AP Calculus BC

Name: answers/solutions

4.1 Worksheet (odds and 14)

Find the value of the derivative (if it exists) at each indicated extremum. Check your answers by graphing the function in your calculator and comparing the slope of the tangent line at these points.

1. $f(x) = \frac{x^2}{x^2+4}$ at (0,0)

$f'(0) = \boxed{0}$

(can use derivative shortcuts)

2. $f(x) = 4 - |x|$ at (0,1)

$|x|$, can't use shortcuts $\Rightarrow$ limit definition

LH: $f'(0) = \lim\limits_{x \to 0^-} \frac{[4-|x|]-[4-|0|]}{x-0}$ for $x<0$, $|x| = -x$

$f'(0) = \lim\limits_{x \to 0^-} \frac{[4-(-x)]-4}{x} = \lim\limits_{x \to 0^-} \frac{x}{x} = \lim\limits_{x \to 0^-} (1) = \underline{\underline{1}}$

RH: $f'(0) = \lim\limits_{x \to 0^+} \frac{[4-|x|]-[4-|0|]}{x}$ for $x>0$, $|x| = x$

$f'(0) = \lim\limits_{x \to 0^+} \frac{[4-x]-[4]}{x} = \lim\limits_{x \to 0^+} \frac{-x}{x} = \lim\limits_{x \to 0^+} (-1) = \underline{\underline{-1}}$

$\neq$ so $\boxed{f'(0) \text{ DNE}}$

Find the critical numbers of the given function.

3. $g(t) = t\sqrt{4-t}$ $(t < 3)$

$\boxed{t = 8/3}$

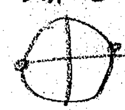
4. $h(x) = \sin^2 x + \cos x$ $(0 < x < 2\pi)$

$h'(x) = 2\sin x \cos x - \sin x$

$\underline{h'(x) = 0}$

$\sin x(2\cos x - 1) = 0$

$\sin = 0$

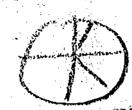


$x = 0, \pi$

not in interval

$2\cos x - 1 = 0$

$\cos x = 1/2$



$x = \pi/3, 5\pi/3$

$\underline{h'(x) \text{ DNE}}$

N/A

$\boxed{x = \pi/3, \pi, 5\pi/3}$

5. $f(x) = x^2 \log_2(x^2+1)$

$\boxed{x = 0}$

6. $g(t) = 2t\ln t$

$g'(t) = 2t\left(\frac{1}{t}\right) + \ln t(2) = 2 + 2\ln t$

$\underline{g'(t) = 0}$

$2 + 2\ln t = 0$

$2\ln t = -2$

$\ln t = -1$

$t = e^{-1}$

$t = \frac{1}{e}$

$\underline{g'(t) \text{ DNE}}$

for $t \leq 0$

but that is because $t \leq 0$ is not in domain of $g(t)$

$\boxed{t = \frac{1}{e}}$

Find the absolute extrema of the function on the closed interval.

7. $f(x) = x^3 - \frac{3}{2}x^2, \quad [-1, 2]$

f has an absolute minimum at $(-1, -2.5)$

and an absolute max at $(2, 2)$

over $[-1, 2]$.

8. $y = 3x^{2/3} - 2x, \quad [-1, 1]$

$y' = 2x^{-1/3} - 2$ $\quad$ $\underline{y' = 0}$ $\quad$ $\underline{y' \text{ DNE}}$

$\frac{2}{\sqrt[3]{x}} - 2 = 0$ $\quad$ $x = 0$

$\frac{2}{\sqrt[3]{x}} = 2$

$\sqrt[3]{x} = 1$

$x = 1$

test:

x	f(x)
0	0
1	1
-1	5

y has an absolute minimum of $(0, 0)$

and an absolute maximum of $(-1, 5)$

over $[-1, 1]$

9. $f(x) = \sin x, \quad [\frac{5\pi}{6}, \frac{11\pi}{6}]$

f has an absolute minimum of $(\frac{3\pi}{2}, -1)$

and an absolute maximum of $(\frac{5\pi}{6}, \frac{1}{2})$

over $[\frac{5\pi}{6}, \frac{11\pi}{6})$

10. $f(x) = 3\cos x, \quad [0, 2\pi]$

$f'(x) = -3\sin x$ $\quad$ $\underline{f'(x) = 0}$ $\quad$ $\underline{f'(x) \text{ DNE}}$

$-3\sin x = 0$ $\quad$ N/A

$x = 0, \pi$

test:

x	f(x)
0	3
π	-3
2π	3

f has an absolute minimum at $(\pi, -3)$

and absolute maxima at $(0, 3)$ and $(2\pi, 3)$

over $[0, 2\pi]$

Use a graphing calculator to graph the function and find the absolute extrema of the function on the given interval.

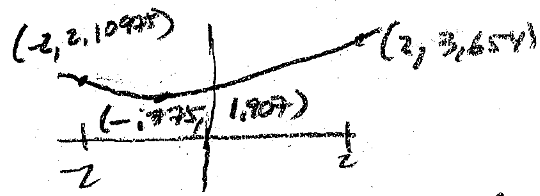
11. $f(x) = \frac{3}{x-1}, \quad (1, 4]$

f has an absolute min of $(4, 1)$

but no absolute maxima

over $(1, 4]$.

12. $f(x) = \sqrt{x+4}\,e^{(x^2/10)}, \quad [-2, 2]$



f has an absolute max at $(2, 3.654)$

and an absolute min at $(-0.775, 1.907)$

over $[-2, 2]$

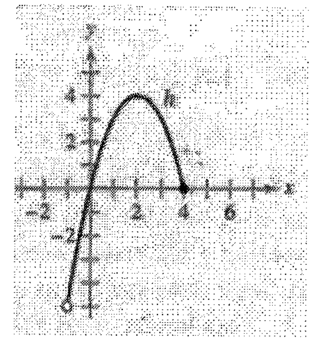
13. (MULTIPLE-CHOICE) Which is the following statements is true about the graph of h?

(A) h has a minimum at (-1, -5) and a maximum at (2, 4).

(B) h has an absolute minimum at (4, 0) and a maximum at (2, 4).

(C) h has a maximum at (2, 4).

(D) h has no extrema.



14. (FREE-RESPONSE) The function f is defined $f(x) = \frac{(4\ln x)}{x^3}$ for all x > 0.

(a) Find $f'(x)$.

(b) Find an equation of the tangent line to the graph of f at $x = e$.

(c) Find the critical numbers of f. Does the function have a relative maximum, a relative minimum, or neither at each critical number?

(d) Find $\lim_{x\to 0^+} f(x)$.

(a) $f'(x) = \frac{4(1-3\ln x)}{x^4}$

(b) $\left(y - \frac{4}{e^3}\right) = \frac{-8}{e^4}(x - e)$

(c) f has a relative maximum at $x = e^{1/3}$
(f has neither a max nor min at $x=0$
because $x=0$ is not in the domain of $f(x)$)

(d) $\lim_{x \to 0^+} f(x) = -\infty$ (or DNE)

15. (MULTIPLE-CHOICE) The critical numbers for $f(x) = x^2(3x-1)^3$ are:

(A) $x = 0$ and $x = \frac{1}{3}$

(B) $x = 0$ and $x = \frac{2}{15}$

(C) $x = \frac{2}{15}$ and $x = \frac{1}{3}$

(D) $x = 0$, $x = \frac{2}{15}$ and $x = \frac{1}{3}$

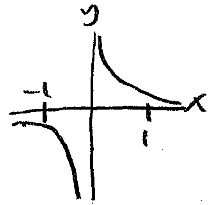
4.2 Worksheet (odds and #16)

Explain why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b)$.

1. $f(x) = 1 - |x - 1|, \quad [0, 2]$

f is not differentiable at $x=0$

2. $f(x) = \left|\frac{1}{x}\right|, \quad [-1, 1]$



f is not differentiable on (-1,1) (at $x=0$)

Determine whether Rolle's Theorem can be applied to f on [a,b]. If Rolle's Theorem can be applied, find all values of c in the (a,b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.

3. $f(x) = (x-1)(x-2)(x-3), \quad [1, 3]$

$\boxed{c = 1.577}$

4. $f(x) = -x^2 + 3x, \quad [0, 3]$

f(x) is a polynomial so f(x) is continuous and differentiable on (0,3)

$f(0)=0$, $f(3)=0$, so $f(0)=f(3)$

Rolle's Theorem applies

$f'(x) = -2x+3 = 0$

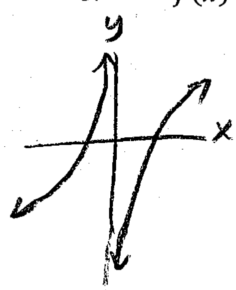
when $x = \boxed{c = \frac{3}{2}}$

5. $f(x) = \tan x, \quad [0, \pi]$

f is not differentiable at $x=\pi/2$

so Rolle's theorem does not apply

6. $f(x) = \frac{x^2-1}{x}, \quad [-1, 1]$



f is not differentiable at $x=0$

so Rolle's theorem does not apply

Use a graphing calculator to graph the function on [a,b]. Determine whether Rolle's Theorem can be applied, and, if so, find all values of c in (a,b) such that $f'(c) = 0$.

7. $f(x) = 2 + \arcsin(x^2 - 1), \quad [-1, 1]$

f is not differentiable at $x=0$

so Rolle's Theorem does not apply.

8. $f(x) = \frac{x}{2} - \sin\frac{\pi x}{6}, \quad [-1, 0]$

f(x) is continuous on (-1,0), $f(-1)=0$, $f(0)=0$

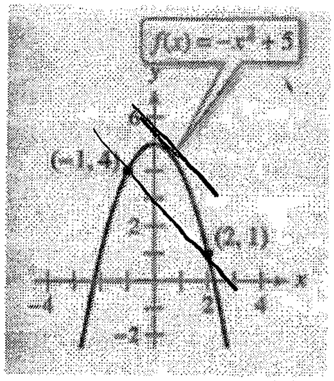
$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi x}{6}\right)$ is also continuous so f is differentiable on (-1,0)

$\frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi x}{6}\right) = 0$

(by calculator graph intercept)

when $x = \boxed{c = -0.576}$

9. Consider the graph of the function $f(x) = -x^2 + 5$:



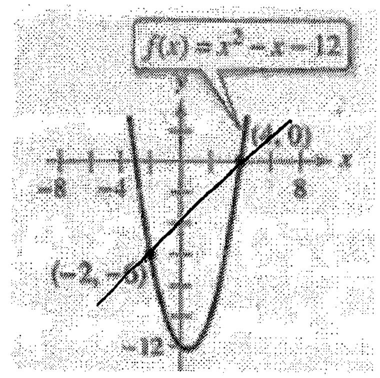
a) Find the equation of the secant line joining the points (-1, 4) and (2, 1).
b) Use the Mean Value Theorem to determine a point c in the interval (-1, 2) such that the tangent line at c is parallel to the secant line.
c) Find the equation of the tangent line through c.

a) $y = -x + 5$

b) $c = \frac{1}{2}$

c) $\left(y - \frac{19}{4}\right) = -\left(x - \frac{1}{2}\right)$

10. Consider the graph of $f(x) = x^2 - x - 12$:



a) Find the equation of the secant line joining the points (-2, -6) and (4, 0).
b) Use the Mean Value Theorem to determine a point c in the interval (-2, 4) such that the tangent line at c is parallel to the secant line.
c) Find the equation of the tangent line through c.

(a) $= m = \frac{-6-0}{-2-4} = \frac{-6}{-6} = 1$

$(y-0) = (x-4)$, $\boxed{y = x-4}$

(b) f(x) is a polynomial, continuous and differentiable on (-2, 4).
So Mean Value Theorem applies

$m_{secant} = 1$

$f'(x) = 2x - 1 = 1$

$2x = 2$

at $x = 1$, so $\boxed{C = 1}$

(c) $f(1) = (1)^2 - (1) - 12 = -12$

$\boxed{(y+12) = 1(x-1)}$

Determine whether the Mean Value Theorem can be applied to f on [a,b] and if it can be applied, find all values of c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

11. $f(x) = x\log_2 x$, [1, 2]

$\boxed{c = \frac{4}{e}}$

12. $f(x) = (x+3)\ln(x+3)$, [-2, -1]

$f(x)$ is continuous on $(-2,-1)$

$f'(x) = (x+3)\frac{1}{x+3}(1) + \ln(x+3)(1) = 1 + \ln(x+3)$

also continuous on $(-2,-1)$ so f is differentiable on $(-2$

Mean Value Theorem applies

$f(-2) = (-2+3)\ln(-2+3) = (1)\ln(1) = 0$

$f(-1) = (-1+3)\ln(-1+3) = 2\ln 2$

need $f'(c) = \frac{2\ln 2 - 0}{-1-(-2)} = \frac{2\ln 2}{1} = 2\ln 2$

$f'(x) = 1 + \ln(x+3) = 2\ln 2$

$\ln(x+3) = 2\ln 2 - 1 = \ln(2^2) - 1 = \ln(4) - 1$

e $\quad x+3 = e^{\ln 4 - 1} = e^{\ln 4}e^{-1} = 4e^{-1} = \frac{4}{e}$

$x = \boxed{c = \frac{4}{e} - 3}$

Use a graphing calculator to (a) graph the function f on [a,b], (b) find and graph the secant line through the endpoints of the interval, and (c) find and graph any tangent lines to the graph of f that are parallel to the secant line.

13. $f(x) = 2e^{(x/4)}\cos(\frac{\pi x}{4})$, [0, 2]

(a) graph)

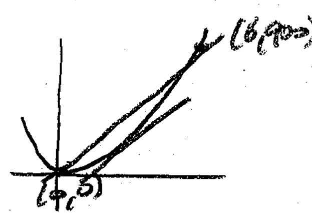
(b) $y = -x+2$

(c) $c = 1.016$

$(y+1) = -(x-1.016)$

14. $f(x) = x^4 - 2x^3 + x^2$, [0, 6]

(a)



$f(0) = 0$

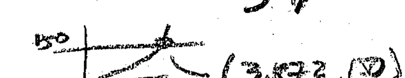
$f(6) = 900$

(b) $m_{sec} = \frac{900-0}{6-0} = 150$

$(y-0) = 150(x-0)$, $\boxed{y = 150x}$

(c) $f'(x) = 4x^3 - 6x^2 + 2x = 150$

(use calculator graph intersection)



$\boxed{c = 3.872}$

$\boxed{(y-150) = 150(x-3.872)}$

15. (MULTIPLE-CHOICE) Consider the function $f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \leq 1 \end{cases}$

Which of the following statements is *false*?

(A) f is differentiable on (0,1).

(B) $f(0) = f(1)$.

(C) f is continuous on [0,1].

(D) The derivative of f is never equal to zero on the interval (0,1).

16. (MULTIPLE-CHOICE) If the Mean Value Theorem is applied to the function $f(x) = x^3 - 4x$ on the interval [0,2], then the number c that must exist in the interval (0,2) is

(A) 0

(B) $\frac{1}{2}$

(C) $\frac{\sqrt{3}}{3}$

(D) $\frac{2\sqrt{3}}{3}$

17. (MULTIPLE-CHOICE) Which of the following functions do not satisfy the conditions of the Mean Value Theorem on the interval [-1, 1]?

(A) $f(x) = \sqrt[5]{x}$

(B) $g(x) = 2xarccosx$

(C) $h(x) = \frac{x}{x-3}$

(D) $p(x) = \sqrt{x+1}$

4.3 Worksheet

Find the open intervals analytically where the function is increasing and decreasing. (You can use your calculator to graph the function and verify your results).

1. $f(x) = \frac{x^3}{4} - 3x$

$\boxed{\begin{array}{l} \text{increasing: } (-\infty,-2) \cup (2,\infty) \\ \text{decreasing: } (-2,2) \end{array}}$

2. $f(x) = x^4 - 2x^2$

$f'(x) = 4x^3 - 4x$

$f'(x) = 0$	$f'(x)$ DNE
$4x^3 - 4x = 0$	N/A
$4x(x^2-1) = 0$	
$4x(x-1)(x+1) = 0$	
$x = 0,\ x = 1,\ x = -1$	

$-\infty \quad -1 \quad 0 \quad 1 \quad \infty$

(-2)	(-0.5)	(0.5)	(2)
$f'(-2)$	$f'(-0.5)$	$f'(0.5)$	$f'(2)$
$(-)(-)(-)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$
$-$	$+$	$-$	$+$

$\boxed{\begin{array}{l} \text{increasing: } (-1,0) \cup (1,\infty) \\ \text{decreasing: } (-\infty,-1) \cup (0,1) \end{array}}$

3. $f(x) = \frac{1}{(x+1)^2}$

$\boxed{\begin{array}{l} \text{increasing: } (-\infty,-1) \\ \text{decreasing: } (-1,\infty) \end{array}}$

4. $f(x) = \frac{x^2}{2x-1}$

$f'(x) = \frac{(2x-1)(2x) - x^2(2)}{(2x-1)^2} = \frac{4x^2-2x-2x^2}{(2x-1)^2}$

$= \frac{2x^2-2x}{(2x-1)^2} = \frac{2x(x-1)}{(2x-1)^2}$

$f'(x) = 0$	$f'(x)$ DNE
$2x(x-1) = 0$	$(2x-1)^2 = 0$
$x = 0,\ x = 1$	$2x-1 = 0$
	$x = 1/2$

$-\infty \quad 0 \quad 1/2 \quad 1 \quad \infty$

(-1)	(0.25)	(0.75)	(2)
$f'(-1)$	$f'(0.25)$	$f'(0.75)$	$f'(2)$
$\frac{(-)(-)}{(+)}$	$\frac{(+)(-)}{(+)}$	$\frac{(+)(-)}{(+)}$	$\frac{(+)(+)}{(+)}$
$+$	$-$	$-$	$+$

$\boxed{\begin{array}{l} \text{increasing: } (-\infty,0) \cup (1,\infty) \\ \text{decreasing: } (0,1/2) \cup (1/2,1) \end{array}}$

5. $f(x) = x^2 - 2x - 8$

$\boxed{\begin{array}{l}\text{increasing: } (1, \infty) \\ \text{decreasing: } (-\infty, 1)\end{array}}$

6. $f(x) = 12x - x^3$

$f'(x) = 12 - 3x^2 = 3(4 - x^2)$

$= -3(-4 + x^2) = -3(x^2 - 4)$

$= -3(x-2)(x+2)$

$\underline{f'(x) = 0}$

$-3(x-2)(x+2) = 0$

$x = 2, x = -2$

$\underline{f'(x) \text{ DNE}}$

N/A

$-\infty \quad -2 \quad 2 \quad \infty$

(-3) $f(-3)$ $(-)(-)(-)$ $-$

(0) $f'(0)$ $(-)(-)(+)$ $+$

(3) $f'(3)$ $(-)(+)(+)$ $-$

$\boxed{\begin{array}{l}\text{increasing: } (-2, 2) \\ \text{decreasing: } (-\infty, -2) \cup (2, \infty)\end{array}}$

7. $f(x) = x\sqrt{16 - x^2}$

$\boxed{\begin{array}{l}\text{increasing: } (-\sqrt{8}, \sqrt{8}) \\ \text{decreasing: } (-4, -\sqrt{8}) \cup (\sqrt{8}, 4)\end{array}}$

8. $f(x) = \frac{\ln x}{\sqrt{x}} = \frac{\ln x}{x^{1/2}}$

$f'(x) = \sqrt{x}\left(\frac{1}{x}\right) - \ln x\left(\frac{1}{2}x^{-1/2}\right)$

$= \frac{\sqrt{x}}{x} - \frac{\ln x}{2\sqrt{x}} = \frac{\sqrt{x}\,2x}{2x\sqrt{x}} - \frac{x\ln x}{2x\sqrt{x}}$

$= \frac{2x - x\ln x}{2x\sqrt{x}} = \frac{x(2 - \ln x)}{2x\sqrt{x}}$

$\underline{f'(x) = 0}$

$x(2 - \ln x) = 0$

$x = 0 \quad 2 - \ln x = 0$

$\ln x = 2$

$x = e^2$

$\underline{f'(x) \text{ DNE}}$

$2x\sqrt{x} = 0$

$x = 0$

$-\infty \quad 0 \quad e^2 \quad \infty$

(-1) (1) e^2 (7.4) (10)

$f'(-1)$ $(-1)(2 - \ln(-1))$ undef not in domain

$f'(1)$ $\frac{(+)(2 - \ln 1)}{(+)(+)}$ $+$

$f'(10)$ $\frac{(+)(-)}{(+)(+)}$ $-$

$\boxed{\begin{array}{l}\text{increasing: } (0, e^2) \\ \text{decreasing: } (e^2, \infty)\end{array}}$

9. $f(x) = x - 2\cos x,\ \ 0 < x < 2\pi$

$\boxed{\begin{array}{l} \text{increasing: } (0, \frac{7\pi}{6}) \cup (\frac{11\pi}{6}, 2\pi) \\ \text{decreasing: } (\frac{7\pi}{6}, \frac{11\pi}{6}) \end{array}}$

10. $f(x) = 3\sin x,\ \ 0 < x < 2\pi$

$f'(x) = 3\cos x$

$f'(x) = 0$ $\qquad$ $f'(x)$ DNE

$3\cos x = 0$ $\qquad$ N/A

$x = \pi/2,\ 3\pi/2$

$0 \quad (\pi/4) \quad \pi/2 \quad (\pi) \quad 3\pi/2 \quad (\frac{7\pi}{4}) \quad 2\pi$

$f'(\pi/4)$ $\qquad$ $f'(\pi)$ $\qquad$ $f'(\frac{7\pi}{4})$

$3(+)$ $\qquad$ $3(-)$ $\qquad$ $3(+)$

$+$ $\qquad$ $-$ $\qquad$ $+$

$\boxed{\begin{array}{l} \text{increasing } (0, \pi/2) \cup (3\pi/2, 2\pi) \\ \text{decreasing } (\pi/2, 3\pi/2) \end{array}}$

For the following functions: (a) find the critical numbers of f, (b) find the open interval(s) on which the function is increasing or decreasing, and (c) apply the First Derivative Test to identify all relative extrema. (You can verify your result by graphing the function in your calculator)

11. $f(x) = 2x^3 + 3x^2 - 12x$

$\boxed{\begin{array}{l} \text{critical numbers: } -2, 1 \\ \text{increasing: } (-\infty, -2) \cup (1, \infty) \\ \text{decreasing: } (-2, 1) \\ \text{relative max at } (-2, 20) \\ \text{relative min at } (1, -7) \end{array}}$

12. $f(x) = \frac{x^5 - 5x}{5} = \frac{1}{5}x^5 - x$

$f'(x) = x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x-1)(x+1)(x^2+1)$

$f'(x) = 0$ $\qquad$ $f'(x)$ DNE

$(x-1)(x+1)(x^2+1) = 0$ $\qquad$ N/A

$x = 1, -1$

local max $\qquad$ local min

$-\infty \quad (-2) \quad -1 \quad (0) \quad 1 \quad (2) \quad \infty$

$f'(-2)$ $\qquad$ $f'(0)$ $\qquad$ $f'(2)$

$(-)(-)(+)$ $\qquad$ $(-)(+)(+)$ $\qquad$ $(+)(+)(+)$

$+$ $\qquad$ $-$ $\qquad$ $+$

$\boxed{\begin{array}{l} \text{critical numbers: } -1, 1 \\ \text{increasing: } (-\infty, -1) \cup (1, \infty) \\ \text{decreasing: } (-1, 1) \end{array}}$

at $x = -1$: $f(-1) = \frac{4}{5}$

$\boxed{\text{relative max at } (-1, 4/5)}$

at $x = 1$: $f(1) = -4/5$

$\boxed{\text{relative min at } (1, \frac{-4}{5})}$

13. $f(x) = e^{-1/(x-2)}$

critical numbers: $x = 2$
increasing: $(-\infty, 2) \cup (2, \infty)$
decreasing: nowhere
at $x = 2$, neither relative max nor min

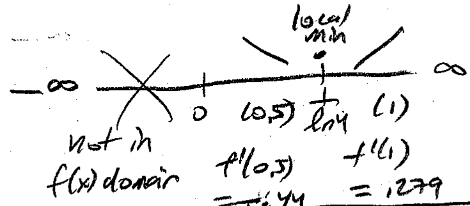
14. $f(x) = x - \log_4(x)$

$f'(x) = 1 - \frac{1}{x\ln 4} = \frac{x\ln 4}{x \ln 4} - \frac{1}{x\ln 4} = \frac{x\ln 4 - 1}{x \ln 4}$

$\underline{f'(x) = 0}$ $\quad$ $\underline{f'(x)\ DNE}$

$x\ln 4 - 1 = 0$ $\quad$ $x \ln 4 = 0$

$x = \frac{1}{\ln 4} \approx 0.721$ $\quad$ $x = 0$



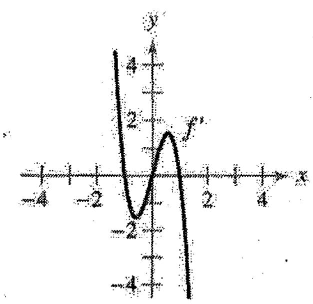
at $x = \frac{1}{\ln 4}$
$f(\frac{1}{\ln 4}) \approx 1.1507$

relative min at $(\frac{1}{\ln 4}, 1.1507)$

critical numbers: $x = \frac{1}{\ln 4}$
increasing: $(\frac{1}{\ln 4}, \infty)$, decreasing $(0, \frac{1}{\ln 4})$

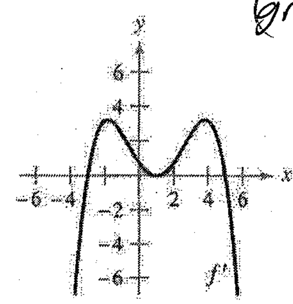
Use the graph of $f'(x)$ to (a) identify the critical number of f, (b) identify the open interval(s) on which f is increasing or decreasing, and (c) determine whether f has a relative maximum, relative minimum, or neither at each critical number.

15.



(a) critical numbers: $x = -1, 0, 1$
(b) f increasing: $(-\infty, -1) \cup (0, 1)$
f decreasing: $(-1, 0) \cup (1, \infty)$
(c) relative max at $x = -1$, $x = 1$
relative min at $x = 0$

16. (Graph of $\underline{\underline{f'}}$)



(a) $\underline{\text{critical numbers}}$ when $f' = 0$ or DNE
at $\boxed{x = -3, 1, 5}$
(b) f $\underline{\text{increasing}}$ when $f' > 0$: $\boxed{(-3, 1) \cup (1, 5)}$
f $\underline{\text{decreasing}}$ when $f' < 0$: $\boxed{(-\infty, -3) \cup (5, \infty)}$
(c) $\underline{\text{relative max}}$ when f' from $+$ $-$
at $\boxed{x = 5}$
$\underline{\text{relative min}}$ when f' from $-$ $+$
$\boxed{\text{at } x = -3}$
(at $x = 1$ neither max nor min)

17. [Multiple-Choice] The derivative h' of a function h is continuous and has exactly two zeros. Selected values of h' are shown in the table:

x	−3	−2	−1	0	1	2	3
$h'(x)$	12	5	0	−3	−4	−3	0

If the domain of h is the set of all real numbers, then h is decreasing on which of the following intervals?

(A) $x \le -1$ or $x > 3$ $\quad$ (B) $0 < x < 2$ $\quad$ (C) $x > -1$ $\quad$ $\boxed{\text{(D) } -1 < x < 3}$

4.4 Worksheet

Find the open intervals analytically where the function is concave up and concave down. (You can use your calculator to graph the function and verify your results).

1. $f(x) = -x^3 + 6x^2 - 9x - 1$

Concave up: $(-\infty, 2)$
Concave down: $(2, \infty)$

2. $h(x) = x^5 - 5x + 2$

$h'(x) = 5x^4 - 5$

$h''(x) = 20x^3$

$h'' = 0$ $\quad$ $h''(x)$ DNE

$20x^3 = 0$ $\quad$ N/A

$x = 0$

$-\infty$ $\quad$ (-1) $\quad$ 0 $\quad$ (1) $\quad$ ∞

$f''(-1) = (+)(-)$ $\quad$ $f''(1) = (+)(+)$

concave up: $(0, \infty)$
concave down: $(-\infty, 0)$

3. $f(x) = \frac{24}{x^2+12}$

Concave up: $(-\infty, -2) \cup (2, \infty)$
concave down: $(-2, 2)$

4. $g(x) = \frac{x^2+4}{4-x^2}$

$$g'(x) = \frac{(4-x^2)(2x) - (x^2+4)(-2x)}{(4-x^2)^2}$$

$$= \frac{8x - 2x^3 + 2x^3 + 8x}{(4-x^2)^2} = \frac{16x}{(4-x^2)^2}$$

$$g''(x) = \frac{(4-x^2)^2(16) - 16x(2)(4-x^2)(-2x)}{(4-x^2)^4}$$

$$= \frac{(4-x^2)[16(4-x^2) + 16(4)(x^2)]}{(4-x^2)^4} = \frac{16(4-x^2+4x^2)}{(4-x^2)^3}$$

$$= \frac{16(4+3x^2)}{(4-x^2)^3}$$

$g''(x) = 0$ $\quad$ $g''(x)$ DNE

$16(4+3x^2) = 0$ $\quad$ $(4-x^2)^3 = 0$

N/A $\quad$ $x^2 = 4$, $x = 2$, $x = -2$

$-\infty$ $\quad$ (-3) $\quad$ -2 $\quad$ (0) $\quad$ 2 $\quad$ (3) $\quad$ ∞

$g''(-3)$ $\quad$ $g''(0)$ $\quad$ $g''(3)$

$\frac{(+)(+)}{(-)}$ $\quad$ $\frac{(+)(+)}{(+)}$ $\quad$ $\frac{(+)(+)}{(-)}$

Concave up $(-2, 2)$
Concave down $(-\infty, -2) \cup (2, \infty)$

5. $y = 2x - \tan x, \quad (-\frac{\pi}{2}, \frac{\pi}{2})$

$\boxed{\begin{array}{l} \text{Concave up } (-\pi/2, 0) \\ \text{Concave down } (0, \pi/2) \end{array}}$

6. $y = x + \frac{2}{\sin x} \quad (-\pi/2, \pi/2)$

$y' = 1 + 2(-\csc x \cot x) = 1 + (-2\csc x)(\cot x)$

$y'' = -2\csc x(-\csc^2 x) + \cot x(-2(-\csc x \cot x))$

$= 2\csc^3 x + 2\cot^2 x \csc x = 2\csc x(\csc^2 x + \cot^2 x)$

$= 2\csc x(\csc^2 x + (\csc^2 x - 1)) = 2\csc x(2\csc^2 x - 1)$

$= 4\csc^3 x - 2\csc x = \frac{4}{(\sin x)^3} - \frac{2(\sin^2 x)}{\sin x(\sin^2 x)} = \frac{4 - 2\sin^2 x}{(\sin x)^3}$

$\underline{y'' = 0}$

$4 - 2\sin^2 x = 0$

$\sin^2 x = 2$, N/A

$\underline{y'' \text{ DNE}}$

$(\sin x)^3 = 0$, $\sin x = 0$

$x = 0$

$-\infty \quad (-\pi/2) \quad 0 \quad (\pi/2) \quad \infty$

$y''\big|_{-\pi/2} = \frac{4 - 2(-1)^2}{(-1)^3} = \frac{(+)}{(-)}$

$y''\big|_{\pi/2} = \frac{4 - 2(1)^2}{(1)^3} = \frac{(+)}{(+)}$

$\boxed{\begin{array}{l} \text{concave up } (0, \pi/2) \\ \text{concave down } (-\pi/2, 0) \end{array}}$

Find the point(s) of inflection and discuss the concavity of the graph of the function.

7. $f(x) = x^3 - 9x^2 + 24x - 18$

$\boxed{\begin{array}{l} \text{inflection pt at } (3, 0) \\ \text{concave up } (3, \infty) \\ \text{Concave down } (-\infty, 3) \end{array}}$

8. $f(x) = -x^3 + 6x^2 - 5$

$f'(x) = -3x^2 + 12x$

$f''(x) = -6x + 12$

$\underline{f''(x) = 0}$

$-6x + 12 = 0$

$6x = 12$

$x = 2$

$\underline{f''(x) \text{ DNE}}$

N/A

$-\infty \quad (0) \quad 2 \quad (3) \quad \infty$

$f''(0) = (+) \qquad f''(3) = (-)$

at $x = 2$, $f(2) = -(2)^3 + 6(2)^2 - 5 = 11$

$\boxed{\begin{array}{l} \text{inflection pt at } (2, 11) \\ \text{concave up } (-\infty, 2) \\ \text{concave down } (2, \infty) \end{array}}$

9. $f(x) = x\sqrt{x+3}$

concave up $(-3, \infty)$
concave down nowhere
no inflection pts

10. $f(x) = x\sqrt{9-x} = x(9-x)^{1/2}$ domain $(-\infty, 9]$

$f'(x) = x\frac{1}{2}(9-x)^{-1/2}(-1) + (9-x)^{1/2}(1) = -\frac{1}{2}x(9-x)^{-1/2} + (9-x)^{1/2}$

$f''(x) = -\frac{1}{2}x\left(-\frac{1}{2}(9-x)^{-3/2}(-1)\right) + (9-x)^{-1/2}\left(-\frac{1}{2}\right) + \frac{1}{2}(9-x)^{-1/2}(-1)$

$= \frac{-x}{4(\sqrt{9-x})^3} - \frac{1}{2\sqrt{9-x}} - \frac{1}{2\sqrt{9-x}} = \frac{-x}{4(\sqrt{9-x})^3} - \frac{1}{\sqrt{9-x}}$

$= \frac{-x - 1(4(9-x))}{4(\sqrt{9-x})^3} = \frac{3x-36}{4(\sqrt{9-x})^3} = \frac{3(x-12)}{4(\sqrt{9-x})^3}$

$f''(x) = 0$
$3(x-12) = 0$
$x = 12$
(not in domain)

$f''(x)$ DNE
$4(\sqrt{9-x})^3 = 0$
$x = 9$

$-\infty \quad (0) \quad 9$

$f''(0) = \frac{(-)}{(+)} = -$

concave up nowhere
concave down $(-\infty, 9)$
no inflection pts

11. $f(x) = \frac{4}{x^2+1}$

concave up $\left(-\infty, \frac{-1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$
concave down $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
inflection pts at $\left(-\frac{1}{\sqrt{3}}, 3\right), \left(\frac{1}{\sqrt{3}}, 3\right)$

12. $f(x) = \frac{x+3}{\sqrt{x}}$ domain: $(0, \infty)$

$f(x) = \frac{x+3}{x^{1/2}} = \frac{x}{x^{1/2}} + \frac{3}{x^{1/2}} = x^{1/2} + 3x^{-1/2}$

$f'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-3/2}$

$f''(x) = -\frac{1}{4}x^{-3/2} + \frac{9}{4}x^{-5/2} = -\frac{1}{4}x^{-3/2} + \frac{9}{4}x^{-3/2}x^{-1}$

$f''(x) = -\frac{1}{4}x^{-3/2}(1 - 9x^{-1}) = \frac{1 - \frac{9}{x}}{-4(\sqrt{x})^3} \cdot \frac{x}{x}$

$f''(x) = \frac{x-9}{-4x x\sqrt{3}} = \frac{x-9}{-4x^2\sqrt{x}}$

$f''(x) = 0$
$x - 9 = 0$
$x = 9$

$f''(x)$ DNE
$-4x^2\sqrt{x} = 0$
$x = 0$
(not in domain)

$0 \quad (1) \quad 9 \quad (10) \quad \infty$

$f(9) = \frac{9+3}{\sqrt{9}} = 4$

$f''(1) = \frac{(-)}{(-)} = +$

$f''(10) = \frac{(+)}{(-)} = -$

concave up $(0, 9)$
concave down $(9, \infty)$
inflection pt at $(9, 4)$

Use the First and Second Derivative Tests to find all relative extrema.

13. $f(x) = \frac{e^x + e^{-x}}{2}$

$\boxed{\text{relative minimum at } (0,1)}$

14. $f(x) = x^2 e^{-x}$ $\quad f'(x) = x^2(-e^{-x}) + e^{-x}(2x)$

$f''(x) = x^2(e^{-x}) + (-e^{-x})(2x) + e^{-x}(2) + (2x)(-e^{-x})$

$f'(x) = 0, \text{DNE}$

$-x^2e^{-x} + 2xe^{-x} = 0 \qquad f(0) = 0,\ f(2) = 2^2 e^{-2} = \frac{4}{e^2}$

$xe^{-x}(-x+2) = 0$

$x = 0,\ x = 2$

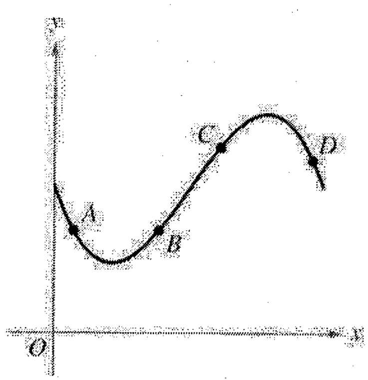
$f''(0) = 0 + 0 + 2 + 0 = 2 > 0$ concave up

$\boxed{\text{So, relative minimum at } (0,0)}$

$f''(2) = 4e^{-2} - 4e^{-2} + 2e^{-2} - 4e^{-2} = -2e^{-2} < 0$ concave down

$\boxed{\text{So, relative maximum at } (2, \frac{4}{e^2})}$

15. [Multiple-Choice] At which of the four points on the graph shown is $\frac{dy}{dx}$ positive and $\frac{d^2y}{dx^2}$ negative?



$\boxed{C}$

16. [Multiple-Choice] The graph of the function $h(x) = 4xe^{-x}$ is

(A) decreasing and concave upward on $(2, \infty)$

(B) increasing and concave downward on $(-\infty, 2)$

(C) increasing and concave upward on $(2, \infty)$

(D) decreasing and concave upward on $(1, \infty)$

$h'(x) = 4x(-e^{-x}) + e^{-x}(4) = -4xe^{-x} + 4e^{-x}$

$h'(x) = 0, \text{DNE}$

$4e^{-x}(-x+1) = 0 \qquad -\infty \quad (0) \quad 1 \quad (2) \quad \infty$

$x = 1 \qquad h'(0)\ (+)(+) \qquad h'(2)\ (+)(-)$

$h''(x) = -4x(-e^{-x}) + e^{-x}(-4) - 4e^{-x}$

$= 4xe^{-x} - 8e^{-x} = 4e^{-x}(x-2)$

$h''(x) = 0, \text{DNE} \qquad -\infty \quad (0) \quad 2 \quad (3) \quad \infty$

$x = 2 \qquad h''(0)\ (+)(-) \qquad h''(3)\ (+)(+)$

$\boxed{A}$

4.5 Worksheet

Evaluate the limit, using L'Hôpital's Rule if necessary.

1. $\lim\limits_{x\to 3} \frac{x^2-2x-3}{x-3}$ $= \boxed{4}$

2. $\lim\limits_{x\to 0} \frac{\sqrt{25-x^2}-5}{x}$ $\frac{0}{0}$

$= \lim\limits_{x\to 0} \frac{\frac{1}{2}(25-x^2)^{-1/2}(-2x)}{1}$

$= \frac{\frac{1}{2}\left(\frac{1}{\sqrt{25}}\right)(-2)(0)}{1} = \frac{0}{1} = \boxed{0}$

3. $\lim\limits_{x\to 0} \frac{\sin(3x)}{\sin(5x)}$ $= \boxed{\frac{3}{5}}$

4. $\lim\limits_{x\to 0^+} \frac{e^x-(1+x)}{x^3}$ $\frac{0}{0}$

$= \lim\limits_{x\to 0^+} \frac{e^x-(0+1)}{3x^2}$ $\frac{0}{0}$

$= \lim\limits_{x\to 0^+} \frac{e^x}{6x}$ $\frac{1}{0} = \infty$ or $-\infty$

$x\to 0^+$ $\frac{e^x}{6x} \to \frac{1}{0^+}$ $\frac{(+)}{(+)} = +$

$\boxed{+\infty}$

5. $\lim\limits_{x\to\infty} \frac{x^3}{e^{x/2}}$ $= \boxed{0}$

6. $\lim\limits_{x\to\infty} \frac{x^3}{e^{x^2}}$ $\frac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \frac{3x^2}{e^{(x^2)}(2x)}$ $\frac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \frac{6x}{e^{(x^2)}(2)+2x(e^{(x^2)}(2x))} = \lim\limits_{x\to\infty} \frac{6x}{2e^{(x^2)}+4x^2(e^{x^2})}$ $\frac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \frac{6}{2e^{(x^2)}(2x)+4x^2(e^{(x^2)}(2x))+(e^{(x^2)}(8x))}$ $\frac{6}{\infty} = \boxed{0}$

7. $\lim\limits_{x\to 0} \frac{\arctan(x)}{\sin(x)}$ $= \boxed{1}$

8. $\lim\limits_{x\to 0} \frac{x}{\arctan(2x)}$ $\frac{0}{0}$

$= \lim\limits_{x\to 0} \frac{1}{\left(\frac{1}{1+(2x)^2}\right)(2)} = \frac{1}{\left(\frac{1}{1+0}\right)(2)} = \boxed{\frac{1}{2}}$

Convert the function in the limit to an indeterminant form, then evaluate the limit using L'Hôpital's Rule.

9. $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \boxed{1}$

10. $\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$ $\infty(0)$

$= \lim_{x\to\infty} \frac{\tan(\frac{1}{x})}{\frac{1}{x}}$ $\frac{0}{0}$

$= \lim_{x\to\infty} \frac{\sec^2(\frac{1}{x})(-x^{-2})}{(-x^{-2})} = \sec^2(0) = \frac{1}{(\cos 0)^2} =$

$= \frac{1}{1} = \boxed{1}$

11. $\lim_{x\to\infty} x^{(1/x)} = \boxed{1}$

12. $y = \lim_{x\to 0^+} (1+x)^{(1/x)}$ 1^∞ (use logarithm)

$\ln y = \ln\left(\lim_{x\to 0^+} (1+x)^{1/x}\right)$

$\ln y = \lim_{x\to 0^+} \left(\frac{1}{x}\ln(1+x)\right)$ $\infty \cdot 0$

$\ln y = \lim_{x\to 0^+} \frac{\ln(1+x)}{x}$ $\frac{0}{0}$

$\ln y = \lim_{x\to 0^+} \frac{\left(\frac{1}{1+x}\right)}{1} = \frac{\left(\frac{1}{1+0}\right)}{1} = \frac{1}{1} = 1$

$\ln y = 1$

$e^{\ln y} = e^1$, $y = e^1 = \boxed{e}$

13. $\lim_{x\to 1^+} (\ln x)^{x-1} = \boxed{1}$

14. $y = \lim_{x\to 4^+} (3(x-4))^{x-4}$ 0^0 (use logarithm)

$\ln y = \ln\left(\lim_{x\to 4^+} 3(x-4)^{x-4}\right) = \lim_{x\to 4^+} \left(\ln(3(x-4))^{x-4}\right)$

$\ln y = \lim_{x\to 4^+} [\ln(3) + \ln(x-4)^{x-4}]$

$\ln y = \ln(3) + \lim_{x\to 4^+} (x-4)\ln(x-4)$ $0(-\infty)$

$\ln y = \ln 3 + \lim_{x\to 4^+} \frac{\ln(x-4)}{\left(\frac{1}{x-4}\right)}$ $\left(\frac{\infty}{\infty}\right) = \ln 3 + \lim_{x\to 4^+} \frac{\ln(x-4)}{(x-4)^{-1}}$

$\ln y = \ln 3 + \lim_{x\to 4^+} \frac{\left(\frac{1}{x-4}\right)}{-(x-4)^{-2}} = \ln 3 + \lim_{x\to 4^+} \frac{-(x-4)^2}{(x-4)}$

$\ln y = \ln 3 + \lim_{x\to 4^+} -(x-4) = \ln 3 - (4-4) = \ln 3$

$y = \boxed{3}$

For each part of 15 and 16, determine if the limit can be evaluated using L'Hôpital's Rule by showing the indeterminant form (but do not evaluate the limit)

15. (i) $\lim_{x\to 2} \frac{x-2}{x^3-x-6}$ $\boxed{\text{yes}}$

16. (i) $\lim_{x\to 0} \frac{x^2-4x}{2x-1}$ $\frac{0}{-1}$ $\boxed{\text{No}}$

(ii) $\lim_{x\to\infty} \frac{x^3}{e^x}$ $\boxed{\text{yes}}$

(ii) $\lim_{x\to 1} \frac{1+x(lnx\quad)}{(x-1)lnx}$ $\frac{0}{0}$ $\boxed{\text{yes}}$

(iii) $\lim_{x\to 1} \frac{\cos(\pi x)}{\ln(x)}$ $\boxed{\text{No}}$

(iii) $\lim_{x\to 3} \frac{e^{x^2}}{x-3}$ $\frac{e^9}{0}$ $\boxed{\text{No}}$

For 17 and 18, show that the indeterminate forms $0^0, \infty^0, and\ 1^\infty$ do not always have a value of 1 by evaluating each limit.

17. $\lim_{x\to 0^+} x^{\left(\frac{ln2}{1+lnx}\right)}$ $=\boxed{2}$

18. $y = \lim_{x\to\infty} x^{\left(\frac{ln2}{1+lnx}\right)}$ (use logarithm)

$\ln y = \ln\left(\lim_{x\to\infty} x^{\left(\frac{\ln 2}{1+\ln x}\right)}\right)$

$\ln y = \lim_{x\to\infty} \left(\frac{\ln 2}{1+\ln x}\right) \ln(x)$

$\ln y = \lim_{x\to\infty} \frac{\ln 2 \ln x}{1+\ln x}$ $\frac{\infty}{\infty}$

$\ln y = \lim_{x\to\infty} \frac{\ln 2 \frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\ln 2}{1} = \ln 2$

$\ln y = \ln 2$

$y = \boxed{2}$

4.6 Worksheet - Odd plus #2 and #4.

Use the First and Second Derivative Tests as well as intercepts and limits to evaluate asymptotes, so fully analyze and sketch the given function. You must show all work and label all intercepts, relative extrema, points of inflection, and asymptotes. (You can verify your work by graphing the function in a calculator).

1. $f(x) = \frac{x^2-6x+12}{x-4}$

Domain $(-\infty,4)\cup(4,\infty)$

y intercept $(0,-3)$

no x intercepts

V.A. $x=4$

no H.A, but slant asymptote $y=x-2$

increasing $(-\infty,2)\cup(6,\infty)$

decreasing $(2,4)\cup(4,6)$

local max at $(2,-2)$

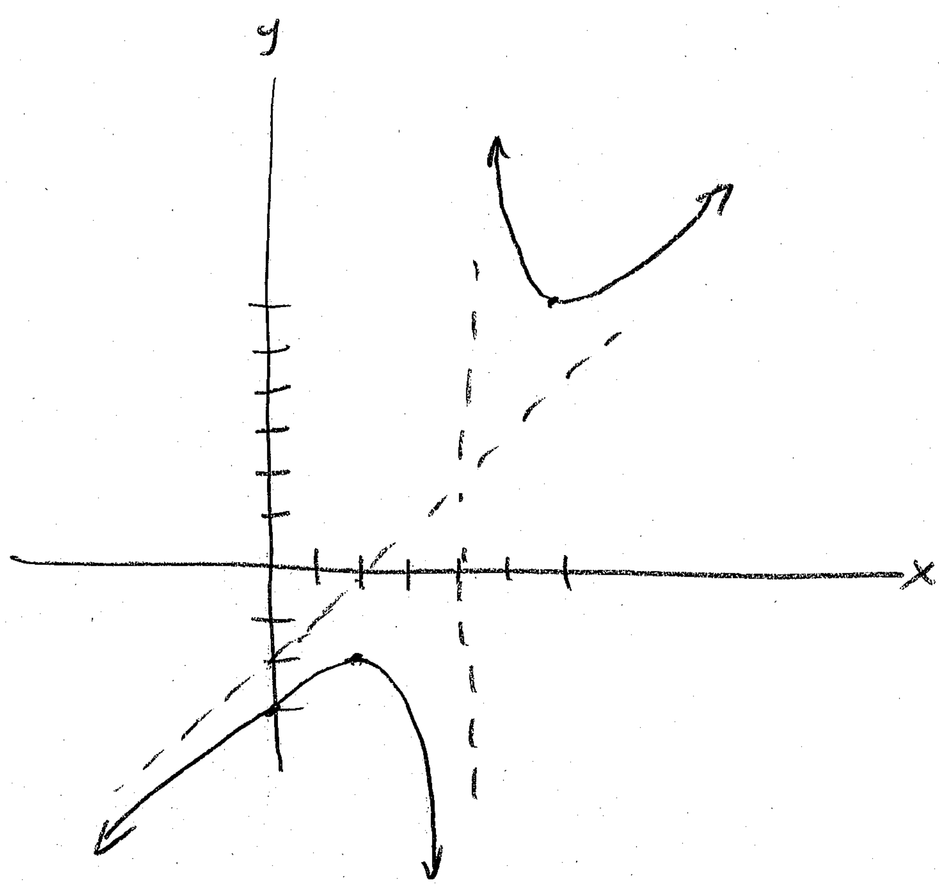
local min at $(6,6)$

concave up $(4,\infty)$

concave down $(-\infty,4)$

no inflection pts

(show all computation work)



2. $f(x) = 3x^{(2/3)} - 2x = 3 \cdot \sqrt[3]{x^2} - 2x$

$\boxed{\text{Domain: } (-\infty, \infty)}$

$\boxed{\text{y int } f(0) = 0 \quad (0,0)}$

x ints: $3\sqrt[3]{x^2} - 2x = 0$

$3\sqrt[3]{x^2} = 2x$

$27x^2 = 8x^3$

$8x^3 - 27x^2 = 0$

$x^2(8x - 27) = 0$

$x = 0,\ x = 27/8$

$\boxed{\text{x ints: } (0,0)\ (27/8, 0)}$

$\boxed{\text{V.A: none}}$

H.A. $\lim\limits_{x \to \infty} 3\sqrt[3]{x^2} - 2x$ $\quad \infty - \infty$

$= \lim\limits_{x \to \infty} 3x^{2/3} - 2x^{2/3}x^{1/3}$

$= \lim\limits_{x \to \infty} x^{2/3}(3 - 2x^{1/3})$

$\infty(-\infty) = -\infty$

$\lim\limits_{x \to -\infty} x^{2/3}(3 - 2x^{1/3})$

$(\infty)(\infty) = \infty$

$\boxed{\text{no H.A. but } \nwarrow \ \searrow}$

$f'(x) = 2x^{-1/3} - 2$

$= \dfrac{2}{\sqrt[3]{x}} - 2$

critical:

$f'(x) = 0$ | $f'(x)$ DNE
$\dfrac{2}{\sqrt[3]{x}} - 2 = 0$ | $x = 0$

$\dfrac{2}{\sqrt[3]{x}} = 2$

$\sqrt[3]{x} = 1$

$x = 1$

$-\infty$ ——— 0 ——— 1 ——— ∞

$f'(-1)$ | $f'(\tfrac{1}{8})$ | $f'(8)$
$\dfrac{2}{-1} - 2$ | $\dfrac{2}{(\frac{1}{2})} - 2$ | $\dfrac{2}{\sqrt[3]{8}} - 2$
$-$ | $+$ | $-$

$\boxed{\begin{array}{l}\text{increasing } (0,1) \\ \text{decreasing } (-\infty, 0) \cup (1, \infty)\end{array}}$

$f(0) = 0$ $\quad \boxed{(0,0) \text{ is local min}}$

$f(1) = 1$ $\quad \boxed{(1,1) \text{ is local max}}$

$f''(x) = -\frac{2}{3}x^{-4/3} = \dfrac{-2}{3\sqrt[3]{x^4}}$

inflection:

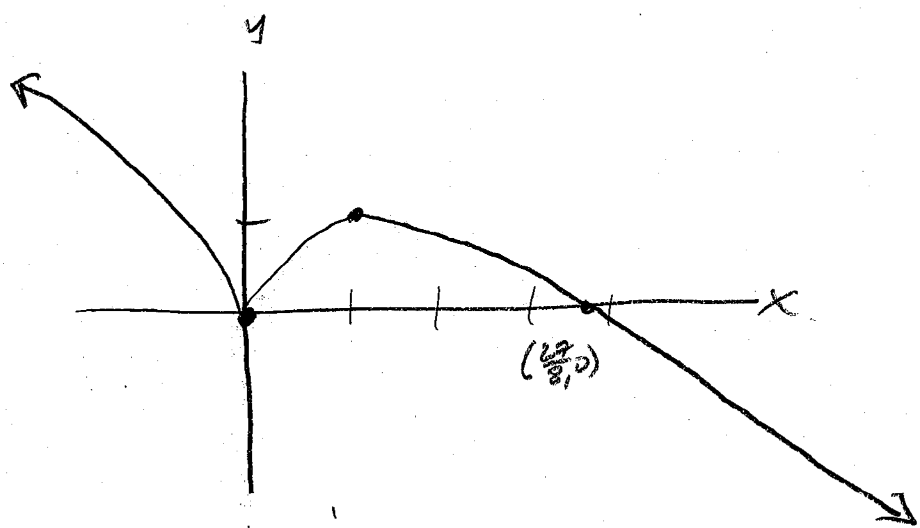
$f''(x) = 0$ | $f''(x)$ DNE
N/A | $x = 0$

$-\infty$ ——— 0 ——— ∞

$f''(-1)$ | $f''(1)$
$\dfrac{(-)}{(+)}$ | $\dfrac{(-)}{(+)}$

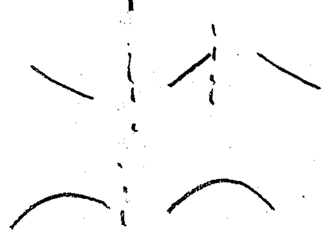
$\boxed{\begin{array}{l}\text{concave up: nowhere} \\ \text{concave down: } (-\infty, 0) \cup (0, \infty)\end{array}}$

$\boxed{\text{no inflection pts}}$



'slope'

concavity



3. $f(x) = 2 - x - x^3$

Domain : $(-\infty, \infty)$

y int : $(0,2)$

x int : $(1,0)$

no V.A.

no H.A. but ↖ ↘

no relative extrema

increasing nowhere

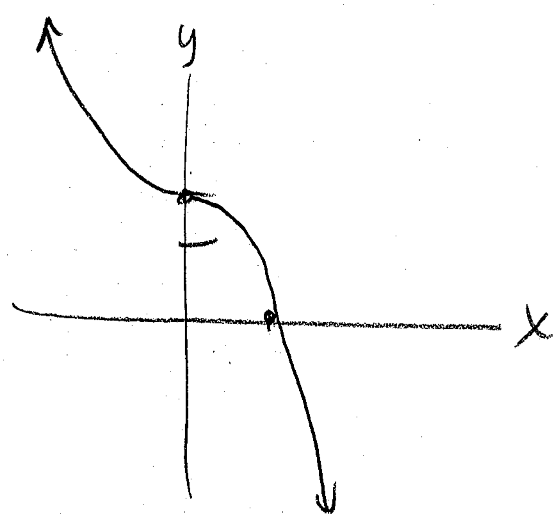
decreasing $(-\infty, \infty)$

concave up $(-\infty, 0)$

concave down $(0, \infty)$

inflection pt $(0,2)$

(show all computation work)



4. $f(x) = x^5 - 5x$

$\boxed{\text{domain: } (-\infty, \infty)}$

$\boxed{\text{y int } f(0)=0 \ (0,0)}$

x ints: $x^5 - 5x = 0$

$x(x^4-5)=0$

$x=0, \quad x^4-5=0$

$x^4=5$

$x = \pm \sqrt[4]{5}$

$\boxed{\text{x ints: } (0,0)(\sqrt[4]{5},0)(-\sqrt[4]{5},0)}$

$\boxed{\text{V.A. none}}$

H.A. $\lim\limits_{x\to\infty} x^5-5x = \infty$

$\lim\limits_{x\to-\infty} x^5-5x = -\infty$

$\boxed{\text{no H.A. but } \swarrow \ \nearrow}$

$f'(x) = 5x^4 - 5 = 5(x^4-1)$

$= 5(x^2-1)(x^2+1)$

$= 5(x-1)(x+1)(x^2+1)$

critical

$f'(x)=0$	$f'(x)$ DNE
$x=1, x=-1$	N/A

$-\infty \quad -1 \quad 1 \quad \infty$

$f'(-2)$	$f'(0)$	$f'(2)$
$(+)(-)(-)(+)$	$(+)(-)(+)(+)$	$(+)(+)(+)(+)$
$+$	$-$	$+$

$\boxed{\begin{array}{l}\text{increasing } (-\infty,-1)\cup(1,\infty) \\ \text{decreasing } (-1,1)\end{array}}$

$f(-1)=4$ $\boxed{(-1,4) \text{ is local max}}$

$f(1)=-4$ $\boxed{(1,-4) \text{ is local min}}$

$f''(x) = 20x^3$

inflection:

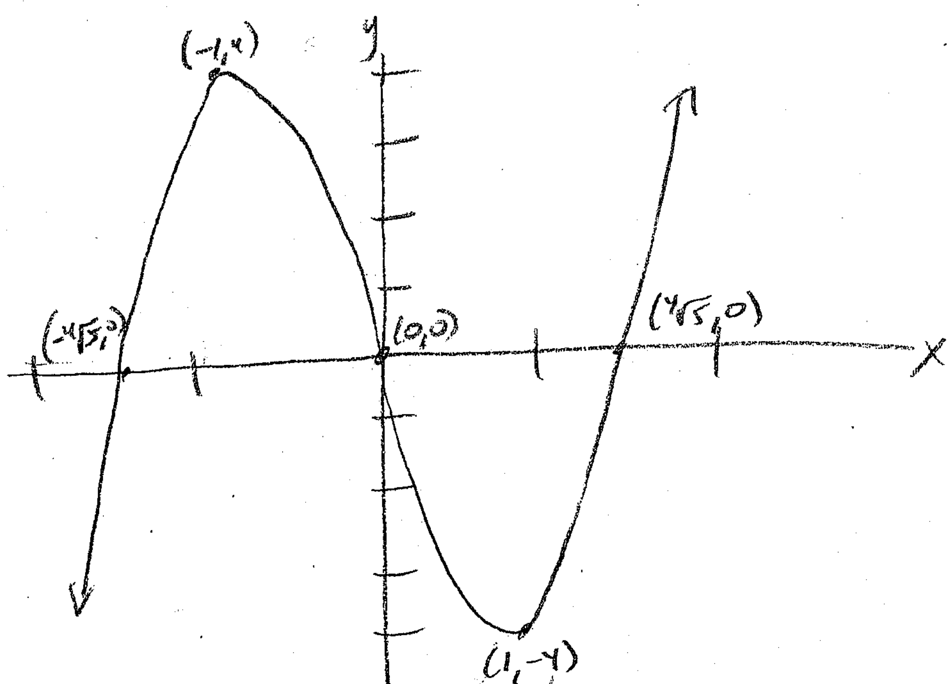
$f''(x)=0$	$f''(x)$ DNE
$20x^3=0$	N/A
$x=0$	

$-\infty \quad 0 \quad \infty$

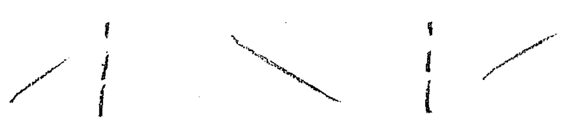
$f''(-1)$	$f''(1)$
$-$	$+$

$\boxed{\begin{array}{l}\text{concave up } (0,\infty) \\ \text{concave down } (-\infty,0)\end{array}}$

$f(0)=0$ $\boxed{(0,0) \text{ is an inflection pt}}$

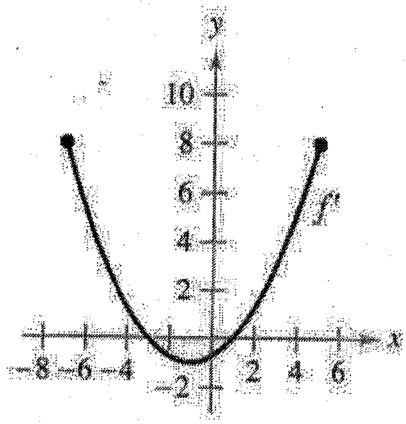


'slope'



Concavity

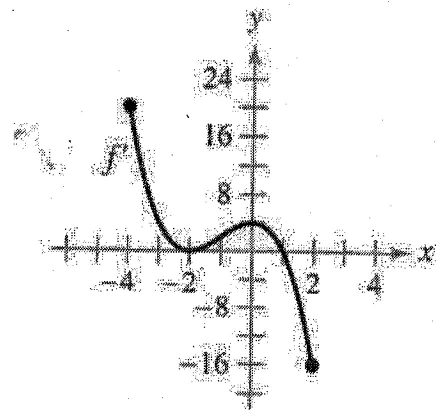
5. The graph of the **first derivative** of a function f on the interval [-7, 5] is shown:



(a) On what interval(s) is f increasing and decreasing?
(b) On what interval(s) is f concave up and down?
(c) At what x-value(s) does f have relative extrema?
(d) At what x-value(s) does f have points of inflection?

(a) incr: $(-7,-3)\cup(1,5)$, decr: $(-3,1)$
(b) up: $(-1,5)$; down $(-7,-1)$
(c) max at $x=-3$, min at $x=1$
(d) infl pt at $x=-1$

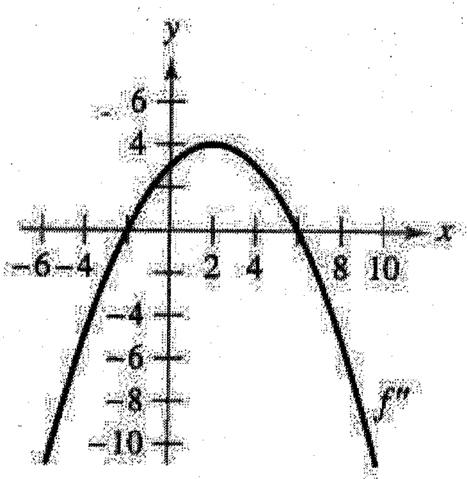
6. The graph of the **first derivative** of a function f on the interval [-4, 2] is shown:



(a) On what interval(s) is f increasing and decreasing?
(b) On what interval(s) is f concave up and down?
(c) At what x-value(s) does f have relative extrema?
(d) At what x-value(s) does f have points of inflection?

(a) f increasing when $f'>0$, f decreasing when $f'<0$
$(-4,-2)\cup(-2,0)\cup(0,1)$ $(1,2)$
(b) f concave up when f' increasing, down when f' decreasing
$(-2,0)$ $(-4,-2)\cup(0,2)$
(c) rel. max when f' from + to −, rel. min when f' from − to +
$x=1$ nowhere
(d) pts of inflection when f' changes direction, at $x=-2$, $x=0$

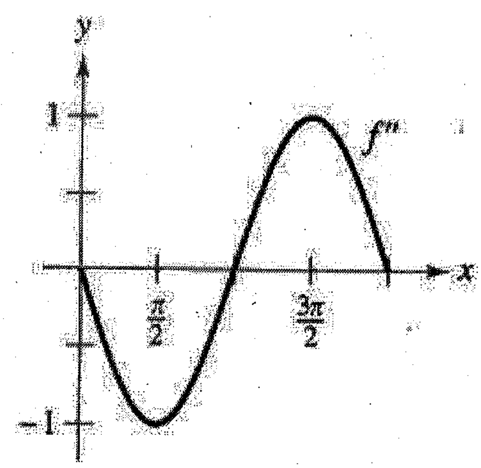
7. The graph of the **second derivative** of a function f is shown:



(a) On what interval(s) is f concave up and down?
(b) On what interval(s) is <u>f'</u> increasing and decreasing?
(c) At what x-value(s) does f have points of inflection?

(a) up: $(-2,6)$; down: $(-6,-2)\cup(6,10)$
(b) f' incr: $(-2,6)$; f' decr: $(-6,-2)\cup(6,10)$
(c) pts of inflection at $x=-2$, $x=6$

8. The graph of the **second derivative** of a function f is shown:



(a) On what interval(s) is f concave up and down?
(b) On what interval(s) is <u>f'</u> increasing and decreasing?
(c) At what x-value(s) does f have points of inflection?

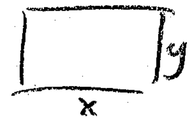
(a) f concave up when $f''>0$, down when $f''<0$
$(\pi,2\pi)$ $(0,\pi)$
(b) f' increasing when $f''>0$, f' decr when $f''<0$
$(\pi,2\pi)$ $(0,\pi)$
(c) f pts of inflection when f'' changes sign: $x=\pi$

4.7 Worksheet - Odds plus #2 and #4.

1. Find two positive numbers such that the product of the numbers is 147 and the sum of the first number plus the three times the second number is a minimum.

$\boxed{\text{numbers are } 21 \text{ \& } 7}$

2. Find the length and width of a rectangle that has a perimeter of 80 m and will maximize the area enclosed.



objective

$A = xy$ (min)

$A = x(40-x)$

$A = 40x - x^2$

$A' = 40 - 2x$

$A' = 0$	A' DNE
$40 - 2x = 0$	N/A

$\boxed{x = 20\text{ m}}$

constraint

$P = 2x + 2y = 80\text{ m}$

$x + y = 40$

$y = 40 - x$

$y = 40 - 20 = 20$, $\boxed{y = 20\text{ m}}$

verify a maximum: $A'' = -2$

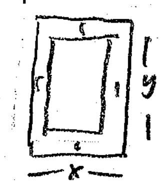
$A''(20) = -2$ concave down

this is a max ✓

3. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 245,000 square meters, and no fencing is needed along the one side which is along the river. What dimensions will require the least amount of fencing?

$\boxed{700\text{m} \times 350\text{m}}$

4. A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the paper such that the least amount of paper is used.



objective

$A_{paper} = xy$ (min)

constraint

$A_{print} = (x-2)(y-2) = 30\text{ in}^2$

$(y-2) = \frac{30}{x-2}$

$y = \frac{30}{x-2} + 2$

$A = x\left(\frac{30}{x-2} + 2\right)$

$A = \frac{30x}{x-2} + 2x$

$A' = \frac{(x-2)(30) - 30x(1)}{(x-2)^2} + 2 = \frac{-60}{(x-2)^2} + 2 = -60(x-2)^{-2} + 2$

$A' = 0$: $\frac{-60}{(x-2)^2} + 2 = 0$, $\frac{-60}{(x-2)^2} = 2$, $(x-2)^2 = 30$

$x - 2 = \sqrt{30}$

$\boxed{x = 2 + \sqrt{30}\text{ in} \quad x \approx 7.477\text{ in}}$

$y = \frac{30}{(2+\sqrt{30}) - 2} + 2$

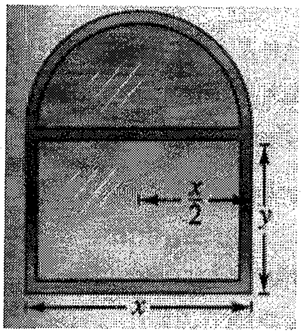
$\boxed{y = 2 + \sqrt{30} \approx 7.477\text{ in}}$

verify min: $A'' = 120(x-2)^{-3} = \frac{120}{(x-2)^3}$

$A''(7.477) = \frac{(+)}{(+)}$ + concave up

this is a min ✓

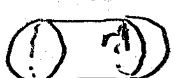
5. A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window. Find the dimensions which provides maximum total window area if the total perimeter is 16 ft (assume the crossbar in the middle is not part of the perimeter).



$$\boxed{\begin{array}{l} x = \frac{8}{1+\frac{\pi}{4}} \approx 4.4808 \text{ ft} \\ y \approx 2.2404 \text{ ft} \end{array}}$$

6. A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the solid must be 14 cm³.
Find the radius of the cylinder that requires minimum surface area.

objective



constraint

(min) $A = 2\pi r h + 4\pi r^2$

$V = \pi r^2 h + \frac{4}{3}\pi r^3 = 14 \text{ cm}^3$

$\pi r^2 h = 14 - \frac{4}{3}\pi r^3$

$A = 2\pi r\left(\frac{14}{\pi r^2} - \frac{4r}{3}\right) + 4\pi r^2$

$h = \frac{14}{\pi r^2} - \frac{4r}{3}$

$A = \frac{28}{r} - \frac{8\pi}{3} r^2 + 4\pi r^2 = 28r^{-1} + \frac{4\pi}{3} r^2$

$A' = -28r^{-2} + \frac{8\pi}{3} r = \frac{-28}{r^2} + \frac{8\pi}{3} r$

$A' = 0$

A' DNE
$r = 0$ (not valid)

$\frac{-28}{r^2} + \frac{8\pi}{3} r = 0$

$\frac{8\pi}{3} r = \frac{28}{r^2}$

$r^3 = \frac{21}{2\pi}$

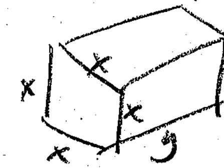
$$\boxed{r = \sqrt[3]{\frac{21}{2\pi}} \approx 1.4951 \text{ cm}}$$

$h = \frac{14}{\pi(1.4951)^2} - \frac{4(1.4951)}{3} = 0$

$\boxed{h = 0 \text{ cm}}$ (a sphere is optimum)

7. A rectangular package to be sent by a postal service can have a maximum combined length and girth of 108 inches (girth is the distance around all sides of a cross-section). Find the dimensions of a package which encloses the maximum possible volume. (assume the cross-section is a square)

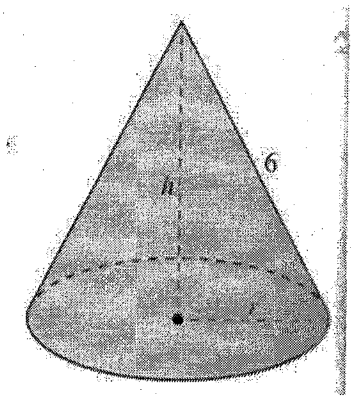
picture:



(girth $= 4x + y$)

$$\boxed{\begin{array}{l} x = 18 \text{ in} \\ y = 36 \text{ in} \end{array}}$$

8. A right cone has a slant height of 6:



(a) write the volume of the cone as a function of the radius. ($V_{cone} = \frac{1}{3}\pi r^2 h$)

(b) What are the dimensions that maximize the volume of the cone?

(a) 'constraint' $r^2+h^2=36$

$h^2=36-r^2,\ h=\sqrt{36-r^2}$

$\boxed{V=\frac{\pi}{3}r^2\sqrt{36-r^2}} = \frac{\pi}{3}r^2(36-r^2)^{1/2}$

$V'=\frac{\pi}{3}r^2\left(\frac{1}{2}(36-r^2)^{-1/2}(-2r)\right)+(36-r^2)^{1/2}\left(\frac{2\pi}{3}r\right)$

$V'=\frac{-\pi r^3}{3\sqrt{36-r^2}}+\frac{2\pi}{3}r\sqrt{36-r^2}$

$V'=0$ $\quad$ V' DNE ($r=6$, not valid cone)

$\frac{2\pi}{3}r\sqrt{36-r^2}=\frac{\pi r^3}{3\sqrt{36-r^2}}$

$2r(36-r^2)=r^3$

$72r-2r^3=r^3$

$3r^3-72r=0$

$3r(r^2-24)=0$

$r\neq 0,\ r^2=24$

$\boxed{r=\sqrt{24}\approx 4.899}$

$h=\sqrt{36-(\sqrt{24})^2}$

$\boxed{h=\sqrt{12}\approx 3.464}$

verify a max

V'' = too hard, so test r value in V':

4.899

$V'(4)$ $\quad$ $V'(5)$

22.479 $\quad$ −4.736

+ $\quad$ −

this is a max ✓

9. A rectangle is bounded by the x-axis and the graph $y=25-x^2$.

(a) What are the dimensions of the rectangle that maximize the rectangle's area?

width $=2\sqrt{\frac{25}{3}}\approx 5.7735$

height $=\frac{50}{3}\approx 16.667$

(b) What are the dimensions of the rectangle that maximize the rectangle's perimeter?

width $=2$

height $=24$

10. (Challenge problem) Let $f(x) = 2 - 2sinx$.

(a) Sketch the graph of f on the interval $\left[0, \frac{\pi}{2}\right]$

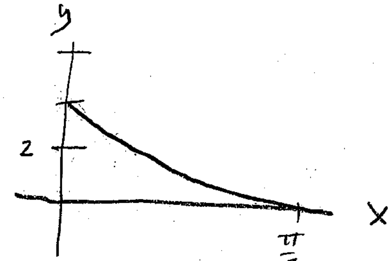
(b) Find the distance from the origin to the y-intercept and the distance from the origin to the x-intercept.

(c) Write the distance d from the origin to a point on the graph f as a function of x. Use your calculator to graph d and find the minimum distance.

(d) Use calculus and the zero feature of your calculator to find the value of x that minimizes the function d on the interval $\left[0, \frac{\pi}{2}\right]$.

What is the minimum distance?

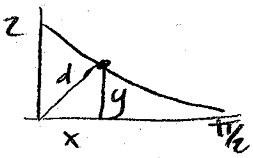
(a)



(b) distance origin to y-intercept $= \boxed{2}$

distance origin to x-intercept $= \boxed{\frac{\pi}{2}}$

(c)



objective	constraint
$d^2 = x^2 + y^2$	$\leftarrow y = 2 - 2\sin x$
$d^2 = x^2 + (2 - 2\sin x)^2$	

d will be minimum when d^2 is minimum

so to make this a little easier

define $g(x) = d^2 = x^2 + (2-2\sin x)^2$

and minimize $g(x)$

$g'(x) = 2x + 2(2-2\sin x)'(-2\cos x)$

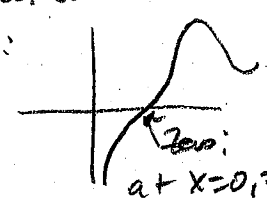
$g'(x) = 0$	$g'(x)$ DNE
	N/A

$2x - 4\cos x(2-2\sin x) = 0$

use calculator to solve

$Y1 = 2X - 4\cos x(2 - 2\sin x)$

find zero:



$g(0.79672) = 0.9595 = d^2$

so $d_{min} = \sqrt{0.9595} = \boxed{0.97954}$

(d) we could instead just use calculator graph of g(x) directly to minimize: