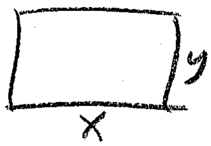
Unit 4 - optimization Problems review

(1)



objective function

(max) $A = xy$

$A = x(160 - x)$

$A = 160x - x^2$

constraint

$L = 2x + 2y = 320$

$2y = 320 - 2x$

$y = 160 - x$

$\underline{A' = 0}$ $\underline{A' \text{ DNE}}$

$160 - 2x = 0$

$2x = 160$

$x = 80$; $y = 160 - 80 = 80$

check to see if min or max:

$A'' = -2$ concave down so max

$\boxed{80 \text{ yds} \times 80 \text{ yds}}$

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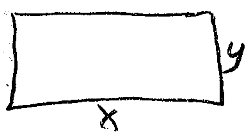
(2)



objective function

$L = 2x + 2y$

$L = 2x + 2\left(\frac{10000}{x}\right)$

$L = 2x + 20000x^{-1}$

constraint

$A = xy = 10000$

$y = \frac{10000}{x}$

$\underline{L' = 0}$ $\underline{L' \text{ DNE}}$

$x = 0$ (not possible)

$2 - 20000x^{-2} = 0$

$\frac{20000}{x^2} = 2$, $x^2 = 10000$

$x = \pm 100$

(− not possible)

check to see if min or max:

$L'' = 40000x^{-3} = \frac{40000}{x^3}$

at $x = 100$ (+) concave up so min

$y = \frac{10000}{100} = 100$

$\boxed{100 \text{ m} \times 100 \text{ m}}$

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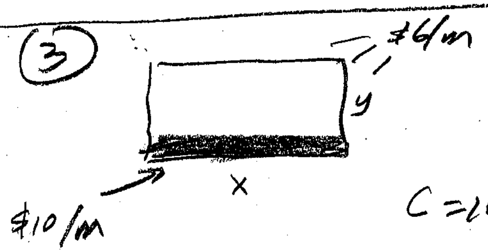
(3)



objective function

$C = 10x + 6(x + 2y)$

$C = 10x + 6x + 12\left(\frac{10000}{x}\right)$

$C = 16x + 120000x^{-1}$

constraint

$A = xy = 10000$

$y = \frac{10000}{x}$

$\underline{C' = 0}$ $\underline{C' \text{ DNE}}$

$x = 0$ (not possible)

$16 - 120000x^{-2} = 0$

$\frac{120000}{x^2} = 16$

$16x^2 = 120000$

$x^2 = 7500$

$x = \pm\sqrt{7500} = 86.6025$

(− not possible)

$y = \frac{10000}{\sqrt{7500}} = 115.4701$

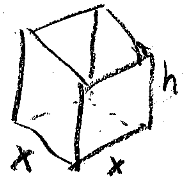
check to see if min or max:

$C'' = 240000x^{-3} = \left.\frac{240000}{x^3}\right|_{x = 86.6}$ is (+)

concave up so min cost

$$\boxed{\begin{array}{l} x \text{ (expensive side)} = \sqrt{7500} \approx 86.6025 \text{ m} \\ y = \frac{10000}{\sqrt{7500}} \approx 115.4701 \text{ m} \end{array}}$$

(4)



objective function

(max) $V = x^2 h$

$V = x^2\left(\frac{1200 - x^2}{4x}\right)$

$V = \frac{1}{4}x(1200 - x^2)$

$V = 300x - \frac{1}{4}x^3$

$V' = 0$ V' DNE

$300 - \frac{3}{4}x^2 = 0$

$\frac{3}{4}x^2 = 300$

$3x^2 = 1200$

$x^2 = 400$

$x = \pm 20$: $h = \frac{1200 - 20^2}{4(20)} = 10$

(− not possible

constraint

$S = 4xh + x^2 = 1200$

$4xh = 1200 - x^2$

$h = \frac{1200 - x^2}{4x}$

check if min or max:

$V'' = -\frac{3}{2}x \Big|_{at\ x=20}$ (−) concave down so max volume

$$\boxed{\begin{array}{c} V_{max} = (20)^2(10) = 4000\,cm^3 \\ \text{with } x = 20\,cm,\ h = 10\,cm \end{array}}$$

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(5) $P = \frac{100I}{I^2 + I + 4}$ $P' = \frac{(I^2+I+4)(100) - (100I)(2I+1)}{(I^2+I+4)^2}$

$P' = 0$

$\frac{100I^2 + 100I + 400 - 200I^2 - 100I}{(I^2+I+4)^2}$

$\frac{-100I^2 + 400}{(I^2+I+4)^2} = 0$

$100I^2 = 400$

$I^2 = 4$

$I \pm 2$

(−2 not possible)

$\underline{\underline{I = 2}}$

P' DNE

denominator = zero?

$\frac{-1 \pm \sqrt{1 - 4(1)(4)}}{2(1)}$ no real zeros

check if min or max

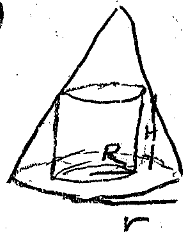
I'' difficult, so check I' values:

−∞ (0) 2 (3) ∞ max P

$\frac{(+)}{(+)}$ $\frac{(-)}{(+)}$

P_{max} at $\boxed{I = 2 \text{ (thousand footcandles)}}$

(6)



objective function

h (max) $V = \pi R^2 H$

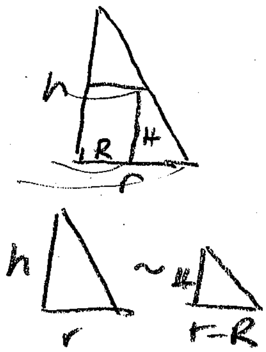
$$V = \pi R^2 \left(\frac{h(r-R)}{r}\right)$$

constraint

$$\frac{h}{r} = \frac{H}{r-R}$$

$$h(r-R) = Hr$$

$$H = \frac{h(r-R)}{r}$$



$$V = \pi R^2 \frac{h}{r}(r-R)$$

$$V = (\pi h)R^2 - (\pi \tfrac{h}{r})R^3 \quad \leftarrow r \,\&\, h \text{ are } \underline{\text{constants}}$$

$$V' = 2(\pi h)R - 3(\pi \tfrac{h}{r})R^2$$

$\underline{V' = 0}$ $\quad$ $\underline{V' \text{ DNE}}$

$$(2\pi h)R - (3\pi \tfrac{h}{r})R^2 = 0$$

$$R\left((2\pi h) - (3\pi \tfrac{h}{r})R\right) = 0$$

$R = 0$ (not possible)

$$(2\pi h) - (3\pi \tfrac{h}{r})R = 0$$

$$(3\pi \tfrac{h}{r})R = (2\pi h)$$

$$R = \frac{2\pi h}{3\pi \frac{h}{r}} = \frac{\pi h(2)}{\pi h(\frac{3}{r})} = \frac{2r}{3}$$

<u>check if min or max</u>

$$V'' = 2\pi h - 6\pi \frac{h}{r} R$$

when $R = \frac{2r}{3}$:

$$V'' = 2\pi h - 6\pi \frac{h}{r} \frac{2r}{3}$$

$$= 2\pi h - 4\pi^2 \frac{h}{r} < 0 \text{ for } +h, +r$$

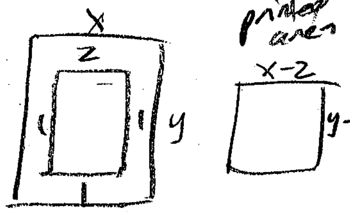
concave down

so max V

$$H = \frac{h(r-R)}{r} = \frac{h(r - \frac{2r}{3})}{r} = \frac{h\left(\frac{3r}{3} - \frac{2r}{3}\right)}{r} = \frac{h(\frac{1}{3}r)}{r} = \frac{h}{3}$$

$$V_{max} = \pi \left(\frac{2r}{3}\right)^2 \frac{h}{3} = \boxed{\frac{4\pi}{27} r^2 h}$$

⑦



objective function

(max) $A_{print} = (x-2)(y-3)$

constraint

$A_{total} = xy = 180$

$y = \frac{180}{x}$

$A_{print} = (x-2)\left(\frac{180}{x} - 3\right)$

$A_p = 180 - 3x - \frac{360}{x} + 6$ $(-360x^{-1})$

$A'_p = -3 + \frac{360}{x^2} = -3 + 360x^{-2}$

$A' = 0$

$-3 + \frac{360}{x^2} = 0$

$\frac{360}{x^2} = 3$

$3x^2 = 360$

$x^2 = 120$

$x = \pm\sqrt{120} = 10.9544$ (− not possible)

$y = \frac{180}{x} = \frac{180}{\sqrt{120}} = 16.4317$

A' DNE

$x = 0$ (not possible)

check if min or max

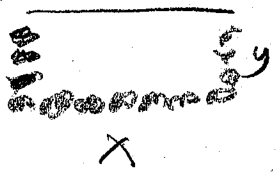
$A'' = -720x^{-3} = \frac{-720}{x^3}$ $\frac{(-)}{(+)}$ −

concave down

max A_p

$$\boxed{\begin{aligned} x &= \sqrt{120} = \sqrt{4}\sqrt{30} = 2\sqrt{30} \text{ in} \\ y &= \frac{180}{\sqrt{120}} = \frac{180}{2\sqrt{30}} = \frac{90}{\sqrt{30}} \text{ in} \end{aligned}}$$

(8)



objective function

(min) $C = 15(x+2y) + 10(x)$

$C = 15\left(x + 2\left(\frac{3000}{x}\right)\right) + 10x$

$C = 15x + \frac{90000}{x} + 10x$

$C = 25x + \frac{90000}{x} = 25x + 90000x^{-1}$

$C' = 25 - 90000x^{-2} = 25 - \frac{90000}{x^2}$

constraint

$A = xy = 3000$

$y = \frac{3000}{x}$

$\underline{C' = 0}$

$25 - \frac{90000}{x^2} = 0$

$\frac{90000}{x^2} = 25$

$25x^2 = 90000$

$x^2 = 3600$

$x = \pm 60$; $y = \frac{3000}{60} = 50$

(− not possible)

$\underline{C' \text{ DNE}}$

$x = 0$ (not possible)

check if min or max

$C'' = 180000x^{-3} = \frac{180000}{x^3}$

(+) for +x

concave up

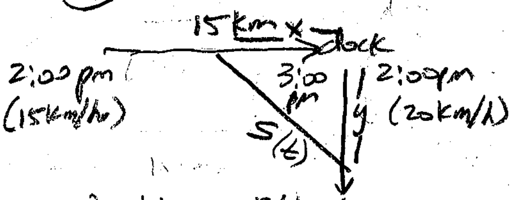
so min cost

$C_{min} = 15(60) + 15(50) + 15(50) + 10(60)$

$C_{min} = \$3000$

when $x = 60$ ft, $y = 50$ ft

(9)



in 1 hour, 2nd boat travels $\frac{15km}{hr} \cdot 1hr$

15km

objective function

$s^2 = x^2 + y^2$

constraint

$x = 15 - 15t$

$\leftarrow y = 20t$

(t in hours)

$s^2 = (15-15t)^2 + (20t)^2$

find min of $\underline{s^2}$ (easier)

$s^2 = 225 - 450t + 225t^2 + 400t^2$

min: $D = s^2 = 625t^2 - 450t + 225$

$D' = 1250t - 450$

$\underline{D' = 0}$ $\qquad$ $\underline{D' \text{ DNE}}$

$1250t - 450 = 0$

$1250t = 450$

$t = \frac{450}{1250} = \frac{9}{25} = .36 \text{ hrs}$

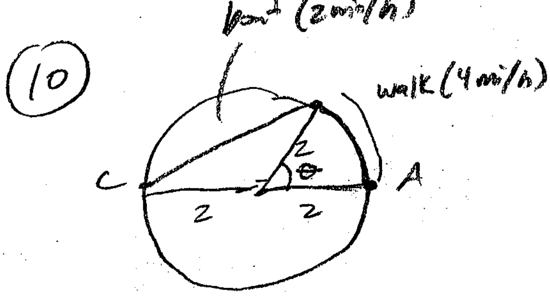
$.36 \text{hrs} \frac{60 \text{ min}}{1 \text{ hr}} = \boxed{21.6 \text{ min after } 2{:}00\text{pm}}$

verify if min or max

$D'' = 1250$ (+) $\cup$

concave up so min (distance)^2

(10)



objective function

(min) $t = t_{walk} + t_{boat}$

$$t = \frac{2\theta}{4} + \frac{\sqrt{8-8\cos\alpha}}{2}$$

need in terms of one variable, let's use α

so $\theta = \pi - \alpha$

$$t = \frac{2(\pi-\alpha)}{4} + \frac{\sqrt{8-8\cos\alpha}}{2}$$

$$t = \frac{1}{2}\pi - \frac{1}{2}\alpha + \frac{1}{2}(8-8\cos\alpha)^{1/2}$$

$$t' = -\frac{1}{2} + \frac{1}{4}(8-8\cos\alpha)^{-1/2}(8\sin\alpha)$$

$$t' = -\frac{1}{2} + \frac{2\sin\alpha}{\sqrt{8-8\cos\alpha}}$$

distance = (rate) time

so time, $t = \frac{dist}{rate}$

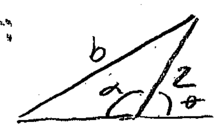
define θ (radians) around to point where she changes from walking to boating

walk: arc length $= r\theta_{rad}$

distance $= 2\theta$

so $t_{walk} = \frac{2\theta}{4}$

boat:



$\alpha = \pi - \theta$

$(180° - \theta)$

law of cosines:

$b^2 = 2^2 + 2^2 - 2(2)(2)\cos\alpha$

$b^2 = 8 - 8\cos\alpha$

$b = \sqrt{8-8\cos\alpha}$

so $t_{boat} = \frac{\sqrt{8-8\cos\alpha}}{2}$

$t' = 0$

$$-\frac{1}{2} + \frac{2\sin\alpha}{\sqrt{8-8\cos\alpha}} = 0$$

$$\frac{2\sin\alpha}{\sqrt{8-8\cos\alpha}} = \frac{1}{2}$$

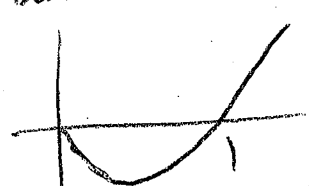
$$\frac{1}{2}\sqrt{8-8\cos\alpha} = 2\sin\alpha$$

$$(\sqrt{8-8\cos\alpha})^2 = (4\sin\alpha)^2$$

$$8 - 8\cos\alpha = 16\sin^2\alpha$$

$y_1 = 8 - 8\cos\alpha - 16(\sin\alpha)^2 = 0$

use calculator to find zeros from $\alpha = 0$ to π



$\alpha = 0$ $\quad$ $\alpha = 2.0943$ radians

t' DNE

$\sqrt{8-8\cos\alpha} = 0$

$8 - 8\cos\alpha = 0$

$8\cos\alpha = 8$

$\cos\alpha = 1$

$\alpha = 0$

2 critical values for α

try both and see which is minimum time

$\alpha = 0$ $\quad t = \frac{2(\pi-0)}{4} + \frac{\sqrt{8-8\cos 0}}{2}$

$= 1.570796$ hrs

$\alpha = 2.0943$

$t = \frac{2(\pi - 2.0943)}{4} + \frac{\sqrt{8-8\cos(2.0943)}}{2}$

$= 2.25564$

Minimum time is if $\alpha = 0$

(she walks all the way around and doesn't use the boat at all)