Unit 3 Review Solutions

① $g(x) = \dfrac{x^2+4x^{1/2}}{x^2} = 1+4x^{-3/2}$

$g'(x) = \boxed{4(-3/2)x^{-5/2}} = \boxed{\dfrac{-6}{x^{5/2}}}$

② $s(t) = \dfrac{1-2t}{t^{1/2}} = t^{-1/2} - 2t^{1/2}$

$s'(t) = \boxed{-\frac{1}{2}t^{-3/2} - 2\left(\frac{1}{2}t^{-1/2}\right)}$

$= \boxed{\dfrac{-1}{2t^{3/2}} - \dfrac{1}{\sqrt{t}}}$

③ $H(t) = \underline{\sin t}\ \underline{\sec^2 t}$

$\boxed{H'(t) = \sin t\,(2\sec t)(\sec t\tan t) + \sec^2 t\,(\cos t)}$

$H'(t) = 2\sin t\sec^2 t\tan t + \sec^2 t\cos t$

$= \boxed{\sec^2 t\,(2\sin t\tan t + \cos t)}$

④ $y = \sqrt{x^4+1} = (x^4+1)^{1/2}$

$\boxed{y' = \frac{1}{2}(x^4+1)^{-1/2}(4x^3)}$

$= \boxed{\dfrac{2x^3}{\sqrt{x^4+1}}}$

⑤ $y = (x^2+1)\sqrt[3]{x^2+2} = (x^2+1)(x^2+2)^{1/3}$

$\boxed{y' = (x^2+1)\left(\frac{1}{3}(x^2+2)^{-2/3}(2x)\right) + (x^2+2)^{1/3}(2x)}$

$= \dfrac{2x(x^2+1)}{3(x^2+2)^{2/3}} + \dfrac{2x(x^2+2)^{1/3}}{1}$

$= \dfrac{2x(x^2+1) + 2x(x^2+2)^{1/3}\,3(x^2+2)^{2/3}}{3(x^2+2)^{2/3}}$

$= \dfrac{2x\left[x^2+1+3(x^2+2)\right]}{3(x^2+2)^{2/3}}$

$\boxed{\dfrac{2x(4x^2+7)}{3(x^2+2)^{2/3}}}$

⑥ $y = e^{\cos t}$

$\boxed{y' = e^{(\cos t)}(-\sin t)}$

⑦ $y = 2^{\sin(\pi x)}$

$\boxed{y' = 2^{(\sin(\pi x))}\ln 2 \cdot \cos(\pi x)\cdot \pi}$

⑧ $y = \cot^7(x^5) = (\cot(x^5))^7$

$\boxed{y' = 7(\cot(x^5))^6(-\csc^2(x^5)\cdot 5x^4) = -35x^4\cot^6(x^5)\csc^2(x^5)}$

⑨ $f(t) = \sin^2(e^{\sin^2 t}) = (\sin(e^{(\sin^2 t)}))^2$

$\boxed{f'(t) = 2\left(\sin(e^{(\sin^2 t)})\right)^1 \cos(e^{(\sin^2 t)})\cdot e^{((\sin t)^2)}\cdot(2(\sin t)^1)\cos t}$

$= \boxed{4\sin t\cos t\, e^{(\sin^2 t)}\sin(e^{(\sin^2 t)})\cos(e^{(\sin^2 t)})}$

(10) $f(\theta) = \ln(\sin\theta)$

$$f'(\theta) = \boxed{\frac{1}{\sin\theta}\cos\theta}$$

$$= \boxed{\cot\theta}$$

(11) $G(s) = \ln^2(s) = (\ln(s))^2$

$$G'(s) = \boxed{2(\ln(s))^1 \cdot \frac{1}{s}}$$

$$= \boxed{\frac{2\ln s}{s}}$$

(12) $f(x) = (5x^2+8)(x^2-4x-6)$

$$f'(x) = \boxed{(5x^2+8)(2x-4) + (x^2-4x-6)(10x)}$$

$$= 10x^3 - 20x^2 + 16x - 32 + 10x^3 - 40x^2 - 60x$$

$$= 20x^3 - 60x^2 - 44x - 32$$

$$= \boxed{4(5x^3 - 15x^2 - 11x - 8)}$$

(13) $y = x^{\sqrt{x}}$ (logarithmic differentiation)

$$\ln y = \ln(x^{\sqrt{x}}) = \underbrace{\sqrt{x}\,\ln(x)}_{\text{product rule}}$$

$$\frac{1}{y}\frac{dy}{dx} = \sqrt{x}\,\frac{1}{x} + \ln x\left(\tfrac{1}{2}x^{-1/2}\right)$$

$$\frac{dy}{dx} = \left[\frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}}\right]y$$

$$\boxed{\frac{dy}{dx} = \left[\frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}}\right]x^{\sqrt{x}}} = \left[\frac{\sqrt{x}\,2\sqrt{x}}{x\,2\sqrt{x}} + \frac{x\ln x}{x\,2\sqrt{x}}\right]x^{\sqrt{x}}$$

$$= \left[\frac{2x + x\ln x}{2x\sqrt{x}}\right]x^{\sqrt{x}} = \frac{\cancel{x}\,(2+\ln x)x^{\sqrt{x}}}{\cancel{x}\,(2\sqrt{x})} = \boxed{\frac{(2+\ln x)x^{\sqrt{x}}}{2\sqrt{x}}}$$

(14) $y = \sin^{-1}(7x)$

$$y' = \boxed{\frac{1}{\sqrt{1-(7x)^2}}(7)} = \boxed{\frac{7}{\sqrt{1-(7x)^2}}}$$

(15) $y = x^{(\sec x)}$ (logarithmic differentiation)

$$\ln y = \ln(x^{\sec x}) = \underbrace{\sec x\,\ln(x)}_{\text{product rule}}$$

$$\frac{1}{y}\frac{dy}{dx} = \sec x\,\frac{1}{x} + \ln x(\sec x\tan x)$$

$$\frac{dy}{dx} = \left[\frac{\sec x}{x} + \ln x\sec x\tan x\right]y$$

$$\boxed{\frac{dy}{dx} = \left[\frac{\sec x}{x} + \ln x\sec x\tan x\right]x^{\sec x}}$$

$$= \frac{\sec x + x\ln x\sec x\tan x}{x}\,x^{\sec x}$$

$$= \boxed{\frac{\sec x(1 + x\ln x\tan x)}{x}\,x^{\sec x}}$$

(16) $y = \tan\sqrt{1+\csc\theta}$

$$y' = \sec^2(\sqrt{1+\csc\theta})\cdot\frac{d}{d\theta}\left[(1+\csc\theta)^{1/2}\right]$$

$$= \boxed{\sec^2(\sqrt{1+\csc\theta})\cdot\left(\tfrac{1}{2}(1+\csc\theta)^{-1/2}(-\csc\theta\cot\theta)\right)}$$

$$= \boxed{\frac{-\csc\theta\cot\theta\sec^2(\sqrt{1+\csc\theta})}{2\sqrt{1+\csc\theta}}}$$

(17) $y = \csc(a+\ln x)$ (a is a constant)

$$y' = -\csc(a+\ln x)\cot(a+\ln x)\cdot\frac{d}{dx}[a+\ln x]$$

$$= \boxed{-\csc(a+\ln x)\cot(a+\ln x)\left(\frac{1}{x}\right)}$$

(18) $y =$

(18) $y = e^x \cos x$ (tan line at $x=0$)
product rule

$y' = e^x(-\sin x) + (\cos x)e^x$

$y'\big|_{x=0} = e^0(-\sin 0) + (\cos 0)e^0$

$= 1(0) + (1)(1) = 1 = m$

$y\big|_{x=0} = e^0 \cos 0 = (1)(1) = 1$

pt $(0,1)$

$\boxed{(y-1) = 1\,(x-0)}$

(19) $y = \sin(\sin x)$ tan line at $(\pi, 0)$

$y' = \cos(\sin x) \cdot \cos x$

$y'\big|_{x=\pi} = \cos(\sin \pi) \cdot \cos \pi = \cos(0) \cdot \cos \pi$

$= (1)(-1) = -1 = m$

$\boxed{(y-0) = -(x-\pi)}$

(20) $x^2 + x \arctan y = y - 1$ tan line at $(-\pi/4, 1)$
(implicit differentiation)

$\frac{d}{dx}[x^2] + x\frac{d}{dx}[\arctan y] + \arctan y \frac{d}{dx}[x] = \frac{d}{dx}[y] - \frac{d}{dx}(1)$

$2x + x\frac{1}{1+y^2}\frac{dy}{dx} + \arctan y\,(1) = \frac{dy}{dx} - 0$

$\frac{dy}{dx}\left(\frac{x}{1+y^2} - 1\right) = -2x - \arctan y$

$\frac{dy}{dx} = \frac{-2x - \arctan y}{\frac{x}{1+y^2} - 1}\Bigg|_{(-\pi/4,1)} = \frac{-2(-\frac{\pi}{4}) - \arctan(1)}{\frac{(-\pi/4)}{1+(1)^2} - 1}$

$\frac{dy}{dx} = \frac{\pi/2 - \pi/4}{-\frac{\pi}{8} - 1} = \frac{\pi/4}{-\frac{\pi}{8} - 1}\frac{(8)}{(8)} = \frac{2\pi}{-\pi - 8}$

$\boxed{(y-1) = \left(\frac{2\pi}{-\pi-8}\right)\left(x + \frac{\pi}{4}\right)}$

(21) $y - x^2y^2 = 6$
(implicit diff.)

$\frac{d}{dx}[y] - \left(x^2\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x^2]\right) = \frac{d}{dx}[6]$

$\frac{dy}{dx} - x^2 2y\frac{dy}{dx} - y^2 2x = 0$

$\frac{dy}{dx}(1 - 2x^2y) = 2xy^2$

$\boxed{\frac{dy}{dx} = \frac{2xy^2}{1-2x^2y}}$

(22) Find y'': $x^4 - 2xy + y^4 = 16$ (implicit differentiation)
product rule

$\frac{d}{dx}[x^4] - \left(2x\frac{d}{dx}[y] + y\frac{d}{dx}[2x]\right) + \frac{d}{dx}[y^4] = \frac{d}{dx}[16]$

$4x^3 - 2x\frac{dy}{dx} - 2y + 4y^3\frac{dy}{dx} = 0$

$\frac{dy}{dx}(4y^3 - 2x) = 2y - 4x^3$

$\frac{dy}{dx} = \frac{2y-4x^3}{4y^3-2x} = \frac{y-2x^3}{2y^3-x}$

now...

$\frac{d^2y}{dx^2} = \frac{(2y^3-x)\frac{d}{dx}[y-2x^3] - (y-2x^3)\frac{d}{dx}[2y^3-x]}{(2y^3-x)^2}$

$= \frac{(2y^3-x)\left(\boxed{\frac{dy}{dx}} - 6x^2\right) - (y-2x^3)\left(6y^2\boxed{\frac{dy}{dx}} - 1\right)}{(2y^3-x)^2}$

$\boxed{= \frac{(2y^3-x)\left(\left[\frac{y-2x^3}{2y^3-x}\right] - 6x^2\right) - (y-2x^3)\left(6y^2\left[\frac{y-2x^3}{2y^3-x}\right] - 1\right)}{(2y^3-x)^2}}$

(don't try to simplify this one ☺)

(23) $x = y^3 - 7y^2 + 2$ Find $\frac{dy}{dx}$ at $(-4,1)$

$\frac{d}{dx}[x] = \frac{d}{dx}[y^3] - \frac{d}{dx}[7y^2] + \frac{d}{dx}[2]$

$1 = 3y^2\frac{dy}{dx} - 14y\frac{dy}{dx} + 0$

$\frac{dy}{dx}(3y^2 - 14y) = 1$

$\frac{dy}{dx} = \frac{1}{3y^2 - 14y}\Bigg|_{(-4,1)} = \frac{1}{3(1)^2 - 14(1)} = \boxed{\frac{1}{-11}}$

(24) $\sqrt{8.5}$

$f(x)=\sqrt{x} \quad a=9$

$f(x)\approx f(a)+f'(a)(x-a)$

$f(x)=\sqrt{x} \quad f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

$f(9)=\sqrt{9}=3 \quad f'(9)=\frac{1}{2\sqrt{9}}=\frac{1}{6}$

$f(x)\approx 3+\left(\frac{1}{6}\right)(x-9)$

$f(8.5)\approx 3+\frac{1}{6}(8.5-9)=\boxed{\frac{35}{12}}$

(25) $(2.001)^5$

$f(x)=x^5 \quad f'(x)=5x^4$

$f(2)=2^5 \quad f'(2)=5(2)^4$

$=32 \quad =80$

$f(x)\approx f(a)+f'(a)(x-a)$

$f(x)\approx 32+80(x-2)$

$f(2.001)\approx 32+80(2.001-2)=\boxed{32.080=\frac{802}{25}}$

(26) $e^{-0.015}$

$f(x)=e^x \quad f'(x)=e^x$

$f(0)=e^0 \quad f'(0)=e^0$

$=1 \quad =1$

$f(x)\approx f(a)+f'(a)(x-a)$

$f(x)\approx 1+(1)(x-0)$

$f(-0.015)\approx 1+(1)(-0.015-0)=\boxed{0.985}$

(27) $f(x)=x^{3/4} \quad a=16$

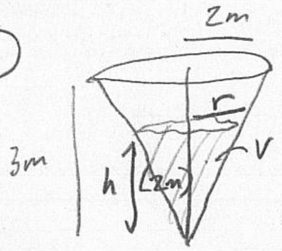
$f(x)=x^{3/4} \quad f'(x)=\frac{3}{4}x^{-1/4}$

$f(16)=8 \quad f'(16)=\frac{3}{8}$

$f(x)\approx f(a)+f'(a)(x-a)$

$\boxed{f(x)\approx 8+\frac{3}{8}(x-16)}$

(28)

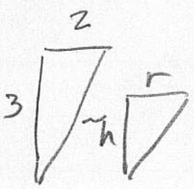


$\frac{dV}{dt}=+2 \text{ m}^3/\text{min}$ ← "flows in"

Find $\frac{dh}{dt}$

$V_{cone}=V=\frac{1}{3}\pi r^2 h$

← 3 variables (need another equation)



$\frac{2}{3}=\frac{r}{h}$ sub back in

$2h=3r$

$r=\frac{2h}{3}$

$V=\frac{1}{3}\pi\left(\frac{2h}{3}\right)^2 h$

$V=\frac{1}{3}\pi\frac{4}{9}h^3$

$V=\frac{4\pi}{27}h^3$

$\frac{d}{dt}[V]=\frac{d}{dt}\left[\frac{4\pi}{27}h^3\right]$

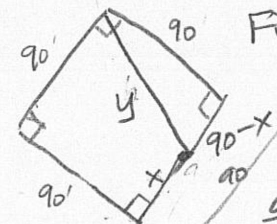
$\frac{dV}{dt}=\frac{4\pi}{9}h^2\frac{dh}{dt}$

$(2)=\frac{4\pi}{9}(2)^2\frac{dh}{dt}$

$\boxed{\frac{dh}{dt}=\frac{9}{8\pi}=0.358 \text{ m/min}}$

* look at 3.7 homework for more related rates problem examples *

(29) (a)



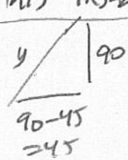
Find $\frac{dy}{dt}$, $\frac{dx}{dt}=+30$ ft/sec

$(90-x)^2+90^2=y^2$

$\frac{d}{dt}\left[(90-x)^2\right]+\frac{d}{dt}\left[90^2\right]=\frac{d}{dt}\left[y^2\right]$

$2(90-x)^1(-1)\frac{dx}{dt}+0=2y\frac{dy}{dt}$

at this instant: $x=\frac{1}{2}90=45$ ft

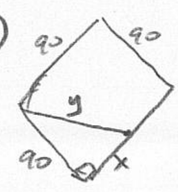


$y=\sqrt{90^2+45^2}=\sqrt{10125}$

$2(90-45)(-1)(30)=2(\sqrt{10125})\frac{dy}{dt}$

$\frac{dy}{dt}=\frac{-1350}{\sqrt{10125}}=\boxed{-13.416 \text{ ft/sec}}$

(b)



Find $\frac{dy}{dt}$, $\frac{dx}{dt}=+30$ ft/sec

$x^2+90^2=y^2$

$\frac{d}{dt}[x^2]+\frac{d}{dt}[90^2]=\frac{d}{dt}[y^2]$

$2x\frac{dx}{dt}+0=2y\frac{dy}{dt}$

$x\frac{dx}{dt}=y\frac{dy}{dt}$

At this instant: $x=\frac{1}{2}90=45$ ft

$y=\sqrt{45^2+90^2}=\sqrt{10125}$

$(45)(30)=(\sqrt{10125})\frac{dy}{dt}$

$\frac{dy}{dt}=\frac{1350}{\sqrt{10125}}=\boxed{+13.416 \text{ ft/sec}}$

30 $C(x)=x^2+2$ (thousands of \$)
$R(x)=2x^2-x$ x in units

(a) marginal cost $= \boxed{C'(x)=2x}$

(b) marginal revenue $= \boxed{R'(x)=4x-1}$

(c) Profit = revenue − cost $= 2x^2-x-(x^2+2) = \boxed{x^2-x-2}$

(d) Breakeven when $C=R$

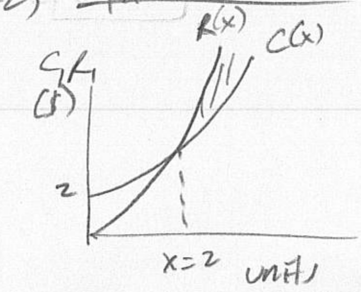
$x^2+2=2x^2-x$

$x^2-x-2=0$

$(x+1)(x-2)=0$

$\cancel{x=-1}$, $x=2$

can't have negative units



For these functions, when $x>2$ Revenue > Cost (this is a positive profit)

(e) $C'(3)=2(3)=6$ (= \$6000/unit)

When 3 units have been made, the cost to make the next unit is \$6000.
(Must be an expensive item, such as an automobile).

31 $s(t)=2t^3-4t^2+t$

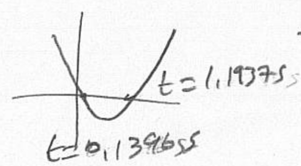
(a) $\boxed{v(t)=s'(t)=6t^2-8t+1}$

(b) $\boxed{a(t)=v'(t)=12t-8}$

(c) $v=0$ graph in calculator

$6t^2-8t+1=0$

$\boxed{t=0.1396\text{ sec } \& \ t=1.1937\text{ sec}}$



or quadratic formula:

$t=\dfrac{8\pm\sqrt{64-4(6)(1)}}{2(6)}$

$=\dfrac{8\pm\sqrt{40}}{12} = 1.1937, 0.1396$ sec

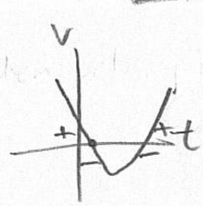
(d) $a=0$

$12t-8=0$

$12t=8$

$t=\dfrac{8}{12}=\boxed{\dfrac{2}{3}=0.667\text{ sec}}$

(e)



at $t=0.1396$ sec the velocity is going from + to − meaning the particle was moving in the positive direction but it has stopped and is changing to move in the negative direction.

at $t=1.1937$ sec, the velocity is going from − to + meaning the particle was moving in the negative direction but it has stopped and is changing to move in the positive direction.