Unit 10 Part 1 REVIEW — SOLUTIONS

① $\sum_{n=1}^{\infty}\left(\frac{2n^3+1}{n^3-1}\right)^n$

~~root test~~

(b/c n^n)

$\lim_{n\to\infty}\sqrt[n]{|a_n|}<1$?

$\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n^3+1}{n^3-1}\right)^n}$

$\lim_{n\to\infty}\frac{2n^3+1}{n^3-1}$ $\frac{\infty}{\infty}$

L'Hop.

$\lim_{n\to\infty}\frac{6n^2}{3n^2}=\lim_{n\to\infty}\frac{12n}{6n}$

$=\lim_{n\to\infty}\frac{12}{6}=2>1$

$\boxed{\text{diverges}}$

② $\sum_{n=1}^{\infty}\frac{8^n}{5^n}=\sum_{n=1}^{\infty}\left(\frac{8}{5}\right)^n$

geometric $r=\frac{8}{5}$

$\left|\frac{8}{5}\right|>1$ $\boxed{\text{diverges}}$

③ $\sum_{n=1}^{\infty}\frac{3^n}{4^n-1}$

limit comparison

compare to $\frac{3^n}{4^n}=b_n$

$\lim_{n\to\infty}\frac{a_n}{b_n}>0$?

$\lim_{n\to\infty}\frac{\left(\frac{3^n}{4^n-1}\right)}{\left(\frac{3^n}{4^n}\right)}=\lim_{n\to\infty}\frac{4^n}{4^n-1}$

L'Hop doesn't help, but for large n, $4^n-1\approx 4^n$

so $\lim_{n\to\infty}\frac{4^n}{4^n-1}=\lim_{n\to\infty}\frac{4^n}{4^n}=1>0$

now check $\sum_{n=1}^{\infty}\frac{3^n}{4^n}=\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$

geometric w/ $r=\frac{3}{4}$

$\left|\frac{3}{4}\right|<1$ converges

so $\sum_{n=1}^{\infty}\frac{3^n}{4^n-1}$ $\boxed{\text{converges}}$

note: direct compare doesn't work because

$\frac{3^n}{4^n}<\frac{3^n}{4^n-1}$

so the fact the this converges doesn't force the RHS to converge (because it is beneath it)

④ $\sum_{n=1}^{\infty}\frac{n}{\ln(n)}$

try nth term (not matching other forms)

$\lim_{n\to\infty}a_n=0$?

$\lim_{n\to\infty}\frac{n}{\ln(n)}$ $\frac{\infty}{\infty}$

L'Hop

$\lim_{n\to\infty}\frac{1}{\frac{1}{n}}=\lim_{n\to\infty}n=\infty$

$\neq 0$ so d...

$\sum_{n=1}^{\infty}\frac{n}{\ln(n)}$ $\boxed{\text{diverges}}$

⑤ $\sum_{n=1}^{\infty}\frac{n!}{3^n}$ ratio test (w/ n!, try ratio 1st)

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$?

$\lim_{n\to\infty}\left|\frac{(n+1)!}{3^{n+1}}\frac{3^n}{n!}\right|$

$\lim_{n\to\infty}\left|\frac{(n+1)n!\,3^n}{3\cdot 3^n\,n!}\right|$

$\lim_{n\to\infty}\left|\frac{n+1}{3}\right|=\infty$ $\boxed{\text{diverges}}$

(6) $\sum_{n=1}^{\infty} \frac{1}{5^n+1}$

direct comparison w/ $\frac{1}{5^n} = b_n$

(denom ±1)

$0 < a_n < b_n$?

$\frac{1}{5^n+1} < \frac{1}{5^n}$ ✓

check $\sum_{n=1}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \frac{1^n}{5^n}$

$= \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$

geometric w/ $r = \frac{1}{5}$

Converges

so $\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ also $\boxed{\text{converges}}$

(terms are $< \frac{1}{5^n}$)

(7) $\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$ nth term = 0, not geometric, not p-series, not good for root, not alternating, not easy to compare

try integral test

- terms positive? yes ✓
- decreasing? check derivative

$f(x) = \frac{4x}{2x^2+1}$ $f'(x) = \frac{(2x^2+1)(4)-(4x)(4x)}{(2x^2+1)^2}$

$= \frac{8x^2+4-16x^2}{(2x^2+1)^2}$

$= \frac{-8x^2+4}{(2x^2+1)^2}$ $\frac{(-)}{(+)}$ −

decreasing ✓

now $\int_1^{\infty} \frac{4x}{2x^2+1} dx$

$u = 2x^2+1$

$\frac{du}{dx} = 4x$, $du = 4x\,dx$

$\int_3^{\infty} \frac{1}{u} du = \lim_{b \to \infty} \int_3^b \frac{1}{u} du$

$= \lim_{b \to \infty} [\ln|u|]_3^b$

$= \lim_{b \to \infty} \ln|b| - \ln|3|$

$\infty - 3 = \infty$ diverges

so $\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$ also $\boxed{\text{diverges}}$

(8) $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$

p-series w/ $p = 2 > 1$ $\boxed{\text{Converges}}$

(11) $\sum_{n=0}^{\infty} 5 \frac{2^n}{3^n} = \sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$

geometric w/ $r = \frac{2}{3}$

$\left|\frac{2}{3}\right| < 1$

$\boxed{\text{Converges}}$

(9) $\sum_{n=1}^{\infty} \frac{2n^3}{n^3+4}$

nth term

$\lim_{n \to \infty} \frac{2n^3}{n^3+4} \frac{\infty}{\infty}$

L'Hop

$= \lim_{n \to \infty} \frac{6n^2}{3n^2} = \lim_{n \to \infty} \frac{12n}{6n}$

$= \lim_{n \to \infty} \frac{12}{6} = 2 \neq 0$

$\boxed{\text{Diverges}}$

(10) $\sum_{n=1}^{\infty} (-1)^n \frac{5n-1}{4n+1}$

alternating series test

$a_{n+1} < a_n$?

$\frac{5(n+1)-1}{4(n+1)+1} < \frac{5n-1}{4n+1}$

$\frac{5n+4}{4n+5} < \frac{5n-1}{4n+1}$

not obvious, so let's use derivative:

$f(x) = \frac{5x-1}{4x+1}$

$f'(x) = \frac{(4x+1)(5)-(5x-1)(4)}{(4x+1)^2}$

$= \frac{20x+5-20x+4}{(4x+1)^2} = \frac{9}{(4x+1)^2}$ $\frac{+}{+}$ +

not true that $a_{n+1} < a_n$

also... $\lim_{n \to \infty} \frac{5n-1}{4n+1} \frac{\infty}{\infty}$

L'Hop $\lim_{n \to \infty} \frac{5}{4} \neq 0$

So $\boxed{\text{diverges}}$

(12) $\sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi}{2}\right)$

~~doesn't fit~~ patterns, try writing out some terms...

$\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{2}\right) + \sin\left(\frac{5\pi}{2}\right) + \ldots$

$1 + (-1) + (1) + (-1)$

oscillating between 1 & -1

$\boxed{\text{diverges}}$

(13) $\sum_{n=1}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \frac{1^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$

geometric w/ $r = \frac{1}{5}$ $\left|\frac{1}{5}\right| < 1$ $\boxed{\text{converges}}$

could also do ratio test:

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1?$

$\lim_{n\to\infty} \left|\frac{1}{5^{n+1}} \cdot \frac{5^n}{1}\right| = \lim_{n\to\infty} \left|\frac{5^n}{5 \cdot 5^n}\right| = \lim_{n\to\infty} \left|\frac{1}{5}\right|$

$= \frac{1}{5} < 1$ (converges)

(14) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

direct comparison w/ $\frac{1}{2n}$

but use $a_n = \frac{1}{2n}$, $b_n = \frac{1}{2n-1}$

$0 < a_n \le b_n$ ✓

now check $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$

p-series w/ $p=1$ diverges

and since this is under $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

$\sum_{n=1}^{\infty} \frac{1}{2n-1}$ also $\boxed{\text{diverges}}$

could also do limit comparison w/ $\frac{1}{2n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$? (w/ $a_n = \frac{1}{2n-1}$)

$\lim_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{2n}} = \lim_{n\to\infty} \frac{2n}{2n-1}$ $\frac{\infty}{\infty}$

L'Hop $= \lim_{n\to\infty} \frac{2}{2} = 1 > 0$

now check $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$

p-series w/ $p=1$

diverges

so $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ also (diverges)

(15) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ (ln ... usually integral test)

- terms positive? yes ✓
- decreasing? $f(x) = \frac{\ln x}{x^2}$ $f'(x) = \frac{x^2\left(\frac{1}{x}\right) - \ln x(2x)}{x^4}$

$= \frac{x - 2x\ln x}{x^4}$

$= \frac{x(1-2\ln x)}{x^4}$ (for $x > 2$) $\frac{(-)}{(+)}$

decreasing ✓

now $\int_1^{\infty} \frac{\ln x}{x^2}\,dx$

by parts: $u = \ln x$ $dv = x^{-2}dx$

$\frac{du}{dx} = \frac{1}{x}$ $\int dv = \int x^{-2}dx$

$du = \frac{1}{x}dx$ $v = -x^{-1} = \frac{-1}{x}$

$uv - \int v\,du$

$\left(\frac{1}{x}\right)\left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right)\frac{1}{x}dx$

$-\frac{1}{x^2} + \int x^{-2}dx$

$\lim_{b\to\infty} \left[-\frac{1}{x^2} - \frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left[-\frac{1}{b^2} - \frac{1}{b}\right] - \left[-\frac{1}{(1)^2} - \frac{1}{(1)}\right]$

$0 - 0 - [-1 - 1]$

$-[-2] = 2$

converges

so $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ also $\boxed{\text{converges}}$

(16) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

limit comparison w/ $\frac{n}{n^2}$, $\frac{1}{n} = b_n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\left(\frac{n}{n^2+1}\right)}{\frac{1}{n}}$

$\lim_{n\to\infty} \frac{n^2}{n^2+1} \quad \frac{\infty}{\infty}$

L'Hop

$\lim_{n\to\infty} \frac{2n}{2n} = \lim_{n\to\infty} \frac{2}{2} = 1 > 0$

now check $\sum_{n=1}^{\infty} \frac{1}{n}$

p-series w/ $p=1$

diverges

so $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ also $\boxed{\text{diverges}}$

(17) $\sum_{n=1}^{\infty} (-2)^n = -2 + (-2) + (-2) \cdots = -\infty$

$\boxed{\text{diverges}}$

(18) $\sum_{n=1}^{\infty} \frac{5n^4}{n^4+n^2+7}$

nth term $\lim_{n\to\infty} \frac{5n^4}{n^4+n^2+7} \quad \frac{\infty}{\infty}$

L'Hop!: $\lim_{n\to\infty} \frac{20n^3}{4n^3+2n} = \lim_{n\to\infty} \frac{60n^2}{12n^2+2}$

$= \lim_{n\to\infty} \frac{120n}{24n} = \lim_{n\to\infty} \frac{120}{2} = 60 \neq 0$

$\boxed{\text{diverges}}$

(19) $\sum_{n=1}^{\infty} \frac{9^n}{n^5}$

ratio test

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$?

$\lim_{n\to\infty} \left|\frac{9^{n+1}}{(n+1)^5} \frac{n^5}{9^n}\right|$

$\lim_{n\to\infty} \left|\frac{9 \cdot 9^n}{(n+1)^5} \frac{n^5}{9^n}\right|$

$\lim_{n\to\infty} \left|\frac{9n^5}{(n+1)^5}\right| \quad \frac{\infty}{\infty}$

L'Hop $\lim_{n\to\infty} \frac{45n^4}{5(n+1)^4} \quad \frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{180n^3}{20(n+1)^3} = \lim_{n\to\infty} \frac{540n^2}{60(n+1)^2}$

$= \lim_{n\to\infty} \frac{1080n}{120(n+1)} = \lim_{n\to\infty} \frac{1080}{120} = 9 > 1$

$\boxed{\text{diverges}}$

(20) $\sum_{n=1}^{\infty} \frac{5}{n^{0.4}} = 5 \sum_{n=1}^{\infty} \frac{1}{n^{0.4}}$

p-series w/ $p = 0.4$

$\boxed{\text{diverges}}$

(21) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

root test

$\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$?

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n}$

$\lim_{n\to\infty} \frac{n}{2n+1} \quad \frac{\infty}{\infty}$

L'Hop

$\lim_{n\to\infty} \frac{1}{2} < 1$

$\boxed{\text{converges}}$

(22) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{3^n}$

alternating series test

$a_{n+1} < a_n$?

$\frac{1}{3^{n+1}} < \frac{1}{3^n}$ ✓

$\lim_{n\to\infty} \frac{1}{3^n} = 0$ ✓

$\boxed{\text{converges}}$

(23) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ / $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n!}$, $\sum_{n=1}^{\infty} \left|(-1)^{n+1}\frac{1}{n!}\right| = \sum_{n=1}^{\infty} \frac{1}{n!}$

<u>alternating series test</u>

$a_{n+1} < a_n$?

$\frac{1}{(n+1)!} < \frac{1}{n!}$ ✓

$\lim_{n\to\infty} \frac{1}{n!} = 0$ ✓

<u>converges</u>

<u>ratio test</u>

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$?

$\lim_{n\to\infty} \left|\frac{1}{(n+1)!} \cdot \frac{n!}{1}\right|$

$\lim_{n\to\infty} \left|\frac{1}{(n+1)n!} \cdot \frac{n!}{1}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$

<u>converges</u>

Because $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n!}$ and $\sum_{n=1}^{\infty} \left|(-1)^{n+1}\frac{1}{n!}\right|$ both converge,

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ $\boxed{\text{converges absolutely}}$

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(24) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}}$, $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}}$, $\sum_{n=1}^{\infty} \left|(-1)^{n+1}\frac{1}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

<u>alternating series test</u>

$a_{n+1} < a_n$?

$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ ✓

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

<u>converges</u>

<u>p-series</u> w/ $p = 1/2$

<u>diverges</u>

Because $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}}$ converges but $\sum_{n=1}^{\infty} \left|(-1)^{n+1}\frac{1}{\sqrt{n}}\right|$ diverges,

$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}}$ $\boxed{\text{converges conditionally}}$

(25) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{(n+1)^2}$, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{(n+1)^2}$

alternating series test

$a_{n+1} < a_n$?

use derivative: $f(x) = \frac{x^2}{(x+1)^2}$ $f'(x) = \frac{(x+1)^2(2x) - (x^2)(2(x+1)'(1))}{(x+1)^4}$

$= \frac{2x(x+1)^2 - 2x^2(x+1)}{(x+1)^4}$

also, $\lim_{n\to\infty} \frac{n^2}{(n+1)^2}$ $\frac{\infty}{\infty}$

$= \frac{2x(x+1)[x+1-x]}{(x+1)^4} = \frac{2x(x+1)}{(x+1)^4}$ $\frac{+}{+}$ +

L'Hop $= \lim_{n\to\infty} \frac{2n}{2(n+1)} = \lim_{n\to\infty} \frac{2}{2} = 1 \neq 0$

not decreasing

diverges

so $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{(n+1)^2}$ $\boxed{\text{diverges}}$

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(26) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n^3-1}$ approx w/error < 0.0005

$|R_N| < a_{N+1}$

$\frac{1}{2(N+1)^3 - 1} < 0.0005$

$2(N+1)^3 - 1 > \frac{1}{0.0005} = 2000$

$2(N+1)^3 = 2001$

$(N+1)^3 = \frac{2001}{2}$

$N+1 = \sqrt[3]{\frac{2001}{2}}$

$N = \sqrt[3]{\frac{2001}{2}} - 1 = 9.0016$ ↗

need $\boxed{10 \text{ terms}}$