AP Calculus BC Name: Solutions

Unit2 Review Period: ______

These problems provide an overview of the unit but we recommend that you also review all homework problems from the unit.

#1) If $f(x)=\dfrac{x}{1+2x}$, find the slope of the tangent line at $\left(-\dfrac{1}{4},-\dfrac{1}{2}\right)$ and use it to write the equation of the tangent line to the curve.

$$m=f'(-\tfrac{1}{4})=\lim_{x\to-\frac{1}{4}}\frac{f(x)-f(-\frac{1}{4})}{x-(-\frac{1}{4})}$$

$$=\lim_{x\to\frac{1}{4}}\frac{\frac{x}{1+2x}-\frac{(-\frac{1}{4})}{1+2(-\frac{1}{4})}}{x+\frac{1}{4}}$$

$$=\lim_{x\to-1/4}\frac{\frac{x}{1+2x}+\frac{1}{2}}{x+\frac{1}{4}}\cdot\frac{4(1+2x)}{4(1+2x)}$$

$$=\lim_{x\to-1/4}\frac{4x+2(1+2x)}{(4x+1)(1+2x)}$$

$$f'(-\tfrac{1}{4})=\lim_{x\to-1/4}\frac{8x+2}{(4x+1)(1+2x)}$$

$$=\lim_{x\to-1/4}\frac{2(4x+1)}{(4x+1)(1+2x)}$$

$$=\lim_{x\to-1/4}\frac{2}{1+2x}$$

$$=\frac{2}{1+2(-\frac{1}{4})}$$

$$\boxed{m=4}$$

tangent line:

$$\boxed{(y+\tfrac{1}{2})=4(x+\tfrac{1}{4})}$$

#2) Find the slope of the tangent line to the curve $y=9-2x^2$ at the point $(2,1)$.

Find an equation of this tangent line.

$$m=f'(2)=\lim_{x\to2}\frac{f(x)-f(2)}{x-2}$$

$$=\lim_{x\to2}\frac{[9-2x^2]-[9-2(2)^2]}{x-2}$$

$$=\lim_{x\to2}\frac{-2x^2+8}{x-2}$$

$$=\lim_{x\to2}\frac{-2(x^2-4)}{x-2}$$

$$=\lim_{x\to2}\frac{-2\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$$

$$=\lim_{x\to2}-2(x+2)$$

$$=-2((2)+2)$$

$$\boxed{m=-8}$$

tangent line:

$$\boxed{(y-1)=-8(x-2)}$$

#3) The displacement (in meters) of an object moving in a straight line is given by $s = 1 + 2t + \frac{t^2}{4}$, where t is measured in seconds.

(a) Find the average velocity over the following time periods:

(i) [1,3] (ii) [1,2] (iii) [1,1.5] (iv) [1,1.1]

(b) Find the instantaneous velocity when $t = 1$.

(a)(i) $V_{avg} = \frac{f(3)-f(1)}{3-1} = \frac{9.25-3.25}{3-1} = \frac{6}{2} = \boxed{3 \text{ m/s}}$

(ii) $V_{avg} = \frac{f(2)-f(1)}{2-1} = \frac{6-3.25}{2-1} = \frac{2.75}{1} = \boxed{2.75 \text{ m/s}}$

(iii) $V_{avg} = \frac{f(1.5)-f(1)}{1.5-1} = \frac{4.5625-3.25}{1.5-1} = \frac{1.3125}{.5} = \boxed{2.625 \text{ m/s}}$

(iv) $V_{avg} = \frac{f(1.1)-f(1)}{1.1-1} = \frac{3.5025-3.25}{1.1-1} = \frac{0.2525}{0.1} = \boxed{2.525 \text{ m/s}}$

(b) $f'(1) = \lim_{t\to1} \frac{f(t)-f(1)}{t-1}$

$= \lim_{t\to1} \frac{[1+2t+\frac{t^2}{4}]-[1+2(1)+\frac{(1)^2}{4}]}{t-1}$

$= \lim_{t\to1} \frac{\frac{1}{4}t^2+2t-\frac{9}{4}}{t-1}$

$f'(1) = \lim_{t\to1} \frac{1}{4}\,\frac{t^2+8t-9}{t-1}$

$= \lim_{t\to1} \frac{1}{4}\,\frac{(t-1)(t+9)}{(t-1)}$

$= \lim_{t\to1} \frac{1}{4}(t+9) = \frac{1}{4}((1)+9) = \boxed{\frac{10}{4} \text{ m/s}}$

(2.5)

#4) Find values for a and b that will make f continuous everywhere, if $f(x) = \begin{cases} 3x+1, & x<2 \\ ax+b, & 2 \le x < 5 \\ x^2, & 5 \le x \end{cases}$

at $x=2$

$\lim_{x\to2^-} 3x+1 = \lim_{x\to2^+} ax+b$

$3(2+1) \quad a(2)+b$

$7 = 2a+b$

at $x=5$

$\lim_{x\to5^-} ax+b = \lim_{x\to5^+} x^2$

$a(5)+b = 5^2$

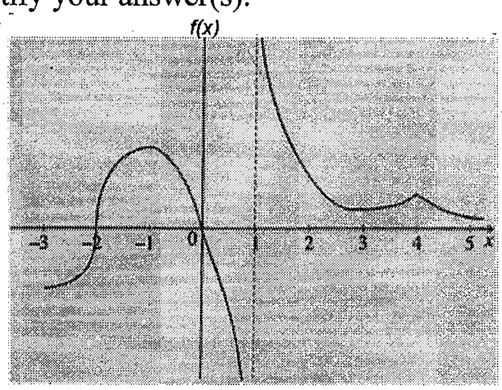
$5a+b=25$

system: $\begin{cases} 2a+b=7 \\ 5a+b=25 \end{cases}$

$5a+b=25$
$-2a-b=-7$
$3a=18$, $\boxed{a=6}$

$2(6)+b=7$
$\boxed{b=-5}$

#5) The graph of $f(x)$ is given below. For which value(s) of x is $f(x)$ not differentiable? Justify your answer(s).



at: $x=-2$ (vertical tangent)

$x=1$ (discontinuity/vertical asymptote)

$x=4$ (abrupt change, 'cusp')

#6) Given:

$$f(x)=\begin{cases} 1, & x\le -1 \\ -x, & -1<x<0 \\ 1, & x=0 \\ -x, & 0<x<1 \\ 1, & x\ge 1 \end{cases}$$

(a) $\lim_{x\to -1^-} 1 = \boxed{1}$ $\lim_{x\to -1^+} -x = \boxed{1}$

$\lim_{x\to 0^-} -x = \boxed{0}$ $\lim_{x\to 0^+} -x = \boxed{0}$

$\lim_{x\to 1^-} = -x = \boxed{-1}$ $\lim_{x\to 1^+} 1 = \boxed{1}$

(a) Find the right-hand and left-hand limits of *f* at *x* = -1, 0, and 1.
(b) Does *f* have a limit as *x* approaches -1? 0? 1?
If so, what is it? If not, why not?
(c) Is *f* continuous at *x* = -1? 0? 1? Explain.

(b) $\lim_{x\to -1} f(x) = \boxed{1}$ $\lim_{x\to 0} f(x) = \boxed{0}$ $\lim_{x\to 1} f(x)\ \boxed{DNE}$ (because LH, RH are not equal)

(c)

at -1

✓ 1) $f(-1)=1$ exists

✓ 2) $\lim_{x\to -1} f(x) = 1$ exists

✓ 3) $f(-1) \stackrel{?}{=} \lim_{x\to -1} f(x)$

$1 = 1$

$\boxed{\text{yes, } f \text{ is continuous at } x=-1}$

at 0

✓ 1) $f(0)=1$ exists

✓ 2) $\lim_{x\to 0} f(x) = 0$ exists

x 3) $f(0) \stackrel{?}{=} \lim_{x\to 0} f(x)$

$1 \neq 0$

$\boxed{\text{No, } f \text{ is discontinuous at } x=0}$

at 1

✓ 1) $f(1)=1$ exists

x 2) $\lim_{x\to 1} f(x)$ DNE

$\boxed{\text{No, } f \text{ is discontinuous at } x=1}$

#7) Find the <u>average</u> rate of change of $f(x)=1+\sin x$ over the interval $[0, \frac{\pi}{2}]$.

$$\frac{f(\frac{\pi}{2})-f(0)}{\frac{\pi}{2}-0}$$

$$\frac{[1+\sin(\frac{\pi}{2})]-[1+\sin(0)]}{\frac{\pi}{2}}$$

$$\frac{[1+1]-[1+0]}{\frac{\pi}{2}}$$

$$\frac{1}{(\frac{\pi}{2})} = \boxed{\frac{2}{\pi}}$$

#8) Let $f(x)=x^2-3x$ and $P=(1, f(1))$.

Find (a) the slope of the curve $y=f(x)$ at P,

(b) an equation of the tangent at P,

(c) an equation of the normal at P.

(a) $m=f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$

$=\lim_{x\to 1}\frac{[(x)^2-3(x)]-[(1)^2-3(1)]}{x-1}$

$=\lim_{x\to 1}\frac{x^2-3x+2}{x-1}$

$=\lim_{x\to 1}\frac{\cancel{(x-1)}(x-2)}{\cancel{x-1}}$

$=\lim_{x\to 1} x-2=(1)-2=\boxed{-1}$

b) $y=f(1)=(1)^2-3(1)=-2$

$\boxed{(y+2)=-(x-1)}$

c) 'normal' means $\perp$ to the tangent line

so $m=-\frac{1}{(-1)}=1$

$\boxed{(y+2)=1(x-1)}$

#9) Is there a number that is exactly 4 more than its cube? $x \overset{?}{=} x^3+4$

$f(x)=x^3-x+4=0$

use the Intermediate Value Theorem

$f(0)=(0)^3-(0)+4=4$

$f(-2)=(-2)^3-(-2)+4=-2$

Yes, because $f(x)$ is a polynomial, it is continuous over $(-\infty,\infty)$ and since $f(0)=4$ and $f(-2)=-2$ by the I.V.T. there must exist a c, $-2\le c\le 0$ such that $f(c)=0$ b/c $-2\le f(c)\le 4$.

#10) Which of the following values is the average rate of change of $f(x)=\sqrt{x+1}$ over the interval [0, 3] ?

(multiple choice): a) -3 b) -1 c) -1/3 $\boxed{\text{d) } 1/3}$ e) 3

$\frac{f(3)-f(0)}{3-0}$

$\frac{\sqrt{3+1}-\sqrt{0+1}}{3}$

$\frac{2-1}{3}$

$\boxed{\frac{1}{3}}$

#11) Which of the following statements is false for the function

$$f(x)=\begin{cases} \frac{3}{4}x, & 0\le x<4 \\ 2, & x=4 \\ -x+7, & 4<x\le 6 \\ 1, & 6<x<8 \end{cases}$$

a) $\lim_{x\to 4^-} \frac{3}{4}x \quad \lim_{x\to 4^+} -x+7$
$3 = 3$ true

b) $f(4)=2$ true

c) $\lim_{x\to 6^-} -x+7 \quad \lim_{x\to 6^+} 1$
$1 = 1$ true

d) $\lim_{x\to 8^-} 1 = 1$ true

e) ✓ 1) $f(4)=2$ exists
✓ 2) $\lim_{x\to 4} f(x) = 3$ exists
✗ 3) $f(4) \stackrel{?}{=} \lim_{x\to 4} f(x)$ no,
$2 \neq 3$ not continuous

(multiple choice):

a) $\lim_{x\to 4} f(x)$ exists

b) $f(4)$ exists

c) $\lim_{x\to 6} f(x)$ exists

d) $\lim_{x\to 8^-} f(x)$ exists

e) f is continuous at $x=4$

#12) Which of the following is an equation for the tangent line to $f(x)=9-x^2$ at $x=2$?

(multiple choice):

a) $y=\frac{1}{4}x+\frac{9}{2}$

b) $y=-4x+13$

c) $y=-4x-3$

d) $y=4x-3$

e) $y=4x+13$

$$m=f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$$
$$=\lim_{x\to 2}\frac{[9-(x)^2]-[9-(2)^2]}{x-2}$$
$$=\lim_{x\to 2}\frac{-x^2+4}{x-2}$$
$$=\lim_{x\to 2}\frac{-(x^2-4)}{x-2}$$
$$=\lim_{x\to 2}\frac{-\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$$
$$=\lim_{x\to 2} -(x+2) = -(2+2) = -4$$

$y=f(2)=9-(2)^2=5$

tan line: $(y-5)=-4(x-2)$
$y-5=-4x+8$
$\boxed{y=-4x+13}$

#13) (a) If $f(x)=e^{-x^2}$, estimate the value of $f'(1)$ ~~graphically~~ and numerically.

(b) Find an approximate equation of the tangent line to the curve $f(x)=e^{-x^2}$ at the point where $x = 1$

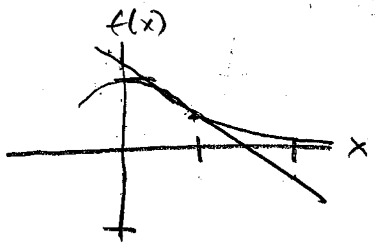
(c) Illustrate part (b) by graphing the curve and the tangent line on the same axes.

(a) numerically: pick a small interval $[1, 1.1]$

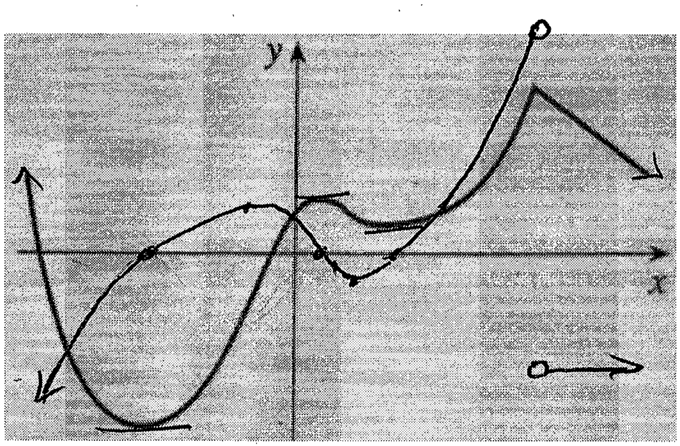
avg change $\approx f'(x) = \dfrac{f(1.1)-f(1)}{1.1-1} = \dfrac{e^{-(1.1)^2}-e^{-(1)^2}}{1.1-1} = \dfrac{-0.0696821618}{0.1} = \boxed{-0.6968}$

(b) $f(1) \approx .36788$ $\boxed{(y-.3679) = -0.6968(x-1)}$

(c)

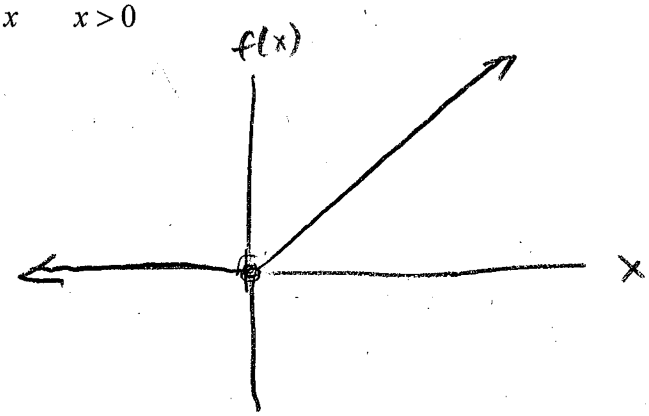


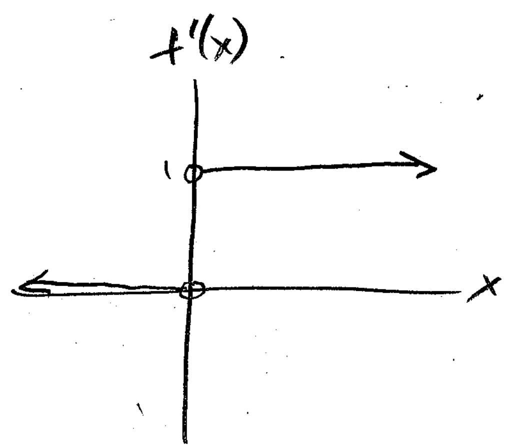
#14) Given the graph of f, sketch a graph of its derivative.



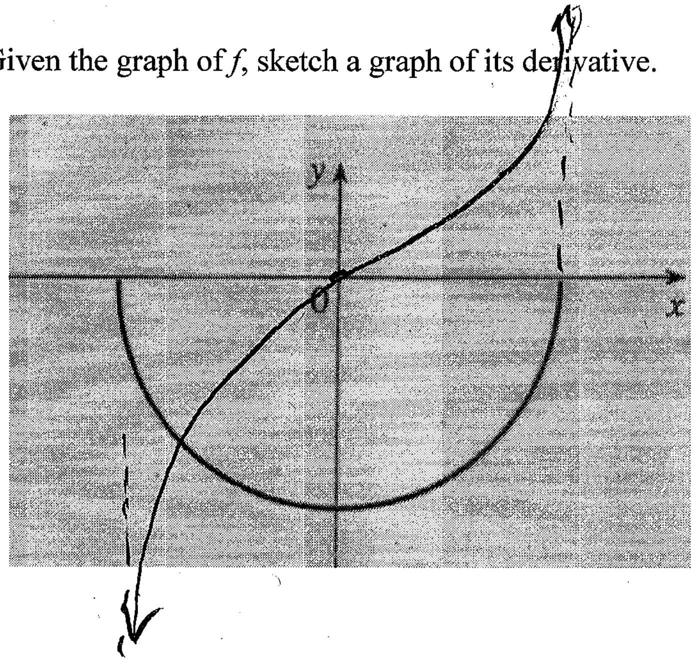
#15) Sketch the graph of f, then sketch the derivative of f.

$$f(x)=\begin{cases} 0, & x \le 0 \\ x & x > 0 \end{cases}$$





#16) Given the graph of f, sketch a graph of its derivative.



#17) Given the graph of f, sketch a graph of its derivative.