = 1 + 2c$

$3 = 2c$

$c = \frac{3}{2}$

$\frac{y^2}{2} = 3\ln|x| + \frac{1}{2}x^2 + \frac{3}{2}$

$y^2 = 6\ln|x| + x^2 + 3$

$y = \pm\sqrt{6\ln|x| + x^2 + 3}$

$y(1) = 2$ is $+$

so

$\boxed{y = \sqrt{6\ln|x| + x^2 + 3}}$

(4) $\frac{dP}{dt} = kP \quad P(0) = 20$

$\frac{1}{P} dP = k\,dt$

$\int \frac{1}{P} dP = \int k\,dt$

$\ln|P| = kt + C_1$

$\ln|20| = k(0) + C_1$

$C_1 = \ln|20|$

$\ln|P| = kt + \ln|20|$

$P = e^{(kt + \ln|20|)} = e^{kt} e^{\ln|20|} = e^{kt}(20)$

$\boxed{P = 20e^{kt}}$

the initial population

so $\boxed{P(t) = P_0 e^{kt}}$ in general (memorize this!)

---

(5) $\frac{dy}{dt} = 0.5y - 0.004y^2 \quad y(0) = 30$

$\frac{dy}{dt} = 0.5\left(y - \frac{.004}{0.5}y^2\right) = 0.5\left(y - \frac{1}{125}y^2\right)$

$\frac{1}{y - \frac{1}{125}y^2} dy = 0.5\,dt$

$\int \frac{1}{y(1 - \frac{1}{125}y)} dy = \int 0.5\,dt \quad \ldots (1) \int \frac{1}{y} dy + \left(\frac{1}{125}\right) \int \frac{1}{1 - \frac{1}{125}y} dy = \int 0.5\,dt$

partial fractions:

$\frac{1}{y(1 - \frac{1}{125}y)} = \frac{A}{y} + \frac{B}{1 - \frac{1}{125}y}$

$A\left(1 - \frac{1}{125}y\right) + By = 1$

$\left(-\frac{1}{125}A + B\right)y + A = (0)y + 1$

system: $\begin{cases} -\frac{1}{125}A + B = 0 \\ A = 1 \end{cases}$

$A = 1$

$B = \frac{1}{125}$

$u = 1 - \frac{1}{125}y$

$\frac{du}{dy} = -\frac{1}{125}$

$du = -\frac{1}{125} dy$

$dy = -125\,du$

$\ln|y| + \left(\frac{1}{125}\right)(-125) \underbrace{\int \frac{1}{u} du}_{\ln|1 - \frac{1}{125}y|} = 0.5t + C_1$

log properties...

$\ln\left|\frac{y}{1 - \frac{1}{125}y}\right| = 0.5t + C_1$

$\frac{y}{1 - \frac{1}{125}y} = e^{(0.5t + C_1)} = C_2 e^{0.5t}$

$y = C_2 e^{0.5t}\left(1 - \frac{1}{125}y\right) = C_2 e^{0.5t} - C_2 e^{0.5} \frac{1}{125} y$

$y\left(1 + C_2 e^{0.5} \frac{1}{125}\right) = C_2 e^{0.5t}$

$y = \frac{C_2 e^{0.5t}}{1 + C_2 e^{0.5} \frac{1}{125}}$ $\longrightarrow$ on next page

5 continued

$$y = \frac{C_2 e^{0.5t}}{1 + C_2 e^{0.5t} \cdot \frac{1}{125}} = \frac{C_2 e^{0.5t}\, 125}{125 + C_2 e^{0.5t}} \left(\frac{e^{-0.5t}}{e^{-0.5t}} \right)$$

$$y = \frac{125 C_2}{125 e^{-0.5t} + C_2} \left(\frac{\frac{1}{C_2}}{\frac{1}{C_2}} \right)$$

$$y = \frac{125}{\frac{125}{C_2} e^{-0.5t} + 1}$$ define $C = \frac{125}{C_2}$ (one final constant)

$$\boxed{y = \frac{125}{1 + C e^{-0.5t}}}$$ ← 125 is the "carrying capacity"

memorize these forms!

if you encounter $\frac{dy}{dt} = 0.5y - 0.004y^2$

rewrite: $\frac{dy}{dt} = 0.5y \left(1 - \frac{0.004}{0.5} y\right) = 0.5y \left(1 - \frac{y}{125}\right)$ ← carrying capacity

exponential constant

memorized solution form: $y(t) = \frac{125}{1 + C e^{-0.5t}}$ ← (don't forget the negative)

⑥ $\frac{dA}{dt} = kA$

$\int \frac{1}{A} dA = \int k\,dt$

$\ln|A| = kt + C_1$

$A = e^{(kt+C_1)} = e^{kt} \underbrace{e^{C_1}}_{C} = Ce^{kt}$

$A = Pe^{rt}$ (P = principle)
(r = annual interest rate)

$A = Pe^{rt}$ ← (should memorize)

$A = 1000e^{.12t}$

(a) $A(20) = 1000e^{.12(20)} = \boxed{\$11023.18}$

(b) $2000 = 1000e^{.12t}$

$e^{.12t} = \frac{2000}{1000} = 2$

$.12t = \ln(2)$

$t = \frac{\ln(2)}{.12} = \boxed{5.7762 \text{ yrs}}$

---

⑦ radioactivity solution is also of form $Q = Q_0 e^{kt}$

(a) $\boxed{\frac{dQ}{dt} = kQ}$

(b) $Q = Q_0 e^{kt}$

$Q = 50e^{kt}$

$30 = 50e^{k(75)}$

$e^{75k} = \frac{30}{50}$

$75k = \ln(\frac{30}{50})$

$k = \frac{\ln(\frac{30}{50})}{75} = -0.0068110083$

$\boxed{Q = 50e^{-0.0068110083t}}$

(c) $Q(100) = 50e^{-0.0068110083(100)} = \boxed{25.30298\text{g}}$

(d) reduced by 20% means 80% remains $0.8(50)$

$0.8(50) = 50e^{-0.0068110083t}$

$e^{-0.0068110083t} = \frac{0.8(50)}{50} = 0.8$

$t = \frac{\ln(0.8)}{-0.0068110083} = \boxed{32.76219 \text{ days}}$

(8) (a) $\boxed{\frac{dP}{dt} = kP}$

(b) $P = P_0 e^{kt}$

$P = 500 e^{kt}$

$1200 = 500 e^{k(10)}$

$e^{10k} = \frac{1200}{500}$

$10k = \ln\left(\frac{1200}{500}\right)$ $k = \frac{\ln\left(\frac{1200}{500}\right)}{10} = 0.0875468737$

$\boxed{P(t) = 500 e^{0.0875468737t}}$

(c) $P(50) = 500 e^{0.0875468737(50)} = 3981.312$

$\boxed{3981 \text{ rabbits}}$

(d) $900 = 500 e^{0.0875468737t}$

$t = \frac{\ln\left(\frac{900}{500}\right)}{0.0875468737} = \boxed{\cancel{3.501} \text{ days}}$

6.714

---

(9) (a) M = minutes to build

t = days experience

$\boxed{\frac{dM}{dt} = k \frac{1}{t}}$

(b) $\frac{1}{k} dM = \frac{1}{t} dt$

$\frac{1}{k} \int 1 \, dM = \int \frac{1}{t} dt$

$\frac{1}{k} M = \ln|t| + C$

$\frac{1}{k}(45) = \ln|1| + C$ $C = \frac{45}{k}$

$\frac{1}{k} M = \ln|t| + \frac{45}{k}$

$M = k \ln|t| + 45$ now $M = 10$ when $t = 30$

$10 = k \ln|30| + 45$

$k \ln|30| = -35$

$k = \frac{-35}{\ln|30|} = -10.29049363$

$\boxed{M(t) = (-10.29049363) \ln|t| + 45}$

(c) $M(10) = -10.29049363 \ln|10| + 45$

$= \boxed{21.30526 \text{ minutes}}$

(d) $6 = -10.29049363 \ln|t| + 45$

$-10.29049363 \ln|t| = -39$

$\ln|t| = \frac{-39}{-10.29049363} = 3.789905655$

$t = e^{3.789905655} = \boxed{44.2522 \text{ days}}$

(10) $\frac{dy}{dt} = 0.5y - 0.004y^2$ (see #5)

$\frac{dy}{dt} = 0.5y\left(1 - \frac{y}{125}\right)$ carrying capacity $= 125$
exp. constant $= 0.5$

(a) size of population = carrying capacity = $\boxed{125}$

(b) solution (memorized form): $y(t) = \frac{125}{1+Ce^{-0.5t}}$ $y(0) = 1$

$$1 = \frac{125}{1+Ce^{-0.5(0)}} = \frac{125}{1+C}$$

$$1+C = 125$$

$$C = 124$$

$$\boxed{y(t) = \frac{125}{1+124e^{-0.5t}}}$$

(c) logistic equations have max rate of increase at half carrying capacity (memorized fact)

So when $\frac{125}{2} = 62.5$ $\boxed{(62) \text{ people}}$

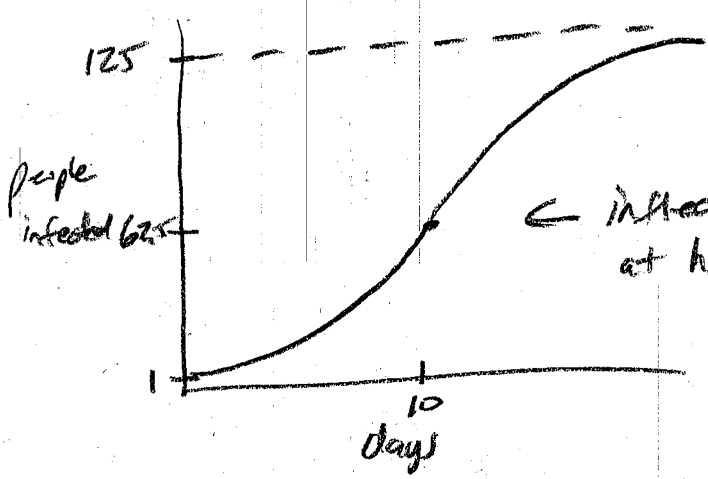
(d) $62.5 = \frac{125}{1+124e^{-0.5t}}$

$1+124e^{-0.5t} = \frac{125}{62.5} = 2$ → $124e^{-0.5t} = 1$

$e^{-0.5t} = \frac{1}{124}$

$-0.5t = \ln\left(\frac{1}{124}\right)$, $t = \frac{\ln\left(\frac{1}{124}\right)}{-0.5} = 9.164$ $\boxed{10 \text{ days}}$

(e)



← inflection point at half carrying capacity

(11) unlike #8 (unlimited growth, $P = P_0 e^{kt}$) now we have growth limited by environment, logistic growth

(a) carrying capacity $= 40000$

D.E. form: $\boxed{\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{40000}\right)}$

(b) Solution form: $P(t) = \dfrac{40000}{1 + Ce^{kt}}$ 2 data points:

t	P
0	500
10	2000

use $P(0) = 500$:

$500 = \dfrac{40000}{1 + Ce^{k(0)}} = \dfrac{40000}{1 + C}$

$500 + 500C = 40000$

$C = \dfrac{39500}{500} = 79$

$P = \dfrac{40000}{1 + 79e^{kt}}$

now use $P(10) = 2000$:

$2000 = \dfrac{40000}{1 + 79e^{k(10)}}$

$2000 + 2000(79)e^{10k} = 40000$

$2000(79)e^{10k} = 38000$

$e^{10k} = \dfrac{38000}{2000(79)} = \dfrac{19}{79}$

$10k = \ln\left(\frac{19}{79}\right)$, $k = \dfrac{\ln\left(\frac{19}{79}\right)}{10} = -0.1425008873$

$\boxed{P(t) = \dfrac{40000}{1 + 79e^{-0.1425008873t}}}$

(c) $20000 = \dfrac{40000}{1 + 79e^{-0.1425 \cdot t}}$

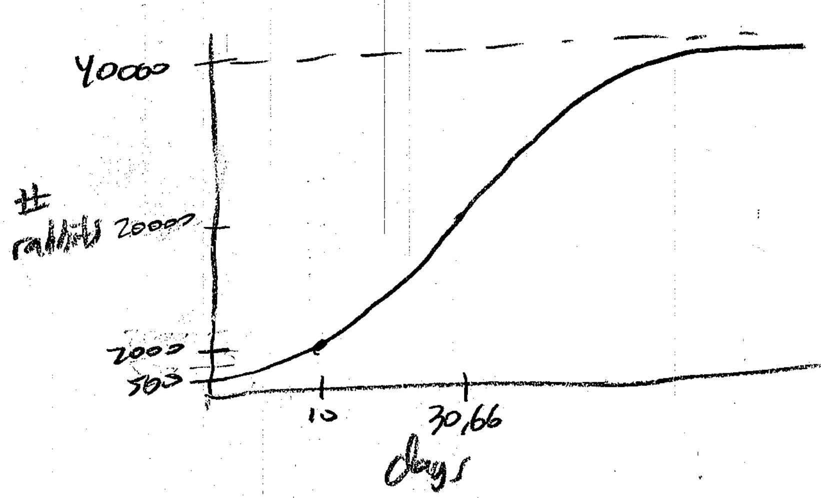
$1 + 79e^{-0.1425t} = \dfrac{40000}{20000} = 2$

$79e^{-0.1425t} = 1$

$t = \dfrac{\ln\left(\frac{1}{79}\right)}{-0.1425008873} = \boxed{30.6626 \text{ days}}$

(d) occurs when at half carrying capacity, which is 20000 so $\boxed{30.6626 \text{ days}}$

(e)



(12) $\frac{dy}{dx} = 2x+3y \quad y(1)=3$

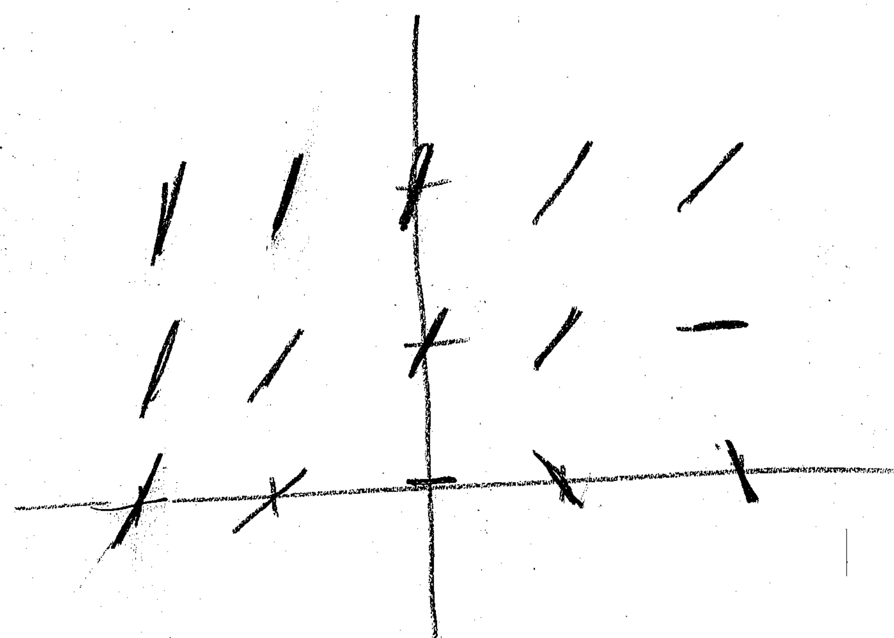
x	y	$\Delta y = (2x+3y)\Delta x$
1	3	$(2(1)+3(3))(0.2) = 2.2$
1.2	5.2	$(2(1.2)+3(5.2))(0.2) = 3.6$
1.4	8.8	$(2(1.4)+3(8.8))(0.2) = 5.84$
1.6	14.64	$(2(1.6)+3(14.64))(0.2) = 9.424$
1.8	24.064	$(2(1.8)+3(24.064))(0.2) = 15.1584$
2	39.2224	

$\boxed{y(2) \approx 39.2224}$

---

(13) $\frac{dy}{dx} = -x+2y$

(x,y)	$\frac{dy}{dx} = -x+2y$
$(0,0)$	$-0+2(0)=0$
$(0,1)$	$-0+2(1)=2$
$(0,2)$	$-0+2(2)=4$
$(1,0)$	$-1+2(0)=-1$
$(1,1)$	$-1+2(1)=1$
$(1,2)$	$-1+2(2)=3$
$(2,0)$	$-2+2(0)=-2$
$(2,1)$	$-2+2(1)=0$
$(2,2)$	$-2+2(2)=2$
$(-1,0)$	$-(-1)+2(0)=1$
$(-1,1)$	$-(-1)+2(1)=3$
$(-1,2)$	$-(-1)+2(2)=5$
$(-2,0)$	$-(-2)+2(0)=2$
$(-2,1)$	$-(-2)+2(1)=4$
$(-2,2)$	$-(-2)+2(2)=6$



(very rough sketch is fine :))