AP Calculus BC

10.7 Worksheet

Name: answers/solutions

Period: ____________

Find the nth MacLaurin polynomial for the given function.

1. $f(x) = e^{4x},\ n = 4$

$\boxed{P_4(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4}$

2. $f(x) = e^{-x/2},\ n = 4,\ f(0) = 1$

$f'(x) = -\frac{1}{2}e^{-1/2x},\ f'(0) = -\frac{1}{2}$

$f''(x) = \frac{1}{4}e^{-1/2x},\ f''(0) = \frac{1}{4}$

$f'''(x) = -\frac{1}{8}e^{-1/2x},\ f'''(0) = -\frac{1}{8}$

$f^{(4)}(x) = \frac{1}{16}e^{-1/2x},\ f^{(4)}(0) = \frac{1}{16}$

$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$

$= 1 + (-\frac{1}{2})x + \frac{(1/4)}{2}x^2 + \frac{(-1/8)}{6}x^3 + \frac{(1/16)}{24}x^4$

$= \boxed{1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4}$

3. $f(x) = \sin(x),\ n = 5$

$\boxed{P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$

4. $f(x) = \cos(\pi x),\ n = 4,\ f(0) = 1$

$f'(x) = -\pi\sin(\pi x),\ f'(0) = 0$

$f''(x) = -\pi^2\cos(\pi x),\ f''(0) = -\pi^2$

$f'''(x) = \pi^3\sin(\pi x),\ f'''(0) = 0$

$f^{(4)}(x) = \pi^4\cos(\pi x),\ f^{(4)}(0) = \pi^4$

$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$

$= 1 + (0)x + \frac{-\pi^2}{2}x^2 + \frac{0}{6}x^3 + \frac{\pi^4}{24}x^4$

$= \boxed{1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4}$

5. $f(x) = xe^x$, $n = 4$

$$\boxed{P_4(x) = x + x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{6}x^4}$$

6. $f(x) = x^2e^{-x}$, $n = 4$, $f(0)=0$

$f'(x) = x^2(-e^{-x}) + e^{-x}(2x) = -x^2e^{-x} + 2xe^{-x}$, $f'(0) = 0$

$f''(x) = (-x^2)(-e^{-x}) + e^{-x}(-2x) + 2x(-e^{-x}) + e^{-x}(2) = x^2e^{-x} - 4xe^{-x} + 2e^{-x}$, $f''(0) = 2$

$f'''(x) = x^2(-e^{-x}) + e^{-x}(2x) + (-4x)(-e^{-x}) + e^{-x}(-4) - 2e^{-x} = -x^2e^{-x} + 6xe^{-x} - 6e^{-x}$, $f'''(0) = -6$

$f^{(4)}(x) = (-x^2)(-e^{-x}) + e^{-x}(-2x) + 6x(-e^{-x}) + e^{-x}(6) + 6e^{-x} = x^2e^{-x} - 8xe^{-x} + 12e^{-x}$, $f^{(4)}(0) = 12$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$= 0 + 0x + \frac{2}{2}x^2 + \frac{-6}{6}x^3 + \frac{12}{24}x^4 = \boxed{x^2 - x^3 + \tfrac{1}{2}x^4}$$

7. $f(x) = \frac{1}{x+1}$, $n = 5$

$$\boxed{P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5}$$

8. $f(x) = \frac{x}{x+1}$, $n = 4$, $f(0) = 0$

$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$, $f'(0) = 1$

$f''(x) = \frac{(x+1)^2(0) - 1(2(x+1))}{(x+1)^4} = \frac{-2}{(x+1)^3}$, $f''(0) = -2$

$f'''(x) = \frac{(x+1)^3(0) - (-2)(3(x+1)^2)}{(x+1)^6} = \frac{6}{(x+1)^4}$, $f'''(0) = 6$

$f^{(4)}(x) = \frac{(x+1)^4(0) - 6(4(x+1)^3)}{(x+1)^8} = \frac{-24}{(x+1)^5}$, $f^{(4)}(0) = -24$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$= 0 + 1x + \frac{-2}{2}x^2 + \frac{6}{6}x^3 + \frac{-24}{24}x^4$$

$$= \boxed{x - x^2 + x^3 - x^4}$$

9. $f(x) = \sec(x),\ n = 2$

$$\boxed{P_2(x) = 1 + \tfrac{1}{2}x^2}$$

10. $f(x) = \tan(x),\ n = 3$, $f(0)=0$

$f'(x) = \sec^2 x$, $f'(0) = 1$

$f''(x) = 2(\sec x)\sec x \tan x$

$= 2\sec^2 x \tan x$, $f''(0) = 0$

$f'''(x) = (2\sec^2 x)(\sec^2 x) + \tan x(4\sec x \sec x \tan x)$

$= 2\sec^4 x + 4\tan^2 x \sec^2 x$, $f'''(0) = 2$

$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$= 0 + 1x + \frac{0}{2}x^2 + \frac{2}{6}x^3$

$= \boxed{x + \tfrac{1}{3}x^3}$

Find the nth Taylor polynomial for the given function centered at c.

11. $f(x) = \sqrt{x},\ n = 3,\ c = 4$

$$\boxed{P_3(x) = 2 + \tfrac{1}{4}(x-4) - \tfrac{1}{64}(x-4)^2 + \tfrac{1}{512}(x-4)^3}$$

12. $f(x) = \sqrt[3]{x},\ n = 3,\ c = 8$

$= x^{1/3}$, $f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$, $f'(8) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3}$, $f''(8) = \frac{-1}{144}$

$f'''(x) = \frac{10}{27}x^{-8/3}$, $f'''(8) = \frac{5}{3456}$

$P_3(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 + \frac{f'''(8)}{3!}(x-8)^3$

$= 2 + \frac{1}{12}(x-8) + \frac{-1/144}{2}(x-8)^2 + \frac{5/3456}{6}(x-8)^3$

$= \boxed{2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2 + \tfrac{5}{20736}(x-8)^3}$

13. $f(x) = \ln(x),\ n = 4,\ c = 2$

$$\boxed{P_4(x) = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4}$$

14. $f(x) = x^2\cos(x),\ n = 2,\ c = \pi$, $f(\pi) = -\pi^2$

$P_2(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2$

$f'(x) = x^2(-\sin x) + \cos x(2x) = -x^2\sin x + 2x\cos x$

$f'(\pi) = -2\pi$

$f''(x) = (-x^2)\cos x + \sin x(-2x) + 2x(-\sin x) + \cos x(2)$

$= -x^2\cos x - 4x\sin x + 2\cos x$

$f''(\pi) = \pi^2 - 2$

$$\boxed{-\pi^2 + (-2\pi)(x-\pi) + \frac{(\pi^2-2)}{2}(x-\pi)^2}$$

Determine the degree of the MacLaurin polynomial required for the error in the approximation of the function at the indicated value to be less than 0.001.

15. $f(x) = \sin(x),\ approximate\ f(0.3)$

$$\boxed{N = 3}$$

See this example ↓ (Probably best to just build the actual polynomial and check the actual error for each truncation)

16. $f(x) = \cos(x),\ approximate\ f(0.1)$

$R_N < \frac{f^{(N+1)}(z)}{(N+1)!}(x-c)^{N+1}$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

max possible $f^{(N+1)}(z) = 1$

& $c = 0$ (Maclaurin)

then; $R_N < 0.001$ guess-and-check

$\frac{(1)}{(N+1)!}(0.1-0)^{N+1} < 0.001$

$R_1 = 0.1$

$R_2 = 0.005$

$R_3 = 1.66 \cdot 10^{-4}$ ✓ $\boxed{N=3}$

{ this is an approximation

But ... for $N=2$ $P_2(x) = 1 - \frac{1}{2}x^2$, $P_2(0.1) = 1 - \frac{1}{2}(0.1)^2 = 0.995$ and $\cos(0.1) = 0.995004165$

so actual error for $N=2$ is $4.165 \cdot 10^{-6}$ ($N=2$ also OK)

(reason this is lower is $f^{(N+1)}(z)$ is not at max of '1')

* this is really the correct answer

17. $f(x) = e^x$, *approximate* $f(0.6)$

$\boxed{N=4}$

18. $f(x) = \ln(x)$, *approximate* $f(1.25)$ whoops! Can't do Maclaurin because ln(0) DNE...

...but we could do a Taylor polynomial centered at $c=1$:

$f(x)=\ln(x)$, $f(1)=\ln(1)=0$

$f'(x)=\frac{1}{x}=x^{-1}$, $f'(1)=1$

$f''(x)=-x^{-2}=\frac{-1}{x^2}$, $f''(1)=-1$

$f'''(x)=2x^{-3}=\frac{2}{x^3}$, $f'''(1)=2$

$f^{(4)}(x)=-6x^{-4}$, $\frac{-6}{x^4}$, $f^{(4)}(1)=-6$ actual: $\ln(1.25)=0.223435513$

$\vdots$

$P=0+1(x-1)-\frac{1}{2!}(x-1)^2+\frac{2}{3!}(x-1)^3-\frac{6}{4!}(x-1)^4+\ldots$

$P_1(1.25)=.25$, error $=.026857$

$P_2(1.25)=.21875$, error $=-.00439$

$P_3(1.25)=.2239583\overline{3}$, error $=8.15 \times 10^{-4}$ ✓

$\boxed{N=3}$

AP Calculus BC

10.8 Worksheet [redacted]

State where the power series is centered.

1. $\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$

$\boxed{C=0}$

2. $\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)}{2^n n!}x^n$

$\boxed{C=0}$

3. $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^3}$

$\boxed{C=2}$

4. $\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi)^{2n}}{(2n)!}$

$\boxed{C=\pi}$

Find the radius of convergence of the power series.

5. $\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^n}{n+1}\right)$

$\boxed{R=1}$

6. $\sum_{n=0}^{\infty}(3x)^n$

$$\lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(3x)^n}\right| = \lim_{n\to\infty}\left|\frac{(3x)(3x)^n}{(3x)^n}\right|$$

$$= \lim_{n\to\infty}|3x| = |3x|<1$$

$$|x|<\frac{1}{3}$$

$\boxed{R=\frac{1}{3}}$

7. $\sum_{n=1}^{\infty}\left(\frac{(4x)^n}{n^2}\right)$

$\boxed{R=\frac{1}{4}}$

8. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{5^n}$

$$\lim_{n\to\infty}\left|\frac{\left(\frac{x^{n+1}}{5^{n+1}}\right)}{\left(\frac{x^n}{5^n}\right)}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{5^{n+1}}\cdot\frac{5^n}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x\,x^n 5^n}{5\cdot 5^n x^n}\right| = \lim_{n\to\infty}\left|\frac{1}{5}x\right|$$

$$\left|\frac{1}{5}x\right|<1$$

$$|x|<5$$

$\boxed{R=5}$

9. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$\boxed{R = \infty}$

10. $\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

$$\lim_{n\to\infty} \left| \frac{\left(\frac{(2n+2)!\,x^{2n+2}}{(n+1)!}\right)}{\left(\frac{(2n)!\,x^{2n}}{n!}\right)} \right| = \lim_{n\to\infty} \left| \frac{(2n+2)!\,x^{2n+2}}{(n+1)!} \frac{n!}{(2n)!\,x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)(2n)!\,x^2 x^{2n}\,n!}{(n+1)n!\,(2n)!\,x^{2n}} \right|$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{n+1} |x^2| = \lim_{n\to\infty} \frac{4n^2+6n+2}{n+1} |x^2|$$

$$\infty \cdot |x^2| < 1$$

$$|x^2| = 0, \ \boxed{R = 0}$$

Find the interval of convergence of the power series.

11. $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

$\boxed{(-4, 4)}$

(ignore endpoints for now)

12. $\sum_{n=0}^{\infty} (2x)^n$

$$\lim_{n\to\infty} \left| \frac{(2x)^{n+1}}{(2x)^n} \right| = \lim_{n\to\infty} \left| \frac{(2x)(2x)^n}{(2x)^n} \right|$$

$$= \lim_{n\to\infty} |2x|, \ |2x| < 1, \ |x| < \frac{1}{2}$$

$-1/2 \quad 0 \quad 1/2$

$\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$

13. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

$\boxed{(-1, 1)}$

14. $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

$$\lim_{n\to\infty} \left| \frac{(n+2)x^{n+1}}{(n+1)x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+2)x \cdot x^n}{(n+1)x^n} \right|$$

$$= \lim_{n\to\infty} \frac{n+2}{n+1} |x| = 1 \cdot |x| < 1, \ |x| < 1$$

$-1 \quad 0 \quad 1$

$\boxed{(-1, 1)}$

15. $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

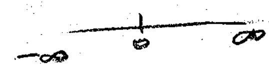
$$\boxed{(-\infty, \infty)}$$

16. $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

$$\lim_{n\to\infty} \left| \frac{(3x)^{n+1}}{(2n+2)!} \frac{(2n)!}{(3x)^n} \right| = \lim_{n\to\infty} \left| \frac{(3x)(3x)^n (2n)!}{(2n+2)(2n+1)(2n)!\,(3x)^n} \right|$$

$$= \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} |3x| = 0 \cdot |3x| < 1$$

$$|3x| < \infty, \quad |x| < \infty$$



$$\boxed{(-\infty, \infty)}$$

17. $\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{3}\right)^n$

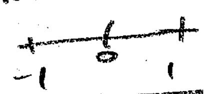
$$\boxed{\text{only } x = 0}$$

18. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+2)(n+3)} \frac{(n+1)(n+2)}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x \cdot x^n (n+1)(n+2)}{(n+2)(n+3)\, x^n} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n+3} |x| = (1)|x| < 1, \quad |x| < 1$$



$$\boxed{(-1, 1)}$$

19. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{6^n}$

$$\boxed{(-6, 6)}$$

20. $\sum_{n=0}^{\infty} \frac{(-1)^n n! (x-5)^n}{3^n}$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(x-5)^{n+1}}{3^{n+1}} \frac{3^n}{n!\,(x-5)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\,n!\,(x-5)(x-5)^n\, 3^n}{3 \cdot 3^n\, n!\,(x-5)^n} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{3} |x-5| = \infty \cdot |x-5| < 1$$

$$|x-5| < 0$$

$$\boxed{\text{only } x = 5}$$

For the remaining problems, individually check endpoint convergence:

21. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{n*9^n}$

$$\boxed{(-5,13]}$$

22. $\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+2}}{(n+2)4^{n+2}}\frac{(n+1)4^{n+1}}{(x-3)^{n+1}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(x-3)(x-3)^{n+1}(n+1)4^{n+1}}{4(x-3)^{n+1}(n+2)4^{n+1}}\right|$$

$$=\lim_{n\to\infty}\frac{n+1}{n+2}\left|\tfrac{1}{4}(x-3)\right| = (1)\left|\tfrac{1}{4}(x-3)\right|<1$$

$$|x-3|<4$$

$-1 \quad 3 \quad 7$

$\underline{x=-1}$: $\sum_{n=0}^{\infty}\frac{(-1-3)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{n+1}\frac{(-4)^{n+1}}{(4)^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{-4}{4}\right)^{n+1}$

$=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n+1}$ alternating series test

- $\lim_{n\to\infty}\frac{1}{n+1}=0$ ✓
- a_{n+1} $\sum a_n$
  $\frac{1}{n+2}<\frac{1}{n+1}$ ✓

converges

$\underline{x=7}$: $\sum_{n=0}^{\infty}\frac{(7-3)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{n+1}\frac{(4)^{n+1}}{(4)^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{4}{4}\right)^{n+1}$

$=\sum_{n=0}^{\infty}\frac{1}{n+1}(1)^{n+1} = \sum_{n=0}^{\infty}\frac{1}{n+1}$ (compare w/ $\sum_{n=0}^{\infty}\frac{1}{n}$)

p-series w/ $p=1$, diverges

limit comparison $\lim_{n\to\infty}\frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{n}{n+1}=1$ (finite, positive) so series are "linked"

$\therefore \sum_{n=0}^{\infty}\frac{1}{n+1}$ also diverges

$$\boxed{[-1,7)}$$

23. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

$\boxed{[0,2]}$

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n}}{n*2^{n}}$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)(x-2)^n n2^n}{2(x-2)(x-2)^n (n+1)2^n} \right|$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \left|\frac{1}{2}(x-2)\right| = (1)\left|\frac{1}{2}(x-2)\right| < 1$$

$$|x-2| < 2$$

$0 \quad 2 \quad 4$

$\underline{x=0}$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(0-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n2^n}(-1)(-1)^n \frac{(-2)^n}{(2)^n}$

$$= \sum_{n=1}^{\infty} (-1)(-1)^n(-1)^n \frac{1}{n2^n} = \sum_{n=1}^{\infty} (-1)(1)^n \frac{1}{n2^n}$$

$$= (-1)\sum_{n=1}^{\infty} \frac{1}{n2^n}$$ ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)2^{(n+1)}}}{\frac{1}{n2^n}} \right| = \lim_{n\to\infty} \left| \frac{n2^n}{(n+1)2^{(n+1)}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n2^n}{(n+1)2 \cdot 2^n} \right| = \lim_{n\to\infty} \frac{1}{2}\frac{n}{n+1} = \frac{1}{2} < 1$$

converges

$\underline{x=4}$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(4-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2)^n}{n(2)^n} = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}(1)^n$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$$ alternating series test

- $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓
- $a_{n+1} \le a_n$

$\frac{1}{n+1} < \frac{1}{n}$ ✓ converges

$\boxed{[0,4]}$

AP Calculus BC

10.8 Worksheet day 2

1. Given the series $f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$, find each of the following series and also the intervals of convergence for each:

(a) $f(x)$

$\boxed{f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n}$ (Given)

$\boxed{(-3, 3)}$

(b) $f'(x)$

$\boxed{f'(x) = \sum_{n=0}^{\infty} \frac{1}{3} n \left(\frac{x}{3}\right)^{n-1}}$

$\boxed{(-3, 3)}$

(c) $\int f(x)dx$

$$\boxed{\int f(x)dx = \sum_{n=0}^{\infty} 3\,\frac{\left(\frac{x}{3}\right)^{n+1}}{n+1}}$$

$$\boxed{[-3,3)}$$

2. Given the series $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$, find each of the following series and also the intervals of convergence for each:

(a) $f(x)$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}} \text{ (Given)}$$

ratio test

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+2}}{(n+2)} \cdot \frac{(n+1)}{(x-1)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(x-1)(x-1)^{n+1}(n+1)}{(n+2)(x-1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n+2} |x-1| = (1)|x-1| < 1$$

$0 \quad 1 \quad 2$

$$x=0: \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(0-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-1)^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

limit comparison w/ $\sum_{n=0}^{\infty} \frac{1}{n}$ (p-series, $p=1$) diverges

$$\lim_{n\to\infty} \left| \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

(finite, positive) so series are "linked"

also diverges

$$x=2: \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(1)^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n+1} \text{ alternating series test}$$

- $\lim_{n\to\infty} \frac{1}{n+1} = 0$ ✓
- $a_{n+1} \overset{?}{\le} a_n$ ✓ converges

$\frac{1}{n+2} < \frac{1}{n+1}$

$$\boxed{(0, 2]}$$

(b) $f'(x)$

$$f'(x) = \sum_{n=0}^{\infty} \frac{(n+1)(x-1)^{n+1}}{n+1} \frac{d}{dx}[x-1]$$
$$(1)$$

$$\boxed{f'(x) = \sum_{n=0}^{\infty} (x-1)^{n+1}}$$

just recheck endpoints:

$0 \quad 1 \quad 2$

$$x=0: \sum_{n=0}^{\infty} (0-1)^{n+1} = \sum_{n=0}^{\infty} (-1)^{n+1}$$

$(-1)+(-1)+(-1)+(-1) \cdots$ diverges

$$x=2: \sum_{n=0}^{\infty} (2-1)^{n+1} = \sum_{n=0}^{\infty} (1)^{n+1}$$

$1+1+1+1+\cdots$ diverges

$$\boxed{(0, 2)}$$

(c) $\int f(x)dx$

$$\int f(x)dx = \sum_{n=0}^{\infty} \int (-1)^{n+1}\frac{(x-1)^{n+1}}{n+1}dx = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}\int (x-1)^{n+1}dx$$ (n is a constant during the integration with respect to x)

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}\int u^{n+1}du$$ now, $u = x-1$, $du = dx$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}\frac{u^{n+2}}{n+2} = \sum_{n=0}^{\infty} (-1)^{n+1}\frac{1}{(n+1)(n+2)}(x-1)^{n+2}$$

$$\boxed{\int f(x)dx = \sum_{n=0}^{\infty} (-1)^{n+1}\frac{1}{(n+1)(n+2)}(x-1)^{n+2}}$$

just recheck endpoints: 0 1 2

$\underline{x=0}$

$$\sum_{n=0}^{\infty} (-1)^{n+1}\frac{1}{(n+1)(n+2)}(0-1)^{n+2}$$

$$\sum_{n=0}^{\infty} (-1)^{n+1}(-1)(-1)^{n+1}\frac{1}{(n+1)(n+2)}$$

$$\sum_{n=0}^{\infty} (-1)((-1)(-1))^{n+1}\frac{1}{(n+1)(n+2)}$$

$$\sum_{n=0}^{\infty} (-1)\frac{1}{(n+1)(n+2)} = (-1)\sum_{n=0}^{\infty}\frac{1}{n^2+3n+2}$$

limit comparison w/ $\sum_{n=0}^{\infty}\frac{1}{n^2}$ (p-series w/ $p=2$ converges)

$$\lim_{n\to\infty}\left|\frac{\left(\frac{1}{n^2+3n+2}\right)}{\left(\frac{1}{n^2}\right)}\right| = \lim_{n\to\infty}\frac{n^2}{n^2+3n+2} = 1$$

(finite, positive)

so series are "linked"

so also converges

$\underline{x=2}$:

$$\sum_{n=0}^{\infty} (-1)^{n+1}\frac{1}{(n+1)(n+2)}(2-1)^{n+2}$$

$$\sum_{n=0}^{\infty} (-1)^{n+1}(1)^{n+2}\frac{1}{(n+1)(n+2)}$$

$$\sum_{n=0}^{\infty} (-1)^{n+1}\frac{1}{(n+1)(n+2)}$$ alternating series ~~test~~

$$\lim_{n\to\infty}\frac{1}{(n+1)(n+2)} = 0$$

$$a_{n+1} \overset{?}{\leq} a_n$$

$$\frac{1}{(n+2)(n+3)} < \frac{1}{(n+1)(n+2)}$$

converges

$$\boxed{[0,2]}$$

AP Calculus BC

10.9 Worksheet

Find the geometric power series for the function, centered at 0.

1. $f(x) = \frac{1}{4-x}$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{x}{4}\right)^n}$$

2. $f(x) = \frac{1}{2+x}$ match $\frac{a}{1-r}$

$\frac{(1/2)}{1-\left(-\frac{x}{2}\right)}$ $a = \frac{1}{2}$, $r = \frac{-x}{2}$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{x}{2}\right)^n}$$

3. $f(x) = \frac{4}{3+x}$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{4}{3}\left(\frac{-x}{3}\right)^n}$$

4. $f(x) = \frac{2}{5-x}$ $\frac{a}{1-r}$

$\frac{(2/5)}{5-\left(\frac{x}{5}\right)}$ $a = 2/5$, $r = \frac{x}{5}$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{2}{5}\left(\frac{x}{5}\right)^2}$$

Find a power series for the function, centered at c, and determine the interval of convergence.

5. $f(x) = \frac{1}{3-x}$, $c = 1$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x-1}{2}\right)^n}$$

$\boxed{(-1, 3)}$

6. $f(x) = \frac{2}{6-x}$, $c = -2$

$x \to (x+2)$

$\frac{2}{6-(x+2)+2} = \frac{2}{8-(x+2)} = \frac{2/8}{1-\frac{x+2}{8}} = \frac{1/4}{1-\frac{x+2}{8}}$

$a = 1/4$, $r = \frac{x+2}{8}$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{x+2}{8}\right)^n}$$ geometric

$\left|\frac{x+2}{8}\right| < 1$, $|x+2| < 8$

$-10 \quad -2 \quad 6$

$\boxed{(-10, 6)}$

(geometric endpts always diverge b/c $|r| = 1$ diverges)

7. $f(x) = \frac{1}{1-3x}$, $c = 0$

$$\boxed{f(x) = \sum_{n=0}^{\infty} 1(3x)^n}$$

$\left(-\frac{1}{3}, \frac{1}{3}\right)$

8. $h(x) = \frac{1}{1-5x}$, $c = 0$ $(x-0) = (x)$

$\frac{1}{1-(5x)}$ $a=1$, $r=5x$

$$\boxed{f(x) = \sum_{n=0}^{\infty} 1(5x)^n}$$

geometric

$|5x| < 1$

$|x| < \frac{1}{5}$

$-1/5$ 0 $1/5$

$$\boxed{\left(-\frac{1}{5}, \frac{1}{5}\right)}$$

9. $f(x) = \frac{2}{1-x^2}$, $c = 0$

$$\boxed{f(x) = \sum_{n=0}^{\infty} 2(x^2)^n}$$

$$\boxed{(-1,1)}$$

10. $f(x) = \frac{5}{5+x^2}$, $c = 0$ $(x-0) = (x)$

$\frac{5}{5+x^2} = \frac{1}{1-\left(-\frac{x^2}{5}\right)}$ $a=1$, $r = \frac{-x^2}{5}$

$$\boxed{f(x) = \sum_{n=0}^{\infty} 1\left(\frac{-x^2}{5}\right)^n}$$

geometric

$\left|\frac{-x^2}{5}\right| < 1$

$|x^2| < 5$

$|x| < \sqrt{5}$

$-\sqrt{5}$ 0 $\sqrt{5}$

$$\boxed{(-\sqrt{5}, \sqrt{5})}$$

Find the power series for the function, centered at 0, and determine the interval for convergence.

11. $h(x) = -\frac{2}{x^2-1} = \frac{1}{1+x} + \frac{1}{1-x}$

$\frac{1}{1-(-x)} + \frac{1}{1-(x)}$

$a=1$, $r=-x$ $a=1$, $r=x$

$$\boxed{\sum_{n=0}^{\infty} 1(-x)^n + \sum_{n=0}^{\infty} 1(x)^n}$$

for both:

$|x| < 1$

-1 0 1

$$\boxed{(-1,1)}$$

notice $h(x)$ in #11 is the same function as $f(x)$ in #9,

How can two different series both be correct? (see next page)

For #9 & #11:

$\frac{2}{1-x^2} = \frac{-2}{x^2-1}$ so these are the same function. It seems two different series can represent the same function.

what's the difference? They may converge at different 'speeds'...

Let's evaluate $f(0.8) = h(0.8) = \frac{2}{1-(0.8)^2} = 5.\overline{555}$

#9 series...

$$\sum_{n=0}^{\infty} 2(x^2)^n = \sum_{n=0}^{\infty} 2((0.8)^2)^n$$

$$= 2 + 1.28 + .8192 + .524288 + .33554432 + \cdots$$

partial sums $\Rightarrow$ 2, 3.28, 4.0992, 4.623488, 4.95903232 ...

#11 series...

$$\sum_{n=0}^{\infty} [(-x)^n + x^n] = \sum_{n=0}^{\infty} [(-0.8)^n + (0.8)^n]$$

$$= 2 + 0 + 1.28 + 0 + .8192 + \cdots$$

partial sums $\Rightarrow$ 2, 2, 3.28, 3.28, 4.0992, ---

both will get to the true value of $5.\overline{555}$
but the #9 series converges faster

12. $f(x) = -\frac{1}{(x+1)^2} = \frac{d}{dx}\left|\frac{1}{x+1}\right|$

$$\frac{1}{x+1} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} 1(-x)^n$$

chain rule

$$f(x) = \frac{d}{dx}\sum_{n=0}^{\infty} 1(-x)^n = \sum_{n=0}^{\infty} n(-x)^{n-1}(-1) = \boxed{(-1)\sum_{n=0}^{\infty} n(-x)^{n-1}}$$

ratio test

$$\lim_{n\to\infty}\left|\frac{(n+1)(-x)^n}{n(-x)^{n-1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(-x)(-x)^{n-1}}{n\,(-x)^{n-1}}\right| = \lim_{n\to\infty}\frac{n+1}{n}|-x| = (1)|-x| < 1$$

-1 0 1

$x=-1$: $\sum_{n=0}^{\infty} n(-(-1))^{n-1}$ $= \sum_{n=0}^{\infty} n(1)^{n-1} = \sum_{n=0}^{\infty} n$ diverges

$x=1$: $\sum_{n=0}^{\infty} n(-1)^{n-1}$ alternating series test $\lim_{n\to\infty} n \to \infty$ diverges

$\boxed{(-1,1)}$

13. $g(x) = \frac{1}{x^2+1} = \frac{1}{1-(-x^2)}$ $a=1$, $r=-x^2$

$$= \boxed{\sum_{n=0}^{\infty} 1(-x^2)^n}$$

geometric

$|-x^2| < 1$

$|x^2| < 1$

$|x| < 1$

$\boxed{(-1,1)}$

Use the series for arctan to approximate the value using $R_N \leq 0.001$.

14. ~~arctan(x) = ¼~~ $\arctan(\frac{1}{4})$ actual $\arctan(\frac{1}{4}) = .2449786631$

$$\arctan(x) \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots \quad [-1,1]$$

$$\arctan(\tfrac{1}{4}) \approx (\tfrac{1}{4}) - \frac{(\frac{1}{4})^3}{3} + \frac{(\frac{1}{4})^5}{5} - \frac{(\frac{1}{4})^7}{7} + \frac{(\frac{1}{4})^9}{9} - \dots$$

partial sums: $\frac{1}{4}$, .2447916..

error: .00502, .000187 ✓

could we $\boxed{\arctan(x) \approx x - \frac{x^3}{3}}$

(see next page)...

10.9 #14 continued

...could also recognize that $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$

and in #13, we found that $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} 1(-x^2)^n$

$$\text{so } \arctan x = \int \frac{1}{1+x^2}dx = \sum_{n=0}^{\infty} \int (-x^2)^n dx$$

$$= \sum_{n=0}^{\infty} \int \underbrace{(-1)^n}_{\text{constant while we are integrating wrt } x} x^{2n} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{2n+1}}{2n+1} + C \right]$$

and ($\arctan(0) = 0$ so $C = 0$)

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\arctan(\tfrac{1}{4}) \approx \frac{(1/4)}{1} - \frac{(1/4)^3}{3} + \frac{(1/4)^5}{5} - \frac{(1/4)^7}{7} \ldots$$

produces the same terms as the first way we worked the problem

Find a power series for the function, centered at 0, and determine the interval of convergence.

15. $f(x) = \frac{1}{(1-x)^2}$

$\boxed{f(x) = \sum_{n=0}^{\infty} n x^{n-1}}$

hint: $\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right]$

$\boxed{(-1,1)}$

Find the sum of the convergent series by using a well-known function. Identify the function and explain how you obtained the sum.

16. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n * n}$ $= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(\frac{1}{2})^n}{n}$ similar form to table: $\ln x = \sum \frac{(-1)^{n-1}(x-1)^n}{n}$

if $x = \frac{3}{2}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n \cdot n} = \boxed{\ln\left(\frac{3}{2}\right) = 0.405465}$

17. $\sum_{n=0}^{\infty} (-1)^n * \frac{1}{2^{2n+1}(2n+1)}$ $= \boxed{\arctan\left(\frac{1}{2}\right) = 0.4636476}$

AP Calculus BC

Name: ______________________

10.10 Worksheet

Period: __________

Find the Taylor Series, centered at c, for the function.

1. $f(x) = \frac{1}{x}$, $c = 1$

$$\boxed{\sum_{n=0}^{\infty} (-1)^n (x-1)^n}$$

can use Taylor Series definition or (easier) make this geometric

2. $f(x) = \frac{1}{1-x}$, $c = 2$ can make geometric centered at 2

$$\frac{1}{1-(x-2)-2} = \frac{1}{-1-(x-2)} = \frac{-1}{1+(x-2)} = \frac{-1}{1-(-(x-2))} \quad a=1,\ r=-(x-2)$$

$$f(x) = (-1)\sum_{n=0}^{\infty} 1\left(-(x-2)\right)^n = (-1)\sum_{n=0}^{\infty} (-1)^n (x-2)^n = \boxed{\sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n}$$

Use the binomial series to find the Maclaurin Series for the function.

3. $f(x) = \sqrt[4]{1+x}$

$$\boxed{f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(1)(3)(7)\cdots(4n-5)}{4^n\, n!} x^n}$$

4. $f(x) = \sqrt{1+x^7}$

table: $(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \ldots$

$\sqrt{1+x^7} = (1+x^7)^{1/2} = 1 + \frac{1}{2}x^{7(1)} + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^{7(2)} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^{7(3)} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^{7(4)}$

$= 1 + \frac{1}{2}x^{7(1)} + (-1)\frac{(1)}{2^2 2!}x^{7(2)} + (1)\frac{(1)(3)}{2^3 3!}x^{7(3)} + (-1)\frac{(1)(3)(5)}{2^4 4!}x^{7(4)}$

$n=2$ $\quad n=3$ $\quad n=4$

$n=2$: $an+b=1$
$a(2)+b=1$
$n=4$: $an+b=5$
$a(4)+b=5$
$\begin{cases} 2a+b=1 \\ 4a+b=5 \end{cases}$
$2a=4,\ a=2$
$2(2)+b=1$
$b=-3$
$2n-3$

$$\boxed{f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(1)(3)(5)\cdots(2n-3)}{2^n n!} x^{7n}}$$

Find the Maclaurin Series for the function. Use the list of power series for elementary functions (in your notes!).

5. $f(x) = e^{x^2/2}$ $= \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}}$

6. $g(x) = e^{-3x}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{(-3x)} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{3^n x^n}{n!}}$

7. $f(x) = \ln(1+x)$ $= \boxed{\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}}$

8. $f(x) = \ln(1 + x^2)$

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$$

$$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(1+x^2-1)^n}{n} = \boxed{\sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n}}$$

9. $f(x) = \cos(4x)$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}}$$

10. $f(x) = \cos(\pi x)$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(\pi x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}}$$

11. $f(x) = 3 + 4e^{x^3}$

$$\boxed{3 + 4 \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}}$$

12. $f(x) = \cos^2(x) = \frac{1+\cos(2x)}{2} = \frac{1}{2} + \frac{1}{2}\cos(2x)$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{1}{2} + \frac{1}{2}\cos(2x) = \boxed{\frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2x}}{(2n)!}}$$

Calculus 2 - Unit 10 Part 2 REVIEW

Find the Maclaurin series for the given function (use the elementary forms table):

1) $g(x)=\dfrac{1}{x^5}$

2) $g(x)=\sin\left(2-x^3\right)$

3) $g(x)=\dfrac{1}{3+x^2}$

Writing first 5 terms of a power expansion for the given function (use the elementary forms table):

4) $f(x)=\cos\left(3x^5\right)$

5) $f(x)=\sin(4x-3)$

Find the specified Maclaurin or Taylor polynomial (for these you must use the definitions):

6) Find the $n=5$ Maclaurin polynomial for the function $f(x)=\sin(3x)$

7) Find the $n=4$ Taylor polynomial centered at $c=2$ for the function $f(x)=\ln(x)$

8) Find the $n=4$ Maclaurin polynomial for the function $f(x)=xe^x$

9) Find the $n=4$ Taylor polynomial centered at $c=9$ for the function $f(x)=\sqrt{x}$

10) Find the $n=5$ Maclaurin polynomial for the function $f(x)=e^{3x}$

11) Determine the degree of the Maclaurin polynomial centered at 0 required to approximate $f(0.4)$ for the function $f(x)=\sin(x)$ for the error to be less than 0.0002.

12) Determine the degree of the Maclaurin polynomial centered at 1 required to approximate $f(1.4)$ for the function $f(x)=\ln(x)$ for the error to be less than 0.0002.

13) Determine the degree of the Maclaurin polynomial centered at 0 required to approximate $f(0.7)$ for the function $f(x)=e^x$ for the error to be less than 0.0004.

14) Find the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}$ (consider the endpoints).

15) Find the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-4)^n}{n9^n}$ (consider the endpoints).

16) Find the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^{n+1}}{n+1}$ (consider the endpoints).

17) If $f(x)=\displaystyle\sum_{n=1}^{\infty}\frac{x^{2n}}{2n+1}$ find the interval of convergence for (a) $f(x)$ (b) $f'(x)$ (c) $\int f(x)\,dx$

Unit 10 Part 2 REVIEW – SOLUTIONS

(1) $g(x)=\frac{1}{x^5}$

from table: $\frac{1}{x}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$

$$g(x)=\boxed{\sum_{n=0}^{\infty}(-1)^n(x^5-1)^n}$$

(2) $g(x)=\sin(2-x^3)$

from table: $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$g(x)=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n(2-x^3)^{2n+1}}{(2n+1)!}}$$

(3) $g(x)=\frac{1}{3+x^2}$

from table: $\frac{1}{1+\underset{(u)}{x}}=\sum_{n=0}^{\infty}(-1)^n x^n$

$1+u=3+x^2$

$u=3+x^2-1=2+x^2$

so $g(x)=\boxed{\sum_{n=0}^{\infty}(-1)^n(2+x^2)^n}$

(4) $f(x)=\cos(3x^5)$

from table: $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots$

$f(x)=1-\frac{(3x^5)^2}{2!}+\frac{(3x^5)^4}{4!}-\frac{(3x^5)^6}{6!}+\frac{(3x^5)^8}{8!}$

$$\boxed{f(x)=1-\frac{9}{2}x^{10}+\frac{27}{8}x^{20}-\frac{81}{80}x^{30}+\frac{729}{4480}x^{40}}$$

(5) $f(x)=\sin(4x-3)$

from table: $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}$

$$\boxed{f(x)=(4x-3)+\frac{1}{6}(4x-3)^3+\frac{1}{120}(4x-3)^5-\frac{1}{5040}(4x-3)^7+\frac{1}{9!}(4x-3)^9}$$

(6) $f(x)=\sin(3x)$ $n=5$ Maclaurin $(c=0)$

$f(x)=\sin(3x)$ $\quad f(0)=0$

$f'(x)=3\cos(3x)$ $\quad f'(0)=3$

$f''(x)=-9\sin(3x)$ $\quad f''(0)=0$

$f'''(x)=-27\cos(3x)$ $\quad f'''(0)=-27$

$f^{(4)}(x)=81\sin(3x)$ $\quad f^{(4)}(0)=0$

$f^{(5)}(x)=243\cos(3x)$ $\quad f^{(5)}(0)=243$

$P_5(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}}{5!}x^5$

$P_5(x)=0+3x+\frac{0}{2}x^2+\frac{(-27)}{6}x^3+\frac{0}{24}x^4+\frac{243}{120}x^5$

$$\boxed{P_5(x)=3x-\frac{9}{2}x^3+\frac{81}{40}x^5}$$

(7) $f(x)=\ln x$ $n=4$ Taylor, $c=2$

$f(x)=\ln x$ $\quad f(2)=\ln 2$

$f'(x)=\frac{1}{x}=x^{-1}$ $\quad f'(2)=\frac{1}{2}$

$f''(x)=-x^{-2}$ $\quad f''(2)=-\frac{1}{4}$

$f'''(x)=2x^{-3}$ $\quad f'''(2)=\frac{1}{4}$

$f^{(4)}(x)=-6x^{-4}$ $\quad f^{(4)}(2)=-\frac{3}{8}$

$P_4(x)=f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2+\frac{f'''(2)}{3!}(x-2)^3+\frac{f^{(4)}(2)}{4!}(x-2)^4$

$P_4(x)=\ln 2+\frac{1}{2}(x-2)+\frac{(-1/4)}{2}(x-2)^2+\frac{(1/4)}{6}(x-2)^3+\frac{(-3/8)}{24}(x-2)^4$

$$\boxed{P_4(x)=\ln 2+\frac{1}{2}(x-2)-\frac{1}{8}(x-2)^2+\frac{1}{24}(x-2)^3-\frac{1}{64}(x-2)^4}$$

(8) $f(x)=xe^x$ $n=4$ Maclaurin

$f(x)=xe^x$ $\quad f(0)=0$

$f'(x)=xe^x+e^x$ $\quad f'(0)=1$

$f''(x)=xe^x+e^x+e^x$
$=xe^x+2e^x$ $\quad f''(0)=2$

$f'''(x)=xe^x+e^x+2e^x$
$=xe^x+3e^x$ $\quad f'''(0)=3$

$f^{(4)}(x)=xe^x+e^x+3e^x$
$=xe^x+4e^x$ $\quad f^{(4)}(0)=4$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x)=0+(1)x+\frac{2}{2}x^2+\frac{3}{6}x^3+\frac{4}{24}x^4$$

$$\boxed{P_4(x)=x+x^2+\tfrac{1}{2}x^3+\tfrac{1}{6}x^4}$$

---

(9) $f(x)=\sqrt{x}$ $n=4$ Taylor $c=9$

$f(x)=x^{1/2}$ $\quad f(9)=3$

$f'(x)=\frac{1}{2}x^{-1/2}$ $\quad f'(9)=\frac{1}{6}$

$f''(x)=-\frac{1}{4}x^{-3/2}$ $\quad f''(9)=-\frac{1}{108}$

$f'''(x)=\frac{3}{8}x^{-5/2}$ $\quad f'''(9)=\frac{1}{648}$

$f^{(4)}(x)=\frac{-15}{16}x^{-7/2}$ $\quad f^{(4)}(9)=\frac{-15}{6}\frac{1}{(\sqrt{9})^7}=\frac{-15}{13122}$

$$P_4(x)=f(9)+f'(9)(x-9)+\frac{f''(9)}{2!}(x-9)^2+\frac{f'''(9)}{3!}(x-9)^3+\frac{f^{(4)}(9)}{4!}(x-9)^4$$

$$P_4(x)=3+\frac{1}{6}(x-9)+\frac{(-1/108)}{2}(x-9)^2+\frac{(1/648)}{6}(x-9)^3+\frac{(-15/13122)}{24}(x-9)^4$$

$$\boxed{P_4(x)=3+\tfrac{1}{6}(x-9)-\tfrac{1}{216}(x-9)^2+\tfrac{1}{3888}(x-9)^3-\tfrac{15}{314928}(x-9)^4}$$

---

(10) $f(x)=e^{3x}$ $n=5$ Maclaurin

$f(x)=e^{3x}$ $\quad f(0)=1$

$f'(x)=3e^{3x}$ $\quad f'(0)=3$

$f''(x)=9e^{3x}$ $\quad f''(0)=9$

$f'''(x)=27e^{3x}$ $\quad f'''(0)=27$

$f^{(4)}(x)=81e^{3x}$ $\quad f^{(4)}(0)=81$

$f^{(5)}(x)=243e^{3x}$ $\quad f^{(5)}(0)=243$

$$P_5(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x)=1+3x+\frac{9}{2}x^2+\frac{27}{6}x^3+\frac{81}{24}x^4+\frac{243}{120}x^5}$$

---

(11) $f(x)=\sin(x)$ $\quad f(0.4)$ $c=0$
error $<.0002$

$f(x)=\sin x$

$f'(x)=\cos x$

$f''(x)=-\sin x$

$f'''(x)=-\cos x$

$f^{(n)}(x)=(-1)^n$ sin or cos

So max $f^{(n+1)}(z)$ will be $\underline{\underline{1}}$

$$\text{error}\le\frac{f^{(n+1)}}{(n+1)!}(x-c)^{n+1}$$
$$\le\frac{[1]}{(n+1)!}(0.4-0)^{n+1}$$
$$\le\frac{1}{(n+1)!}(0.4)^{n+1}$$

try: $n=2$: $\frac{1}{3!}(0.4)^3=.0106$

$n=3$: $\frac{1}{4!}(0.4)^4=.00106$

✓ $n=4$: $\frac{1}{5!}(0.4)^5=.000085$

$\boxed{\text{4 terms}}$

---

(12) $f(x)=\ln x$ $\quad f(1.4)$ $c=1$
error $<.0002$

$f(x)=\ln x$

$f'(x)=\frac{1}{x}=x^{-1}$

$f''(x)=-x^{-2}$

$f'''(x)=2x^{-3}$

$f^{(4)}(x)=(-1)(3)(2)x^{-4}$

$f^{(5)}(x)=(+)(4)(3)(2)x^{-5}$

$f^{(n)}(x)=(-1)^n\frac{(n-1)!}{x^n}$

$f^{(n+1)}(x)=(-1)^{n+1}\frac{n!}{x^{n+1}}$

max will be $\underline{n!}$

$$\text{error}\le\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$
$$\le\frac{[n!]}{(n+1)!}(1.4-1)^{n+1}$$
$$\le\frac{n!}{(n+1)!}(0.4)^{n+1}$$

try:
$n=5$: $\frac{5!}{6!}(0.4)^6=.00068$

$n=6$: $\frac{6!}{7!}(0.4)^7=.000234$

✓ $n=7$: $\frac{7!}{8!}(0.4)^8=.0000819$

$\boxed{\text{7 terms}}$

(13) $f(x)=e^x$ $f(0.7)$ $c=0$
error $< .0004$

$f'(x)=e^x$
$f''(x)=e^x$
$\vdots$
$f^{(n+1)}(x)=e^x$

so max will be e^x over the interval $0 \to 0.7$
max at $x=0.7$ is $e^{0.7}$

$$\text{error} \le \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

$$\le \frac{[e^{0.7}]}{(n+1)!}(0.7-0)^{n+1}$$

$$\le \frac{e^{0.7}}{(n+1)!}(0.7)^{n+1}$$

try $n=4$: $\frac{e^{0.7}}{5!}(0.7)^5 = .0028$

$n=5$: $\frac{e^{0.7}}{6!}(0.7)^6 = .000329$

✓ $n=6$, $\frac{e^{0.7}}{7!}(0.7)^7 = .0000329$

$\boxed{6 \text{ terms}}$

(14) $\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$

ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2 x^{2n}(2n)!}{(2n+2)(2n+1)(2n)!\,x^{2n}}\right|$$

$$= \underline{\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)}}|x|$$

$\frac{1}{\infty}$

$0\,|x| < 1$

try for all x, so
interval of convergence is $\boxed{(-\infty,\infty)}$

(15) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-4)^n}{n9^n}$

ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{(n+1)9^{n+1}}\cdot\frac{n9^n}{(x-4)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-4)(x-4)^n n 9^n}{(n+1)9\cdot9^n(x-4)^n}\right|$$

$$= \lim_{n\to\infty}\frac{n}{n+1}\left|\frac{x-4}{9}\right|$$

L'Hop

$$= \lim_{n\to\infty}\frac{1}{1}\left|\frac{x-4}{9}\right|$$

$\left|\frac{x-4}{9}\right| < 1$

$|x-4| < 9$

$-9 < x-4 < 9$
$+4 \quad +4 \quad +4$

$-5 < x < 13$

now check endpoints

$x=-5$:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{((-5-4))^n}{n9^n}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-9)^n}{n9^n}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\left(\frac{-9}{9}\right)^n$$
$(-1)^n$

$$= \sum_{n=1}^{\infty}(-1)(-1)^n(-1)^n\frac{1}{n}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n}$$ p-series w/ $p=1$ diverges

$x=13$:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(13-4)^n}{n9^n}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(9)^n}{n9^n}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\left(\frac{9}{9}\right)^n$$
$(1)^n$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

alternating series test

$a_{n+1} < a_n$?

$\frac{1}{n+1} < \frac{1}{n}$ ✓

$\lim_{n\to\infty}\frac{1}{n} = 0$ ✓

converges

$\boxed{\text{So interval of convergence is } (-5, 13]}$

(16) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1}$

ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+2}}{(n+2)} \frac{(n+1)}{(x-1)^{n+1}}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n+2}\left|\frac{(x-1)(x-1)^{n+1}}{(x-1)^{n+1}}\right|$$

L'Hop

$$= \lim_{n\to\infty} \frac{1}{1}\,|x-1|$$

$$|x-1|<1$$

$$-1 < x-1 < 1$$

$$+1 \quad +1 \quad +1$$

$$0 < x < 2$$

now check endpoints

$x=0$:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{((0)-1)^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^{n+1}}{(n+1)}$$

$$= \sum_{n=1}^{\infty} (1)^{n+1} \frac{1}{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n+1}$$

Limit comparison w/ $\frac{1}{n} = b_n$

$0 < a_n \leq b_n$?

$\frac{1}{n+1} < \frac{1}{n}$ ✓

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

L'Hop

$$= \lim_{n\to\infty} \frac{1}{1} = 1 > 0$$

now check:

$\sum_{n=1}^{\infty} \frac{1}{n}$ p-series w/ $p=1$ diverges

diverges for $x=0$

(could also use integral test)

$x=2$:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{((2)-1)^{n+1}}{(n+1)}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1)^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+1}$$

alternating series test

$a_{n+1} < a_n$?

$\frac{1}{n+2} < \frac{1}{n+1}$ ✓

$\lim_{n\to\infty} \frac{1}{n+1} = 0$ ✓

converges

$$\boxed{\text{so interval of convergence is } (0, 2]}$$

⑰ $f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{2n+1}$

(a) $f(x)$

ratio ~~test~~

$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{2(n+1)+1} \frac{2n+1}{x^{2n}} \right|$

$= \lim_{n\to\infty} \frac{2n+1}{2n+3} \left| \frac{x^2 x^{2n}}{x^{2n}} \right|$

L'Hop

$\lim_{n\to\infty} \frac{2}{2} |x^2|$

$|x^2| < 1$

$x^2 < 1$

$-1 < x < 1$

now check endpoints:

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2n+1}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{2n+1}$

$= \sum_{n=1}^{\infty} \frac{(1)^n}{2n+1} = \sum_{n=1}^{\infty} \frac{1}{2n+1}$

Limit comparison w/ $\frac{1}{2n} = b_n$

$0 < a_n \leq b_n$?

$\frac{1}{2n+1} < \frac{1}{2n}$ ✓

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{2n}}$

$= \lim_{n\to\infty} \frac{2n}{2n+1}$

L'Hop

$= \lim_{n\to\infty} \frac{2}{2} = 1 > 0$

now: $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$

p-series w/ $p=1$ diverges

so $x=-1$ diverges

$x = 1$: $\sum_{n=1}^{\infty} \frac{(1)^{2n}}{2n+1}$

$= \sum_{n=1}^{\infty} \frac{1}{2n+1}$

same as this so diverges

so $f(x)$ interval of convergence is

$\boxed{(-1, 1)}$

17b $f(x)=\sum_{n=1}^{\infty}\frac{x^{2n}}{2n+1}$

$f'(x)=\sum_{n=1}^{\infty}\frac{2n\,x^{2n-1}}{2n+1}$

ratio test

$\lim_{n\to\infty}\left|\frac{2(n+1)x^{2(n+1)-1}}{2(n+1)+1}\cdot\frac{2n+1}{2n\,x^{2n-1}}\right|$

$=\lim_{n\to\infty}\frac{2(n+1)(2n+1)}{(2n+3)2n}\left|\frac{x^{2n+1}}{x^{2n-1}}\right|$

$=\lim_{n\to\infty}\frac{2(2n^2+2n+n+1)}{4n^2+6n}\left|\frac{x^2x^{2n-1}}{x^{2n-1}}\right|$

$=\lim_{n\to\infty}\frac{4n^2+6n+2}{4n^2+6n}|x^2|$

L'Hop

$=\lim_{n\to\infty}\frac{8n+6}{8n+6}$

$=\lim_{n\to\infty}\frac{8}{8}=1\quad |x^2|<1$

$-1<x<1$

now check endpoints

$x=-1$:

$\sum_{n=1}^{\infty}\frac{2n(-1)^{2n-1}}{2n+1}$

$=\sum_{n=1}^{\infty}(-1)^{2n}(-1)^{-1}\frac{2n}{2n+1}$

$=\sum_{n=1}^{\infty}(-1)^n(-1)^n\frac{1}{-1}\frac{2n}{2n+1}$

$=\sum_{n=1}^{\infty}(1)^n(-1)\frac{2n}{2n+1}$

$=-\sum_{n=1}^{\infty}\frac{2n}{2n+1}$

nth term

$\lim_{n\to\infty}\frac{2n}{2n+1}$

L'Hop

$=\lim_{n\to\infty}\frac{2}{2}=1\neq 0$

diverges

$x=1$:

$\sum_{n=1}^{\infty}\frac{2n(1)^{2n-1}}{2n+1}$

$=\sum_{n=1}^{\infty}\frac{2n}{2n+1}$

nth term

$\lim_{n\to\infty}\frac{2n}{2n+1}$

L'Hop

$=\lim_{n\to\infty}\frac{2}{2}=1\neq 0$

diverges

so for $f'(x)$,
interval of convergence is $\boxed{(-1,1)}$

17c $f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{2n+1}$

$\int f(x)dx = \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)(2n+1)}$ now check endpoint

<u>ratio test</u>

$\lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)(2(n+1)+1)} \cdot \frac{(2n+1)(2n+1)}{x^{2n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)(2n+3)} \frac{(2n+1)(2n+1)}{x^{2n+1}} \right|$

$= \lim_{n\to\infty} \frac{(2n+1)(2n+1)}{(2n+3)(2n+3)} \left| \frac{x^2 x^{2n+1}}{x^{2n+1}} \right|$

$= \lim_{n\to\infty} \frac{4n^2+4n+1}{4n^2+12n+9} \left| x^2 \right|$

L'Hop (x2)

$= \lim_{n\to\infty} \frac{4}{4} = 1 \quad |x^2| < 1$

$-1 < x < 1$

$x = -1$:

$\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)^2}$

$= \sum_{n=1}^{\infty} \frac{(-1)(-1)^n(-1)^n}{(2n+1)^2}$

$= -\sum_{n=1}^{\infty} \frac{(1)^n}{(2n+1)^2}$

$= -\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$

<u>Integral test</u>

terms positive? yes ✓

decreasing?

$g(x) = \frac{1}{(2x+1)^2} = (2x+1)^{-2}$

$g'(x) = -2(2x+1)^{-3}(2)$

$= \frac{-4}{(2x+1)^3} \frac{(-)}{(+)} = -$

- decreasing ✓

$\int_1^{\infty} \frac{1}{(2x+1)^2} dx \quad u = 2x+1$

$\frac{du}{dx} = 2$

$du = 2dx$

$\frac{1}{2}\int_3^{\infty} u^{-2} du$

$\lim_{b\to\infty} \frac{1}{2}\int_3^b u^{-2} du$

$\lim_{b\to\infty} \left[-\frac{1}{2}\frac{1}{u} \right]_3^{\infty}$

$-\frac{1}{2}\frac{1}{(\infty)} - \left[-\frac{1}{2}\frac{1}{3} \right]$

$0 \quad + \frac{1}{6}$

<u>converges</u> at $x = -1$

$x = 1$

$\sum_{n=1}^{\infty} \frac{(1)^{2n+1}}{(2n+1)^2}$

$= \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$

same but not negative so also <u>converges</u> for $x = 1$

So interval of convergence for $\int f(x)dx$ is $\boxed{[-1, 1]}$