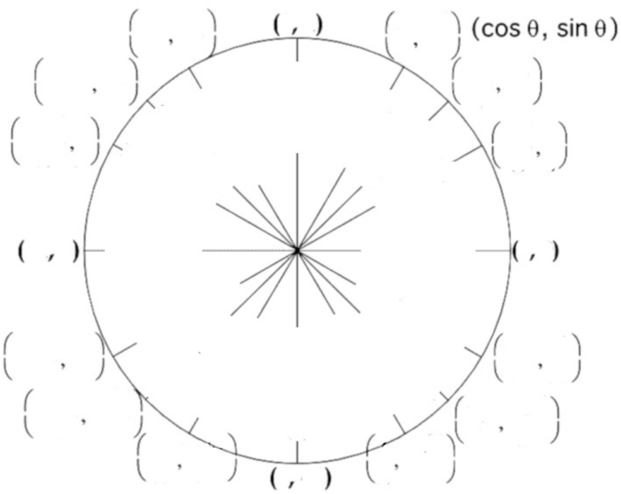
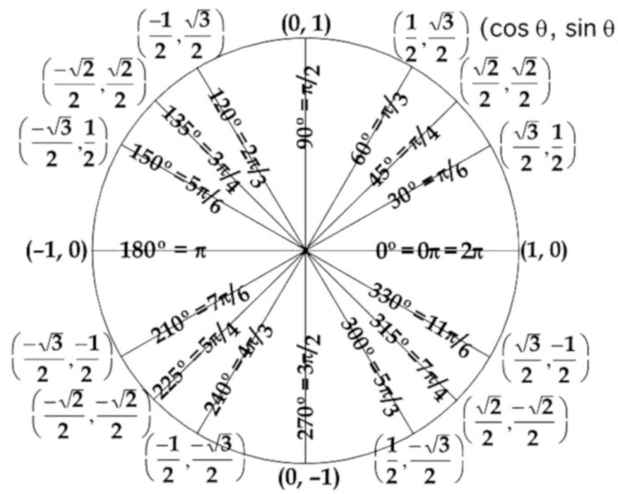
<u>**AP Calculus BC – Study Guide**</u>

Trigonometry…





Reciprocal identities:

$\sin x =$

$\cos x =$

$\tan x =$

$\csc x =$

$\sec x =$

$\cot x =$

$\sin x = \dfrac{1}{\csc x}$

$\cos x = \dfrac{1}{\sec x}$

$\tan x = \dfrac{1}{\cot x}$

$\csc x = \dfrac{1}{\sin x}$

$\sec x = \dfrac{1}{\cos x}$

$\cot x = \dfrac{1}{\tan x}$

Reciprocal identities:

$\tan x =$

$\cot x =$

$\tan x = \dfrac{\sin x}{\cos x}$

$\cot x = \dfrac{\cos x}{\sin x}$

Pythagorean identities:

$\sin^2 x + \cos^2 x =$

$1 + \tan^2 x =$

$1 + \cot^2 x =$

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Power-reducing identities:

$\sin^2(x) =$

$\cos^2(x) =$

$\sin^2(x) = \dfrac{1 - \cos(2x)}{2}$

$\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$

Double-angle identities:

$\sin(2x) =$

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) =$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

Curve shapes (sketch)…

$f(x) = x^2$

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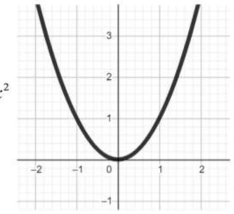

$f(x) = x^3$

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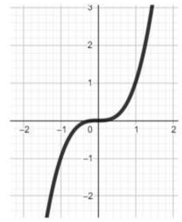

$f(x) = e^x$

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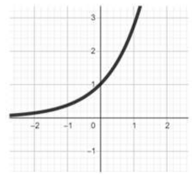

$f(x) = \ln(x)$

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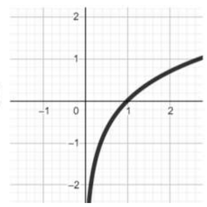

$f(x) = \sqrt{x}$

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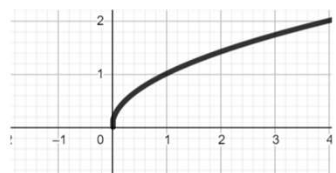
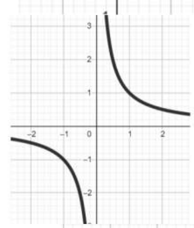


$f(x) = \frac{1}{x}$

$f(x) = \frac{1}{x}$



$f(x) = \sin(x)$

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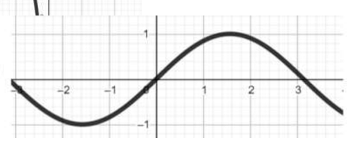
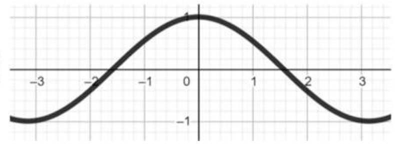


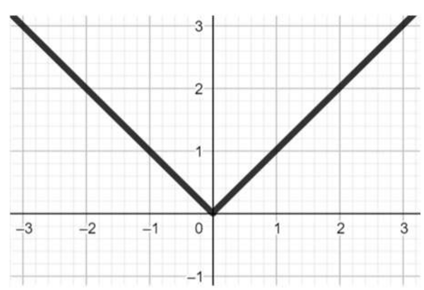
$f(x) = \cos(x)$

$f(x) = \cos(x)$



$f(x) = |x|$

$f(x) = |x|$

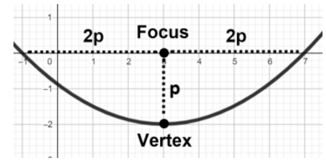


Conic sections…

Convert to standard form and sketch:

$$x^2-6x-8y-7=0$$

Parabola:

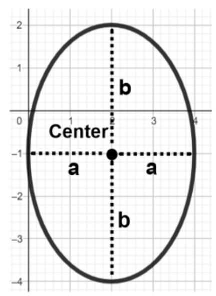


$$x^2-6x-8y-7=0$$
$$x^2-6x+\underline{\underline{9}}=8y+7+\underline{\underline{9}}$$
$$(x-3)^2=8y+16$$
$$(x-3)^2=8(y+2) \qquad (x-h)^2=4p(y-k)$$
$$(h,k)=vertex \qquad p=distance\ vertex\ to\ focus\ and\ directrix$$

$$9x^2+4y^2-36x+8y+4=0$$

Ellipse :

$$9x^2+4y^2-36x+8y+4=0$$
$$9\left(x^2-4x\quad\right)+4\left(y^2+2y\quad\right)=-4$$
$$9\left(x^2-4x+\underline{\underline{4}}\right)+4\left(y^2+2y+\underline{\underline{1}}\right)=-4+\underline{\underline{36}}+\underline{\underline{4}}$$
$$9(x-2)^2+4(y+1)^2=36$$
$$\frac{(x-2)^2}{4}+\frac{(y+1)^2}{9}=1 \qquad \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$
$$(h,k)=center \quad a=dist.\ center\ to\ vertex\ in\ x$$
$$b=dist.\ center\ to\ curve\ in\ y$$



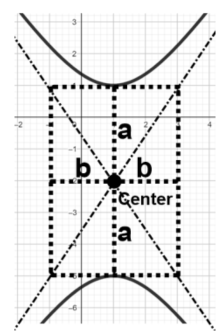
Circle :

special case of ellipse with a=b:

$$\frac{(x-2)^2}{4}+\frac{(y+1)^2}{4}=1$$
$$(x-2)^2+(y+1)^2=4$$

$$9x^2-4y^2-18x-16y+29=0$$

Hyperbola :



$$9x^2-4y^2-18x-16y+29=0$$
$$9\left(x^2-2x\quad\right)-4\left(y^2+4y\quad\right)=-29$$
$$9\left(x^2-4x+\underline{\underline{1}}\right)+4\left(y^2+2y+\underline{\underline{4}}\right)=-4+\underline{\underline{9}}+-\underline{\underline{16}}$$
$$9(x-1)^2-4(y+2)^2=-36$$
$$4(y+2)^2-9(x-1)^2=36$$
$$\frac{(y+2)^2}{9}-\frac{(x+2)^2}{4}=1 \qquad \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$
$$(h,k)=center \quad a=dist.\ center\ to\ vertex\ on\ transverse\ axis$$
$$b=dist.\ center\ to\ edge\ of\ asymptote\ box$$

Limits and Continuity…

What must be true for $\lim_{x \to c} f(x)$ to exist?

$$\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$$
$$\textit{where } L \textit{ is a finite number}$$

What must be true for $f(x)$ to be continuous at *c*?

$$1)\ f(c) \textit{ must exist}$$
$$2)\ \lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$$
$$\textit{limit must exist}$$
$$3)\ f(c) = L$$

Evaluation tactics…(evaluate these limits):

$\lim_{x \to 2} \frac{x-3}{x^2-7}$

<u>Plug in:</u>

$$\lim_{x \to 2} \frac{x-3}{x^2-7} = \frac{(2)-3}{(2)^2-7} = \frac{-1}{-3} = \frac{1}{3}$$

$\lim_{x \to 5} \frac{x^2-25}{x-5}$

<u>Factor and cancel:</u>

$$\lim_{x \to 5} \frac{x^2-25}{x-5} \left(\frac{0}{0}\right) = \lim_{x \to 5} \frac{(x-5)(x+5)}{x-5} = \lim_{x \to 5}(x+5) = 10$$

$\lim_{x \to 9} \frac{x^2-81}{\sqrt{x}-3}$

<u>Rationalize:</u>

$$\lim_{x \to 9} \frac{x^2-81}{\sqrt{x}-3} = \lim_{x \to 9} \frac{(x^2-81)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$$
$$= \lim_{x \to 9} \frac{(x-9)(x+9)(\sqrt{x}+3)}{x-9} = \lim_{x \to 9}(x+9)(\sqrt{x}+3) = (18)(6)$$

What is L'Hopital's Rule?

$$\textit{If } \lim_{x \to c} \frac{f(x)}{g(x)} \textit{ is indeterminant form } \frac{0}{0} \textit{ or } \frac{\pm\infty}{\pm\infty}$$
$$\textit{then } \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Evaluate using L'Hopital's rule:

$$\lim_{x\to\infty}\frac{2x^2-x}{x^2+x}$$

$$\lim_{x\to\infty}\frac{2x^2-x}{x^2+x}\ \left(\frac{\infty}{\infty}\right)$$
$$=\lim_{x\to\infty}\frac{4x-1}{2x+1}=\lim_{x\to\infty}\frac{4}{2}=2$$

$$\lim_{x\to\infty}e^{-x}\sqrt{x}$$

$$\lim_{x\to\infty}e^{-x}\sqrt{x}\ \left(0\bullet\infty\right)$$
$$=\lim_{x\to\infty}\frac{\sqrt{x}}{e^x}\left(\frac{\infty}{\infty}\right)\ =\lim_{x\to\infty}\frac{\frac{1}{2}x^{-\frac{1}{2}}}{e^x}=\lim_{x\to\infty}\frac{1}{2\sqrt{x}e^x}=\frac{1}{\infty}=0$$

$$\lim_{x\to0}\left(1+\frac{x}{2}\right)^{\cot x}$$

function raised to a function? Ln of both sides…

$$y=\lim_{x\to0}\left(1+\frac{x}{2}\right)^{\cot x}$$
$$\ln(y)=\ln\left(\lim_{x\to0}\left(1+\frac{x}{2}\right)^{\cot x}\right)=\lim_{x\to0}\left[\ln\left(\left(1+\frac{x}{2}\right)^{\cot x}\right)\right]$$
$$\ln(y)=\lim_{x\to0}\left[\cot(x)\ln\left(\left(1+\frac{x}{2}\right)\right)\right]\ \left(\frac{\cos 0}{\sin 0}\ln 1=\infty\bullet 0\right)$$
$$\ln(y)=\lim_{x\to0}\left[\frac{\ln\left(\left(1+\frac{x}{2}\right)\right)}{\tan(x)}\right]\ \left(\frac{0}{0}\right)\ l'Hopital's\ rule...$$
$$\ln(y)=\lim_{x\to0}\left[\frac{\frac{1}{1+\frac{x}{2}}\left(\frac{1}{2}\right)}{\sec^2(x)}\right]=\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{(\cos 0)^2}\right)}=\frac{1}{2}$$
$$y=e^{\frac{1}{2}}=\sqrt{e}$$

Special memorized limits:

$$\lim_{x\to0}\frac{\sin x}{x}=$$

$$\lim_{x\to0}\frac{\sin x}{x}=1$$

$$\lim_{x\to0}\frac{1-\cos x}{x}=$$

$$\lim_{x\to0}\frac{1-\cos x}{x}=0$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$$

$$\lim_{x\to\infty}(1+x)^{1/x}=$$

$$\lim_{x\to\infty}(1+x)^{1/x}=e$$

Horizontal asymptotes occur when…

Horizontal asymptotes occur when…
$\lim_{x \to \pm\infty} f(x) = any\ constant$

Vertical asymptotes occur when…

Vertical asymptotes occur when…
$\lim_{x \to c} f(x) = \pm\infty$
(whenever the function's y value is approaching infinity as x approaches a number – usually at uncancelled zeros in the denominator of rational functions)

Derivatives…

Average rate of change of $f(x) =$

(from x = a to x = b)

Average rate of change of $f(x) = \dfrac{f(b) - f(a)}{b - a}$

Instantaneous rate of change of $f(x)$ at x is…

Instantaneous rate of change of $f(x)$ at x $f'(x)$

Limit definition of derivative, $f'(x) =$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative shortcuts…

$$\frac{d}{dx}[c]=$$

$$\frac{d}{dx}\left[x^n\right]=$$

$$\frac{d}{dx}\left[e^x\right]=$$

$$\frac{d}{dx}\left[a^x\right]=$$

$$\frac{d}{dx}\left[\ln(x)\right]=$$

$$\frac{d}{dx}\left[\log_b(x)\right]=$$

$$\frac{d}{dx}\left[\sin(x)\right]=$$

$$\frac{d}{dx}\left[\cos(x)\right]=$$

$$\frac{d}{dx}\left[\tan(x)\right]=$$

$$\frac{d}{dx}\left[\tan(x)\right]=$$

$$\frac{d}{dx}\left[\sec(x)\right]=$$

$$\frac{d}{dx}\left[\csc(x)\right]=$$

$$\frac{d}{dx}\left[\cot(x)\right]=$$

$$\frac{d}{dx}\left[\sin^{-1}(x)\right]=$$

$$\frac{d}{dx}\left[\tan^{-1}(x)\right]=$$

$$\frac{d}{dx}\left[\sec^{-1}(x)\right]=$$

$$\frac{d}{dx}[c]=0$$

$$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$$

$$\frac{d}{dx}\left[e^x\right]=e^x$$

$$\frac{d}{dx}\left[a^x\right]=a^x\ln(a)$$

$$\frac{d}{dx}\left[\ln(x)\right]=\frac{1}{x}$$

$$\frac{d}{dx}\left[\log_b(x)\right]=\frac{1}{x\ln(b)}$$

$$\frac{d}{dx}\left[\sin(x)\right]=\cos(x)$$

$$\frac{d}{dx}\left[\cos(x)\right]=-\sin(x)$$

$$\frac{d}{dx}\left[\tan(x)\right]=\sec^2(x)$$

$$\frac{d}{dx}\left[\tan(x)\right]=\sec^2(x)$$

$$\frac{d}{dx}\left[\sec(x)\right]=\sec(x)\tan(x)$$

$$\frac{d}{dx}\left[\csc(x)\right]=-\csc(x)\cot(x)$$

$$\frac{d}{dx}\left[\cot(x)\right]=-\csc^2(x)$$

$$\frac{d}{dx}\left[\sin^{-1}(x)\right]=\frac{1}{\sqrt{1-x^2}}\quad\left(\frac{d}{dx}\left[\cos^{-1}(x)\right]=\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\frac{d}{dx}\left[\tan^{-1}(x)\right]=\frac{1}{1+x^2}\quad\left(\frac{d}{dx}\left[\cot^{-1}(x)\right]=\frac{-1}{1+x^2}\right)$$

$$\frac{d}{dx}\left[\sec^{-1}(x)\right]=\frac{1}{|x|\sqrt{x^2-1}}\quad\left(\frac{d}{dx}\left[\csc^{-1}(x)\right]=\frac{-1}{|x|\sqrt{x^2-1}}\right)$$

Antiderivative shortcuts…

$\int 0\ dx =$	$\int 0\ dx = C$
$\int c\ dx =$	$\int c\ dx = cx + C$
$\int x^n\ dx =$	$\int x^n\ dx =$
$\int e^x\ dx =$	$\int e^x\ dx = e^x + C$
$\int e^{ax}\ dx =$	$\int e^{ax}\ dx = \frac{e^{ax}}{a} + C$
$\int a^x\ dx =$	$\int a^x\ dx = \frac{a^x}{\ln(a)} + C$
$\int \frac{1}{x}\ dx =$	$\int \frac{1}{x}\ dx = \ln\lvert x\rvert + C$
$\int \sin(x)\ dx =$	$\int \sin(x)\ dx = -\cos(x) + C$
$\int \cos(x)\ dx =$	$\int \cos(x)\ dx = \sin(x) + C$
$\int \sec^2(x)\ dx =$	$\int \sec^2(x)\ dx = \tan(x) + C$
$\int \tan(x)\ dx =$	$\int \tan(x)\ dx = \ln\lvert\sec(x)\rvert + C = -\ln\lvert\cos(x)\rvert + C$
$\int \sec(x)\tan(x)\ dx =$	$\int \sec(x)\tan(x)\ dx = \sec(x) + C$
$\int \csc^2(x)\ dx =$	$\int \csc^2(x)\ dx = -\cot(x) + C$
$\int \cot(x)\ dx =$	$\int \cot(x)\ dx = -\ln\lvert\csc(x)\rvert + C = \ln\lvert\sin(x)\rvert + C$
$\int \csc(x)\cot(x)\ dx =$	$\int \csc(x)\cot(x)\ dx = -\csc(x) + C$
$\int \frac{1}{\sqrt{a^2 - x^2}} =$	$\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{a^2 + x^2} =$	$\int \frac{1}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{x\sqrt{x^2 - a^2}} =$	$\int \frac{1}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

Derivative properties/procedures…

$\frac{d}{dx}[cx]=$

$\frac{d}{dx}[cx]=c\frac{d}{dx}[x]$ (constants can be moved out)

$\frac{d}{dx}[f(x)\pm g(x)]=$

$\frac{d}{dx}[f(x)\pm g(x)]=\frac{d}{dx}[f(x)]\pm\frac{d}{dx}[g(x)]$
(derivative of each term separately)

$\frac{d}{dx}[f(x)g(x)]=$ (product rule)

$\frac{d}{dx}[f(x)g(x)]=f(x)g'(x)+g(x)f'(x)$
(1^{st} times deriv. of 2^{nd} plus 2^{nd} times deriv. of 1st)

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=$ (quotient rule)

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$
(low-dhigh minus high-dlow over low squared)

$\frac{d}{dx}[f(g(x))]=$ (chain rule)

$\frac{d}{dx}[f(g(x))]=f'(g(x))\bullet g'(x)$
(deriv. of outside (with same inside) times deriv. of inside)

1) <u>Implicit differentiation</u>:

ex: Find $\frac{dy}{dx}$ *for* $xy^3+3x^2=4-y^5$

1) <u>Implicit differentiation</u>:

$$x\frac{d}{dx}[y^3]+y^3\frac{d}{dx}[x]+\frac{d}{dx}[3x^2]=\frac{d}{dx}[4]-\frac{d}{dx}[y^5]$$

$$x\left(3y^2\frac{dy}{dx}\right)+y^3(1)+6x=0-5y^4\frac{dy}{dx}$$

$$\frac{dy}{dx}(3xy^2+5y^4)=-6x-y^3$$

$$\frac{dy}{dx}=\frac{-6x-y^3}{3xy^2+5y^4}$$

2) <u>Logarithmic differentiation</u>:

ex: Find $\frac{dy}{dx}$ *for* $y=x^{(5x^3+2x)}$

2) <u>Logarithmic differentiation</u>:

$$\ln(y)=\ln\left(x^{5x^3+2x}\right)$$

$$\ln(y)=(5x^3+2x)\ln(x)$$

$$\frac{d}{dx}[\ln(y)]=\frac{d}{dx}[(5x^3+2x)\ln(x)]$$

$$\frac{d}{dx}[\ln(y)]=(5x^3+2x)\frac{d}{dx}[\ln(x)]+\ln(x)\frac{d}{dx}[(5x^3+2x)]$$

$$\frac{1}{y}\frac{dy}{dx}=(5x^3+2x)\frac{1}{x}+\ln(x)(15x^2+2)$$

$$\frac{dy}{dx}=\left[(5x^3+2x)\frac{1}{x}+\ln(x)(15x^2+2)\right]y$$

$$\frac{dy}{dx}=\left[(5x^3+2x)\frac{1}{x}+\ln(x)(15x^2+2)\right]x^{(5x^3+2x)}$$

Integral properties/procedures…

$\int c\ f(x)\,dx =$

$\int c\ f(x)\,dx = c\int f(x)\,dx$ (constants can be moved out)

$\int\left[f(x)\pm g(x)\right]dx =$

$\int\left[f(x)\pm g(x)\right]dx = \int\left[f(x)\right]dx \pm \int\left[g(x)\right]dx$

(can split into separate integrals for each term)

$\int_b^a f(x)\,dx =$

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

1) <u>u-substitution</u> (integral version of chain rule)

ex: $\int x\cos\left(x^2\right)dx$

1) <u>u-substitution</u> (integral version of chain rule)

$\int x\cos\left(x^2\right)dx \quad u = x^2$

$\frac{du}{dx} = 2x, \quad du = 2xdx, \quad xdx = \frac{1}{2}du$

substitute into original integral:

$\int \cos(u)\frac{1}{2}du = \frac{1}{2}\int\cos(u)\,du = \frac{1}{2}\sin(u) = \frac{1}{2}\sin\left(x^2\right) + C$

2) <u>by parts</u> (integral version of product rule)

ex: $\int x\ln(x)\,dx$

2) <u>by parts</u> (integral version of product rule)

$\int x\ln(x)\,dx \quad u = \ln(x) \qquad dv = x\,dx$

$\frac{du}{dx} = \frac{1}{x} \qquad \int dv = \int x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

substitute into pattern:

$$uv - \int v\,du = \left(\ln(x)\right)\left(\frac{1}{2}x^2\right) - \int\frac{1}{2}x^2\frac{1}{x}dx$$

$$= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$$

3) <u>trigonometric integrals</u>

ex: $\int \sin^3 x\cos^3 x\,dx$

3) <u>trigonometric integrals</u>

$\int\sin^3 x\cos^3\,dx \quad$ (*split off something to form du*)

$\int\sin^3 x\cos^2 x\cos xdx$

$\int\sin^3 x\left(1-\sin^2 x\right)\cos xdx$

$\int\left(\sin^3 x - \sin^5 x\right)\cos xdx$

$\int\sin^3 x\cos xdx - \int\sin^5 x\cos xdx$

$u = \sin x, \frac{du}{dx} = \cos x, \quad \cos xdx = du$

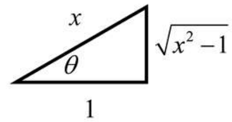
$\int u^3du - \int u^5du$

$\frac{1}{4}u^4 - \frac{1}{6}u^6 + C = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

4) <u>trigonometric substitution</u>

ex: $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$

4) <u>trigonometric substitution</u>



$$\cos\theta = \frac{1}{x} \qquad \tan\theta = \frac{\sqrt{x^2-1}}{1}$$

$$x = \frac{1}{\cos\theta} = \sec\theta \qquad \sqrt{x^2-1} = \tan\theta$$

$$\frac{dx}{d\theta} = \sec\theta\tan\theta \qquad dx = \sec\theta\tan\theta\, d\theta$$

5) <u>partial fraction expansion</u>

ex: $\int \frac{1}{x^2-5x+6}\,dx$

5) <u>partial fraction expansion</u>

$$\int \frac{1}{x^2-5x+6}\,dx \qquad \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$\int \frac{1}{(x-3)(x-2)}\,dx \qquad A(x-2)+B(x-3)=1$$

$$Ax-2A+Bx-3B=1$$

$$(A+B)x+(-2A-3B)=(0)x+(1)$$

$$system: \begin{cases} A+B=0 \\ -2A-3B=1 \end{cases} \qquad A=1,\ B=-1$$

$$1\int \frac{1}{x-3}\,dx - 1\int \frac{1}{x-2}\,dx$$

$$\ln|x-3| - \ln|x-2| + C = \ln\left|\frac{x-3}{x-2}\right| + C$$

6) <u>complete the square to arctan form</u>

ex: $\int \frac{1}{x^2-4x+13}\,dx$

6) <u>complete the square to arctan form</u>

$$\int \frac{1}{x^2-4x+13}\,dx \qquad x^2-4x+\underline{\underline{4}}+13-\underline{\underline{4}}$$

$$(x-2)^2+9$$

$$\int \frac{1}{(x-2)^2+9}\,dx \quad now\ u-sub:\ u=x-2,\ \frac{du}{dx}=1,\ du=dx$$

$$\int \frac{1}{u^2+3^2}\,dx$$

$$\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)+C = \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+C$$

<u>Improper Integrals</u>:

$$\int_1^\infty \frac{1}{x^2}\,dx =$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{b}-\left(-\frac{1}{1}\right)\right] = -\frac{1}{\infty}+1 = 0+1 = 1$$

$$\int_1^\infty \frac{1}{x}\,dx =$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|b| - (\ln|1|)\right] = \infty - 0 = \infty$$

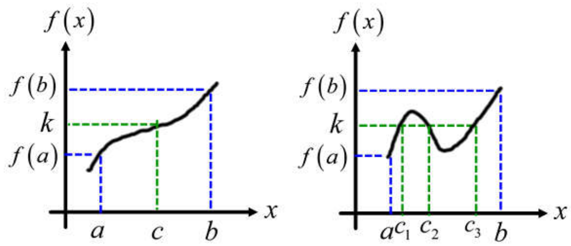
(integral may converge to a number, or diverge)

Theorems…

What is the Intermediate Value Theorem?

<u>Intermediate Value Theorem</u>

If f is continuous on $[a,b]$, $f(a) \neq f(b)$, and k is any numberbetween $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c) = k$.

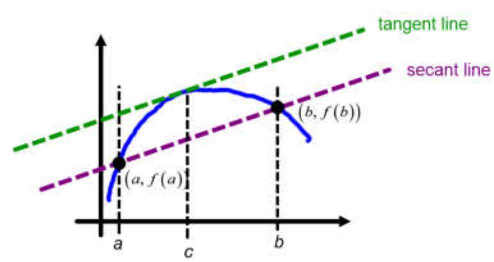


Note: This theorem doesn't provide a method for finding the value(s) *c*, and doesn't indicate the number of *c* values which map to *k*, it only guarantees the existence of at least one number *c* such that f(c) = k.

What is the Mean Value Theorem?

<u>Mean Value Theorem</u>

Let f be continuous on $[a,b]$, and differentiable on (a,b), then there exists a number c in (a,b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



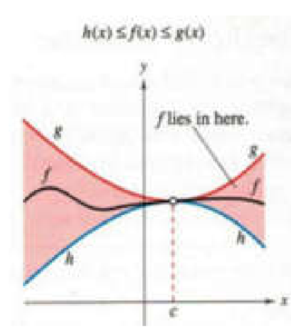
In other words, you can find a mean (average) rate of change across and interval, and there is some input value where the instantaneous rate of change equals the mean rate of change.

(Special case when slope = 0 is called ‘Rolle’s Theorem’)

What is the Squeeze Theorem?

<u>Squeeze Theorem</u>

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$ then $\lim_{x \to c} f(x)$ exists and is equal to L.



Different representational forms of relationships…

Rectangular: $y = f(x)$ *Parametric*: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

Convert to rectangular form and sketch:

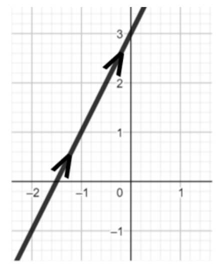
$$\begin{cases} x = 2t \\ y = 4t + 3 \end{cases}$$

$$\begin{cases} x = 1 + 2\cos t \\ y = -2 + 3\sin t \end{cases}$$

Convert to rectangular form:

$$\begin{cases} x = 2t \\ y = 4t + 3 \end{cases} \quad \textit{eliminate parameter by substitution}$$

$$t = \frac{1}{2}x, \quad y = 4\left(\frac{1}{2}x\right) + 3, \quad y = 2x + 3$$



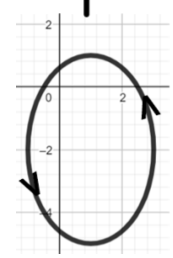
(include direction arrows)

$$\begin{cases} x = 1 + 2\cos t \\ y = -2 + 3\sin t \end{cases} \quad \textit{use } \sin^2 t + \cos^2 t = 1$$

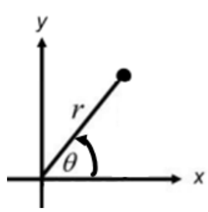
$$\cos t = \frac{x-1}{2}, \quad \sin t = \frac{y+2}{3}$$

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1 \qquad (\textit{ellipse})$$

t	(x,y)
0	(3,-2)
$\frac{\pi}{2}$	(1,1)



Polar:



Formulas for converting polar - rectangular:

$x =$

$y =$

$x^2 + y^2 =$

$\tan\theta =$

Formulas for converting polar - rectangular:

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

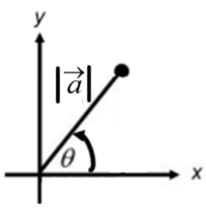
$\tan\theta = \frac{y}{x}$

Convert to rectangular form and sketch:

$r = 8\sin\theta$

$\theta = \frac{5\pi}{6}$

Vectors:



Position: $\vec{a} = \langle a_x, a_y \rangle, \; a \text{ distance of } |\vec{a}| \text{ in direction } \theta$

Formulas for vector $\left(\vec{a} = \langle a_x, a_y \rangle\right)$:

$magnitude\ of\ a = |\vec{a}| =$

$components:$

$a_x =$

$a_y =$

Vectors are equal if…

Find the vector from $(1,3)$ *to* $(9,4)$

Vector-valued functions:
Input is a parameter (typically, *t*), output is a vector
Example: position vector: $\vec{r}(t) = \langle x(t), y(t) \rangle$

Convert to rectangular form:

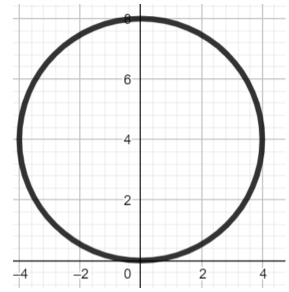
$r = 8\sin\theta$

$r^2 = 8r\sin\theta$

$x^2 + y^2 = 8y$

$x^2 + y^2 - 8y = 0$

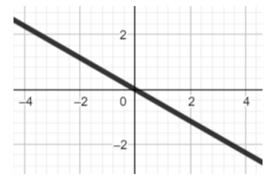
$x^2 + (y-4)^2 + 16$



$\theta = \frac{5\pi}{6}$

$\tan\theta = \tan\left(\frac{5\pi}{6}\right)$

$\frac{y}{x} = \frac{\sin\left(\frac{5\pi}{6}\right)}{\cos\left(\frac{5\pi}{6}\right)} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{-\sqrt{3}}{2}\right)} = \frac{1}{-\sqrt{3}}$



$y = \frac{1}{-\sqrt{3}}x$

Formulas for vectors:

$magnitude\ of\ a = |\vec{a}| = \sqrt{(a_x)^2 + (a_y)^2}$

$components:$

$a_x = |\vec{a}|\cos\theta$

$a_y = |\vec{a}|\sin\theta$

Vectors are equal if…their component values are equal.

$\vec{a} = \langle 9-1,\ 4-3 \rangle = \langle 8,1 \rangle$

Vector-valued functions:
Input is a parameter (typically, *t*), output is a vector
Example: position vector: $\vec{r}(t) = \langle x(t), y(t) \rangle$

Arithmetic operations and properties for different representations…

Multiplication by a constant…

$3\langle 6,-3\rangle =$ $\qquad$ $3\langle 6,-3\rangle = \langle 18,-9\rangle$

$\lim_{x\to 2} 3f(x) =$ $\qquad$ $\lim_{x\to 2} 3f(x) = 3\lim_{x\to 2} f(x)$

$\frac{d}{dx}\left[3f(x)\right] =$ $\qquad$ $\frac{d}{dx}\left[3f(x)\right] = 3\frac{d}{dx}\left[f(x)\right]$

$\int 3f(x)\,dx =$ $\qquad$ $\int 3f(x)\,dx = 3\int f(x)\,dx$

$\sum_{n=1}^{\infty} 3a_n =$ $\qquad$ $\sum_{n=1}^{\infty} 3a_n = 3\sum_{n=1}^{\infty} a_n$

In general, multiplication of objects other than numbers is not straightforward (derivative of function multiplied requires product rule, integral requires integration by parts, multiplication of a vector by another vector not defined for this class, cannot multiply two series in summation form.)

Addition/subtraction…

$\langle 8,1\rangle - \langle 2,5\rangle =$ $\qquad$ $\langle 8,1\rangle - \langle 2,5\rangle = \langle 8-2,\ 1-5\rangle = \langle 6,-4\rangle$

$\lim_{x\to c}\left(x^3 - \frac{1}{x}\right) =$ $\qquad$ $\lim_{x\to c}\left(x^3 - \frac{1}{x}\right) = \lim_{x\to c} x^3 - \lim_{x\to c}\frac{1}{x}$

$\frac{d}{dx}\left[x^3 - \sin(x)\right] =$ $\qquad$ $\frac{d}{dx}\left[x^3 - \sin(x)\right] = 3x^2 - \cos(x)$

$\int\left(x^3 - \cos(x)\right)dx =$ $\qquad$ $\int\left(x^3 - \cos(x)\right)dx = \frac{1}{4}x^4 - \sin(x) + C$

$\sum_{n=1}^{\infty}(a_n \pm b_n) =$ $\qquad$ $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

PEMDAS still applies…

$2\langle 8,1\rangle - 3\langle 2,5\rangle =$

$$2\langle 8,1\rangle - 3\langle 2,5\rangle$$
$$\langle 16,2\rangle - \langle 6,15\rangle \quad \textit{multiplication before addition}$$
$$\langle 16-6,\ 2-15\rangle = \langle 10,-13\rangle$$

For vectors things like limits, derivatives, or integrals apply separately to each term:

$\lim_{t\to 4}\left(\langle t^2,\ t^3\rangle\right) =$ $\qquad$ $\lim_{t\to 4}\left(\langle t^2,\ t^3\rangle\right) = \left\langle \lim_{t\to 4} t^2, \lim_{t\to 4} t^3 \right\rangle$

$\frac{d}{dx}\left[\langle t^2,\ t^3\rangle\right] =$ $\qquad$ $\frac{d}{dx}\left[\langle t^2,\ t^3\rangle\right] = \langle 2t,\ 3t^2\rangle$

$\int\langle t^2,\ t^3\rangle\,dt =$ $\qquad$ $\int\langle t^2,\ t^3\rangle\,dt = \left\langle \frac{1}{3}t^3 + C_1,\ \frac{1}{4}t^4 + C_2\right\rangle = \left\langle \frac{1}{3}t^3,\ \frac{1}{4}t^4\right\rangle + \vec{C}$

For limits:

$\lim_{x \to c}\left[f(x)\right]^n =$ $\lim_{x \to c}\left[f(x)\right]^n = \left[\lim_{x \to c} f(x)\right]^n$

$\lim_{x \to c}\left[\sqrt[n]{f(x)}\right] =$ $\lim_{x \to c}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x \to c} f(x)}$

Derivatives in parametric form $\begin{cases} x = t^2 - 3t \\ y = \sin(t) \end{cases}$:

$\frac{dy}{dx} =$ $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos(t)}{2t-3}$

$\frac{d^2y}{dx^2} =$ $\frac{d^2y}{dx^2} = \frac{\frac{d}{dx}\left[\frac{\cos(t)}{2t-3}\right]}{(2t-3)} = \frac{\frac{(2t-3)(-\sin(t)) - \cos(t)2}{(2t-3)^2}}{2}$

Derivatives in polar form $r = 4\sin(\theta)$:

$\frac{dy}{dx} =$

$x = r\cos\theta = 4\sin\theta\cos\theta \qquad y = r\sin\theta = 4(\sin\theta)^2$

$\frac{dx}{d\theta} = (4\sin\theta)(-\sin\theta) + \cos\theta(4\cos\theta) = -4\sin^2\theta + 4\cos^2\theta$

$\frac{dy}{d\theta} = 8\sin\theta\cos\theta$

$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{8\sin\theta\cos\theta}{-4\sin^2\theta + 4\cos^2\theta}$

$\frac{dy}{dx}$ is the slope of the tangent line on the x-y plane.

$\frac{dy}{dx}$ is the slope of the tangent line on the x-y plane.

Horizontal tangents occur when… Horizontal tangents occur when… $\frac{dy}{dx} = 0$

Vertical tangents occur when… Vertical tangents occur when… $\frac{dy}{dx}$ is undefined

Intersections are always system solutions
(find the intersections):

$$\begin{cases} y = x^2 - 6 \\ y = -x \end{cases}$$

$$\begin{cases} y = x^2 - 6 \\ y = -x \end{cases}$$

$$\begin{array}{lll} x^2 - 6 = -x & x = -3, & x = 2 \\ x^2 + x - 6 = 0 & y = -(-3) & y = -(2) \\ (x+3)(x-2) = 0 & (-3, 3) & (2, -2) \end{array}$$

$$\begin{cases} r = 3(1 + \sin\theta) \\ r = 3(1 - \sin\theta) \end{cases}$$

$$\begin{cases} r = 3(1 + \sin\theta) \\ r = 3(1 - \sin\theta) \end{cases}$$

$$\begin{array}{lll} 3(1+\sin\theta) = 3(1-\sin\theta) & \theta = 0, & \theta = \pi \\ \sin\theta = -\sin\theta & r = 3(1+\sin 0) & r = 3(1+\sin\pi) \\ 2\sin\theta = 0 & r = 3 & r = 3 \\ \sin\theta = 0 & (r,\theta):\ (3,0) & (3,\pi) \end{array}$$

$$\textit{must also graph and check for } r = 0\ (\textit{is an intersection here}):$$

$$\textit{for } r = 3(1+\sin\theta) \rightarrow 0 = 3(1+\sin\theta),\ \sin\theta = -1,\ \theta = \frac{3\pi}{2}$$

$$\textit{for } r = 3(1-\sin\theta) \rightarrow 0 = 3(1-\sin\theta),\ \sin\theta = 1,\ \theta = \frac{\pi}{2}$$

$$\textit{so } \left(0, \frac{3\pi}{2}\right) \textit{ and } \left(0, \frac{\pi}{2}\right) \ (\textit{coincident, but not a 'collision'} - \textit{different } \theta)$$

$$\begin{cases} x_1 = 3\sin t \\ y_1 = 2\cos t \end{cases} \quad \begin{cases} x_2 = 3 + \cos t \\ y_2 = 1 + \sin t \end{cases} \quad 0 \le t < 2\pi$$

$$\begin{cases} x_1 = 3\sin t \\ y_1 = 2\cos t \end{cases} \quad \begin{cases} x_2 = 3 + \cos t \\ y_2 = 1 + \sin t \end{cases} \quad 0 \le t < 2\pi$$

$$\begin{array}{ll} x_1 = x_2 & y_1 = y_2 \\ 3\sin t = 3 + \cos t & 2\cos t = 1 + \sin t \\ \textit{by calculator graph}: & \textit{by calculator graph}: \\ \textit{at } t = 1.5708,\ t = 2.2143 & \textit{at } t = 0.6435,\ t = 4.7123 \end{array}$$

$$(\textit{these are intersections, but not 'collisions'} - \textit{different } t)$$

Applications of derivatives…

<u>What do each of these tell us about f</u> ?

$f(x)$ is

$f'(x)$ is

$f''(x)$ is

Critical points occur when…

Inflection points occur when…

Relative (local) max occurs when…

Relative (local) min occurs when…

What do each of these tell us about f ?

$f(x)\ is\ the\ y-value\ at\ x$

$f'(x)\ is\ the\ instantaneous\ rate\ of\ change\ ('slope')\ at\ x$

$f'(x)>0\quad f\ is\ increasing$

$f'(x)<0\quad f\ is\ decreasing$

$f''(x)\ is\ the\ concavity\ ('curvature')\ at\ x$

$f''(x)>0\quad f\ is\ concave\ up$

$f''(x)<0\quad f\ is\ concave\ down$

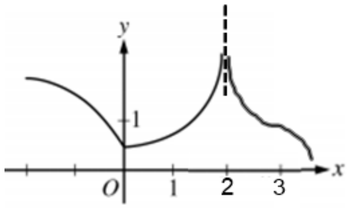
Critical points occur when $f'(x)=0\ or\ DNE$
and the sign of $f'(x)$ changes.

Inflection points occur when $f''(x)=0\ or\ DNE$
and the sign of $f''(x)$ changes.

Relative (local) max occurs when $f'(x)=0\ or\ DNE$
and the sign of $f'(x)$ goes from + to -

Relative (local) min occurs when $f'(x)=0\ or\ DNE$
and the sign of $f'(x)$ goes from - to +

Using a graph of the curve f



Graph of f

Where is f increasing? decreasing?

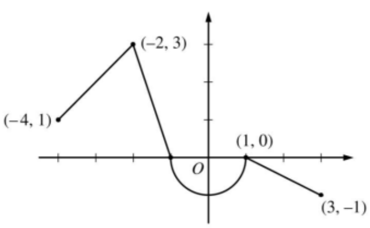
Where is f concave up? concave down?

Where is f continuous?

Where is f differentiable?

Where are the following for f ?
- critical points
- relative maxima
-relative minima
- inflection points

What are the absolute max/min over [-2,1]?

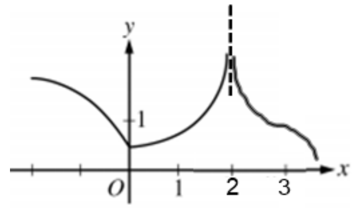


Graph of f

$$\int_{-4}^{3} f(x)\,dx =$$

$$\int_{3}^{-4} f(x)\,dx =$$

Using a graph of the curve f



Graph of f

f *is increasing over* $(0,2)$ [f *going up*]
decreasing over $(-2,0)\cup(2,4)$ [f *going down*]

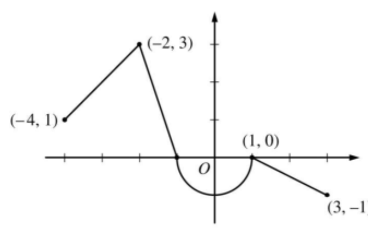
f *is concave up over* $(0,2)\cup(2,3)$
concave down over $(-2,0)\cup(3,4)$

f *is continuous over* $(-2,2)\cup(2,4)$

f *is differentiable over* $(-2,0)\cup(0,2)\cup(2,4)$

Where are the following for f ?
critical points at (-2,2), (0,0.5), (2.5,1)
no relative maxima
relative minima at (0,0.5)
inflection points at (0,0.5), (3,1)

What are the absolute max/min over [-2,1]?
Absolute min at (0,0.5), absolute max at (-2,2)

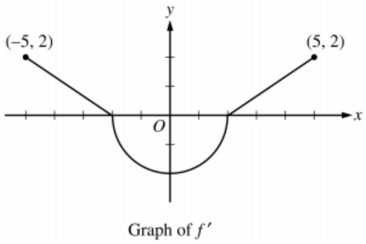


Graph of f

$$\int_{-4}^{3} f(x)\,dx = areas = 4+2-\frac{\pi}{2}-1 = 5-\frac{\pi}{2}$$

$$\int_{3}^{-4} f(x)\,dx = -\int_{-4}^{3} f(x)\,dx = -5+\frac{\pi}{2}$$

<u>Using a graph of the derivative</u> f'



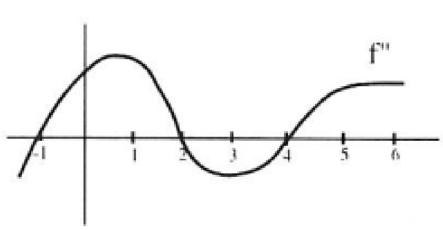
Where is f increasing? decreasing?

Where is f concave up? concave down?

Where are the following for f ?
- critical points
- relative maxima
-relative minima
- inflection points

$If\ f(2)=1,\ then\ f(-5)=$

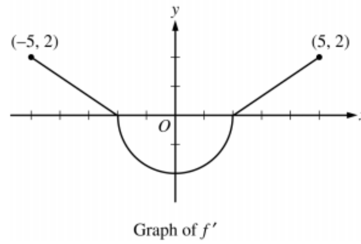
<u>Using a graph of the concavity</u> f''



Where is f concave up? concave down?

Where inflection points for f ?

<u>Using a graph of the derivative</u> f'



f *is increasing over* $(-5,-2)\cup(2,5)$ $[f'>0]$
decreasing over $(-2,2)$ $[f'<0]$

f *is concave up over* $(0,5)$ $[f'\ going\ up]$
concave down over $(-5,0)$ $[f'\ going\ down]$

Where are the following for f ?
critical points at x=-2, x=2 [f' = 0]
relative maxima at x = 2 [f' from – to +]
relative minima at x = -2 [f' from + to -]
inflection point at x =0 [f' graph changing direction]

We can use the Net Change Theorem
(*part of the Fundamental Theorem of Calculus*):

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

evaluate definite integral by plugging limits into antiderivative
This also means an integral of a derivative of something is equal to the accumulation (net change) in the value this is a derivative of :

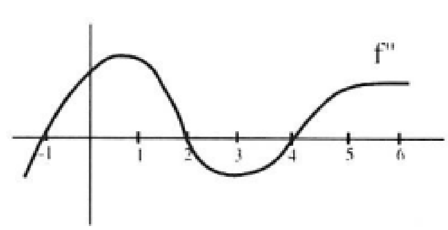
$$\int_a^b f'(x)\,dx = f(b)-f(a)$$

Pick one limit to be what you have and the other what you need:

$\int_{-5}^{2} f'(x)\,dx = f(2)-f(-5)$ and evaluate integral using areas

$$3-\frac{1}{2}\pi(2)^2 = 1-f(-5), \qquad f(-5)=1-3+2\pi=2\pi-2$$

<u>Using a graph of the concavity</u> f''



f *is concave up over* $(-1,2)\cup(4,6)$ $[f''>0]$
concave down over $(-2,-1)\cup(2,4)$ $[f''<0]$

Where inflection points for f ?
at x =-1, x=2, x=4 [f'' = 0 and sign is changing]

Tangent lines…

Rectangular:

$For\ (x-2)^2+(y+3)^2=4$

(a) *Write the equation of the tangent line at* $\left(1,-3+\sqrt{3}\right)$

(b) *Where does this curve have horizontal tangents?*

(c) *Where does this curve have vertical tangents?*

$For\ (x-2)^2+(y+3)^2=4$

(a) *Write the equation of the tangent line at* $\left(1,-3+\sqrt{3}\right)$

$$m=\frac{dy}{dx}\left[\text{use implicit differentiation if needed}\right]:$$

$$2(x-2)(1)+2(y+3)\left(\frac{dy}{dx}\right)=0,\quad \frac{dy}{dx}=\frac{-x+2}{y+3}=\frac{-(1)+2}{\left(-3+\sqrt{3}\right)+3}=\frac{1}{\sqrt{3}}$$

$$\left(y-\left(-3+\sqrt{3}\right)\right)=\frac{1}{\sqrt{3}}(x-1)$$

(b) *Where does this curve have horizontal tangents?*

$$\text{where } \frac{dy}{dx}=0\ \ (\text{numerator}=0),\ \ -x+2=0,\quad \text{at } x=2\ (2\text{ points})$$

(c) *Where does this curve have vertical tangents?*

$$\text{where } \frac{dy}{dx}=DNE\ \ (\text{denominator}=0),\ \ y+3=0,\quad \text{at } y=-3\ (2\text{ points})$$

Parametric:

$$For \begin{cases} x=2t-\pi\sin t \\ y=2-\pi\cos t \end{cases}$$

(a) *Write the equation of the tangent line at* $t=\frac{2\pi}{3}$

(b) *Where does this curve have horizontal tangents?*

(c) *Where does this curve have vertical tangents?*

$$For \begin{cases} x=2t-\pi\sin t \\ y=2-\pi\cos t \end{cases}$$

(a) *Write the equation of the tangent line at* $t=\frac{2\pi}{3}$

$$m=\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\left.\frac{\pi\sin t}{2-\pi\cos t}\right|_{t=\frac{2\pi}{3}}=\frac{\pi\left(\frac{\sqrt{3}}{2}\right)}{2-\pi\left(-\frac{1}{2}\right)}=\frac{\frac{\sqrt{3}\pi}{2}}{\frac{2}{1}+\frac{\pi}{2}}=\frac{\sqrt{3}\pi}{4+\pi}=0.76193$$

$$x=2\left(\frac{2\pi}{3}\right)-\pi\sin\left(\frac{2\pi}{3}\right)=\frac{4\pi}{3}-\pi\left(\frac{\sqrt{3}}{2}\right)=1.4681$$

$$y=2-\pi\cos\left(\frac{2\pi}{3}\right)=2-\pi\left(-\frac{1}{2}\right)=3.5708$$

$$(y-3.5708)=0.76193(x-1.4681)$$

(b) *Where does this curve have horizontal tangents?*

$$\text{where } \frac{dy}{dx}=0\ \ (\text{numerator}=0)\ \ \pi\sin t=0$$

$$t=0,\ t=\pi\ \ (\text{and other values, use calculator})$$

(c) *Where does this curve have vertical tangents?*

$$\text{where } \frac{dy}{dx}=DNE\ \ (\text{denominator}=0)\ \ 2-\pi\cos t=0$$

$$t=-0.8807,\ t=0.8807\ \ (\text{and other values, use calculator})$$

Polar:

$For\ r = 4\sin\theta$

(a) *Write the equation of the tangent line at* $\theta = \frac{\pi}{3}$

(b) *Where does this curve have horizontal tangents?*

(c) *Where does this curve have vertical tangents?*

$For\ r = 4\sin\theta$

(a) *Write the equation of the tangent line at* $\theta = \frac{\pi}{3}$

$$m = \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dy}{d\theta}\right)}$$

and $x = r\cos\theta = 4\sin\theta\cos\theta, \quad y = r\sin\theta = 4(\sin\theta)^2$

$$\frac{dx}{d\theta} = 4\sin\theta(-\sin\theta) + \cos\theta(4\cos\theta) \quad \frac{dy}{d\theta} = 8\sin\theta\cos\theta$$

$$\frac{dx}{d\theta} = 4\left(\cos^2\theta - \sin^2\theta\right) \quad \frac{dy}{d\theta} = 8\sin\theta\cos\theta$$

$$m = \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dy}{d\theta}\right)} = \left.\frac{8\sin\theta\cos\theta}{4\left(\cos^2\theta - \sin^2\theta\right)}\right|_{t=\frac{\pi}{3}} = \frac{8\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{4\left(\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right)}$$

$$= \frac{2\sqrt{3}}{-1} = -2\sqrt{3} = -3.464$$

$$x = 4\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \sqrt{3} = 1.732$$

$$y = 4(\sin\theta)^2 = 4\left(\frac{\sqrt{3}}{2}\right)^2 = 3$$

$$(y - 1.732) = -3.464(x - 3)$$

(b) *Where does this curve have horizontal tangents?*

where $\frac{dy}{dx} = 0$ $(numerator = 0)$ $8\sin\theta\cos\theta = 0$

$\theta = 0, t = \frac{\pi}{2}$ *(and other values, use calculator)*

(c) *Where does this curve have vertical tangents?*

where $\frac{dy}{dx} = DNE$ $(denominator = 0)$ $4\left(\cos^2\theta - \sin^2\theta\right) = 0$

$t = -0.7854, t = 0.7854$ *(and other values, use calculator)*

Position, Velocity (speed), Acceleration…

In 1D:

An object moves in one direction with position x given by $x(t)=t^3-4t^2+3$.

(a) Find velocity as function of time.
(b) What acceleration as a function of time.
(c) What is the position of the particle at *t* = 2?
(d) What is the speed of the particle at *t* = 2?

In 1D:

$$x(t)=t^3-4t^2+3$$
$$(a)\quad v(t)=x'(t)=3t^2-8t$$
$$(b)\quad a(t)=v'(t)=6t-8$$
$$(c)\quad x(2)=(2)^3-4(2)^2+3=-5\quad (\textit{include units if given in problem})$$
$$(d)\quad speed=|v(2)|=\left|3(2)^2-8(2)\right|=|-4|=4$$

An object is launched upward with an initial velocity of $30\ m/s$ from an initial height of $10\ m$ in gravity field with $a(t)=-9.8\ m/s^2$.

(a) Find velocity as a function of time.
(b) Find height as a function of time.
(c) At what time does the object reach maximum height and what is the max height?
(d) At what time does the object hit the ground?

$$a(t)=-9.8$$
$$(a)\quad v(t)=\int a(t)\,dt=\int(-9.8)\,dt=-9.8t+C_1$$
$$v(0)=30,\ so\ 30=-9.8(0)+C_1,\ C_1=30$$
$$v(t)=-9.8t+30$$
$$(b)\quad x(t)=\int v(t)\,dt=\int(-9.8t+30)\,dt=-4.9t^2+30t+C_2$$
$$x(0)=10,\ so\ 10=-4.9(0)^2+30(0)+C_2,\ C_2=10$$
$$x(t)=-4.9t^2+30t+10$$
$$(c)\quad Max\ height\ when\ v=0:\ -9.8t+30=0,\ t=3.06122\text{ sec}$$
$$x(3.06122)=55.91837\ m$$
$$(d)\quad On\ ground\ when\ x=0:\ -4.9t^2+30t+10=0$$
$$at\ t=\frac{-30\pm\sqrt{(30)^2-4(-4.9)(10)}}{2(-4.9)}=\sout{-0.3169},\ 6.439\text{ sec}$$

In 2D (vector/parametric):

An object moves in the xy-plane with:

a velocity vector $\vec{v}(t)=\langle t^3-5t^2,\cos t\rangle$

…or could be given as parametric equations:

$$\begin{cases}x(t)=t^3-5t^2\\ y(t)=\cos t\end{cases}$$

(a) Find the position vector if $\vec{x}(0)=\langle 3,0\rangle$.
(b) Find the acceleration vector.
(c) What is the position, velocity, and acceleration of the object at *t* = 2?
(d) What is the speed of the object at *t* = 2?

In 2D (vector/parametric):

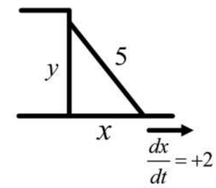
$$\vec{v}(t)=\langle t^3-5t^2,\ \cos t\rangle$$
$$(a)\quad \vec{r}(t)=\left\langle\int(t^3-5t^2)\,dt,\ \int(\cos t)\,dt\right\rangle=\left\langle\frac{1}{4}t^4+C_1,\ \sin t+C_2\right\rangle$$
$$\vec{v}(0)=\langle 3,0\rangle\quad so\quad \langle 3,0\rangle=\left\langle\frac{1}{4}(0)^4+C_1,\ \sin(0)+C_2\right\rangle=\langle C_1,\ C_2\rangle$$
$$C_1=3,\ C_2=0,\quad \vec{r}(t)=\left\langle\frac{1}{4}t^4+3,\ \sin t\right\rangle$$
$$(b)\quad \vec{a}(t)=\left\langle\frac{d}{dx}\left[t^3-5t^2\right],\ \frac{d}{dx}[\cos t]\right\rangle=\langle 3t^2-10t,\ -\sin t\rangle$$
$$(c)\quad \vec{r}(2)=\left\langle\frac{1}{4}(2)^4+3,\ \sin(2)\right\rangle=\langle 7,0.9093\rangle$$
$$\vec{v}(2)=\left\langle(2)^3-5(2)^2,\ \cos(2)\right\rangle=\langle -12,-0.4161\rangle$$
$$\vec{a}(2)=\left\langle 3(2)^2-10(2),\ -\sin(2)\right\rangle=\langle -8,\ -0.9093\rangle$$
$$(d)\ speed=\left|\vec{v}(2)\right|=\sqrt{(-12)^2+(-0.9093)^2}=12.0344$$

NOTE: Polar is similar to vector / parametric, the parameter is just θ instead of t, with $x=r\cos\theta,\ y=r\sin\theta$.

Related Rates Problems…

A 5-foot long ladder is leaning against a building. If the foot of the ladder is sliding away from the building at a rate of 2 ft/sec, how fast is the top of the ladder moving and in what direction when the foot of the ladder is 4 feet from the building?

Draw a picture and assign variables to things which vary then find equations which relate the variables:



$$x^2 + y^2 = 5^2$$

Anything changing is a derivative with respect to time $(+\ \text{if value is increasing})$

$$\frac{dx}{dt} = +2$$

At this snapshot in time, variables have 'snaphot' values:

$$(4)^2 + y^2 = 5^2, \quad y = 3$$

Differentiate implicitly WRT time, plug in values, and solve:

$$x^2 + y^2 = 25$$

$$\frac{d}{dt}\left[x^2\right] + \frac{d}{dt}\left[y^2\right] = \frac{d}{dt}[25]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(4)(2) + 2(4)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{16}{8} = -2\ ft/\sec$$

negative b/c top of ladder is moving so y is decreasing (downward)

Optimization Problems…

A cylindrical can (with circular base) is made with a material for the lateral side which costs \$3/cm^2, and a material for the top and bottom circular sides which costs \$5/cm^2. If the can must enclose a volume of $20\pi\ cm^3$ what should the radius and height be to minimize the material cost?

Need functions for the objective function (what is being optimized) and any constraints.

Objective Function

$$Cost, C = (A_{lateral})\left(\frac{\$3}{cm^3}\right) + (A_{top/bottom})\left(\frac{\$5}{cm^3}\right)$$

$$C = (2\pi rh)(3) + (2)(\pi r^2)(5)$$

$$C = 6\pi rh + 10\pi r^2 \quad \text{cost in terms of } r \text{ and } h$$

Constraint

$$V = 20\pi\ cm^3$$

$$\pi r^2 h = 20\pi$$

Now solve constraint for one variables, substitute into objective function:

$$h = \frac{20\pi}{\pi r^2} = \frac{20}{r^2} \qquad so \qquad C = 6\pi r\left(\frac{20}{r^2}\right) + 10\pi r^2 = 120\pi r^{-1} + 10\pi r^2$$

Now find min *by taking derivative and finding where $C'(r) = 0$*

$$C'(r) = -120\pi r^{-2} + 20\pi r = 0$$

$$20\pi r = \frac{120\pi}{r^2}, \quad r^3 = \frac{120}{20} = 6, \quad r = \sqrt[3]{6} = (6)^{\frac{1}{3}} = 1.81712\ cm$$

Use constraint equation to find other dimension:

$$h = \frac{20}{r^2} = \frac{20}{(1.81712)^2} = 6.057\ cm$$

Should use 2nd – derivative to verify this is a min *not a* max:

$$C''(r) = 240\pi r^{-3} + 20\pi \ \text{ is + for + r, so concave up, so this is a min.}$$

Applications of integrals…

Use Fundamental Theorem of Calculus PT1 to evaluate the definite integral:

$$\int_1^2 x^2\,dx =$$

Use Fundamental Theorem of Calculus PT1 to evaluate the definite integral:

$$\int_1^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^2 = \left(\frac{1}{3}(2)^3\right) - \left(\frac{1}{3}(1)^3\right) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Using the Fundamental Theorem of Calculus PT2 (net change theorem): The rate of change of the altitude of a hot-air balloon is given by $r(t) = t^3 - 4t^2 + 6 \quad (0 \le t \le 8)$. Find the change in altitude of the balloon during the time when the altitude is decreasing.

First graph $r(t)$ *in calculator and find that this rate is negative for* $1.572 < t < 3.514$

Then, since $r(t)$ *is the derivative of altitude:*

$$\int_{1.572}^{3.514} a'(t)\,dt = \int_{1.572}^{3.514} \left(t^3 - 4t^2 + 6\right)dt = a(3.514) - a(1.572)$$

is the change in altitude $= -4.431\ (Math\ 9)$

Using the Fundamental Theorem of Calculus PT2 to find a y-value from another given derivative: If $f'(x) = x^2 - 5x$, *and* $f(1) = 2$ find $f(4)$.

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\int_1^4 \left(3x^2 - \frac{5}{2}x\right)dx = f(4) - f(1)$$

$$\left[x^3 - 5x^2\right]_1^4 = f(4) - 2$$

$$\left((4)^3 - 5(4)^2\right) - \left((1)^3 - 5(1)^2\right) = f(4) - 2$$

$$-12 = f(4) - 2, \quad f(4) = -10$$

Integral as inverse operation of derivative:

$$\frac{d}{dx}\left(\int_2^{3x^2} \left(t^3 - 4t\right)dt\right) =$$

$$\frac{d}{dx}\left(\int_{x^5}^{3x^2} \left(t^3 - 4t\right)dt\right) =$$

Integral as inverse operation of derivative:

$$\frac{d}{dx}\left(\int_a^{b(x)} f(t)\,dt\right) = f(b(x)) \bullet b'(x)\ [\text{chain rule}]$$

$$\frac{d}{dx}\left(\int_2^{3x^2} \left(t^3 - 4t\right)dt\right) = \left(\left(3x^2\right)^3 - 4\left(3x^2\right)\right) \bullet (6x)$$

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\,dt\right) = f(b(x)) \bullet b'(x) - f(a(x)) \bullet a'(x)$$

$$\frac{d}{dx}\left(\int_{x^5}^{3x^2} \left(t^3 - 4t\right)dt\right) = \left(\left(3x^2\right)^3 - 4\left(3x^2\right)\right) \bullet (6x) - \left(\left(x^5\right)^3 - 4\left(x^5\right)\right) \bullet \left(5x^4\right)$$

Average value of a function:
If $f(x) = x^2 - 5x$ find the average value of $f(x)$ *over* $[2,6]$

Average value of a function:

$$\textit{average value of } f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\frac{1}{6-2}\int_2^6 \left(x^2 - 5x\right)dx - \frac{1}{4}\left[\frac{1}{3}x^3 - \frac{5}{2}x^2\right]_2^6 = \frac{8}{3}$$

NOTE: This is different than 'average rate of change of $f(x)$*'*

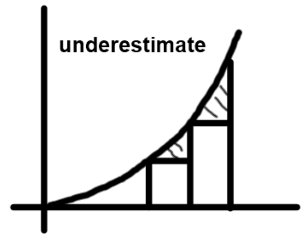
which would instead be: $\dfrac{f(6) - f(2)}{6-2}$

Riemann Sums (approximation of definite integral):

Use a left-endpoint Riemann Sum with two subintervals of equal length to approximate $\int_{2}^{2.4} x^2\,dx$

Does this estimate under- or over-estimate the value?

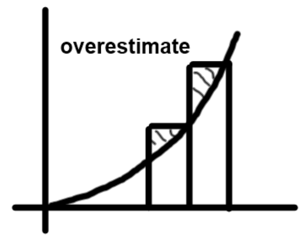
interval	x_i	$f(x_i)\bullet\Delta x = area$
$[2,2.2]$	2	$(2)^2\bullet 0.2 = 0.8$
$[2.2,2.4]$	2.2	$(2.2)^2\bullet 0.2 = \underline{0.968}$
		1.7744



Use a right-endpoint Riemann Sum with two subintervals of equal length to approximate $\int_{2}^{2.4} x^2\,dx$

Does this estimate under- or over-estimate the value?

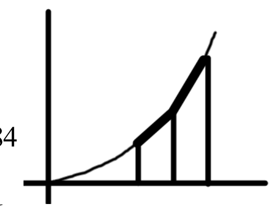
interval	x_i	$f(x_i)\bullet\Delta x = area$
$[2,2.2]$	2.2	$(2.2)^2\bullet 0.2 = 0.968$
$[2.2,2.4]$	2.4	$(2.4)^2\bullet 0.2 = \underline{1.152}$
		2.120



Use the trapezoidal rule with two subintervals of equal length to approximate $\int_{2}^{2.4} x^2\,dx$

$$A_{trapezoid} = \frac{1}{2}\left(f\left(x_{left}\right)+f\left(x_{right}\right)\right)\bullet\Delta x$$

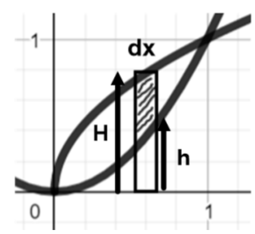
interval	$A_{trapezoid}$
$[2,2.2]$	$\frac{1}{2}\left((2)^2+(2.2)^2\right)\bullet 0.2 = 0.884$
$[2.2,2.4]$	$\frac{1}{2}\left((2.2)^2+(2.4)^2\right)\bullet 0.2 = \underline{1.06}$
	1.944



Area between curves (rectangular):

Find the area enclosed by $f(x)=x^2$ and $g(x)=\sqrt{x}$.

$$A=\int_a^b H\,dx-\int_a^b h\,dx$$
$$=\int_0^1 \sqrt{x}\,dx-\int_0^1 x^2\,dx$$
$$=\int_0^1\left(\sqrt{x}-x^2\right)dx = 0.333$$

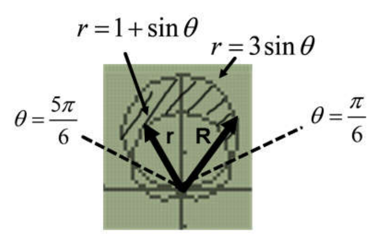


Area between curves (polar):

Find the area inside $r=3\sin\theta$ and outside $r=1+\sin\theta$.

intersections :

$$3\sin\theta = 1+\sin\theta$$
$$2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2}$$
$$\theta=\frac{\pi}{6},\ \theta=\frac{5\pi}{6}$$



$$A=\frac{1}{2}\int_\alpha^\beta R^2\,d\theta-\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$
$$=\frac{1}{2}\int_{\pi/6}^{5\pi/6}(3\sin\theta)^2\,d\theta-\frac{1}{2}\int_{\pi/6}^{5\pi/6}(1+\sin\theta)^2\,d\theta$$
$$=\frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[(3\sin\theta)^2-(1+\sin\theta)^2\right]d\theta = 3.142$$

Volumes:

Find the volume formed by rotating the area enclosed by $f(x) = x^2$ and $g(x) = \sqrt{x}$ around y-axis (disc method).

disc method ('perpendiscular')

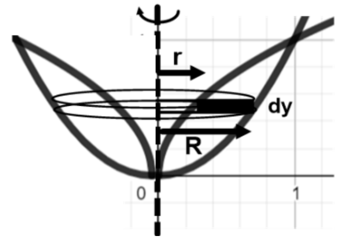
(rectangle ⊥ rotation axis)

$$V = \int_a^b \pi R^2 \, dh - \int_a^b \pi r^2 \, dh$$

$$= \int_a^b \pi R^2 \, dy - \int_a^b \pi r^2 \, dy$$

$$= \int_0^1 \pi \left(\sqrt{y}\right)^2 dy - \int_0^1 \pi \left(y^2\right)^2 dy$$

$$= \pi \int_0^1 \left(y - y^4\right) dy = 0.942$$



Find the volume formed by rotating the area enclosed by $f(x) = x^2$ and $g(x) = \sqrt{x}$ around y-axis (shell method).

shell method ('parashell')

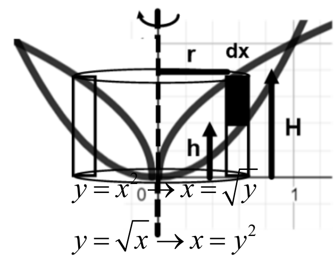
(rectangle || rotation axis)

$$V = \int_a^b 2\pi r H dr - \int_a^b 2\pi r h dr$$

$$= \int_a^b 2\pi r H dx - \int_a^b 2\pi r h dx$$

$$= \int_0^1 2\pi x \left(\sqrt{x}\right) dx - \int_0^1 2\pi x \left(x^2\right) dx$$

$$= 2\pi \int_0^1 \left(x\sqrt{x} - x^3\right) dx = 0.942$$



Find the volume formed by rotating the area enclosed by $f(x) = x^2$ and $g(x) = \sqrt{x}$ around the line $y = 1$.

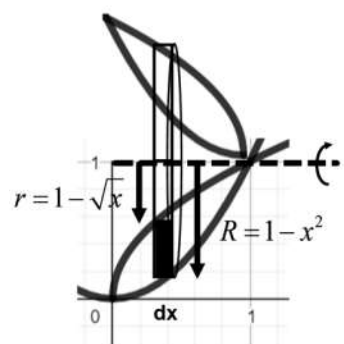
disc method ('perpendiscular')

(rectangle ⊥ rotation axis)

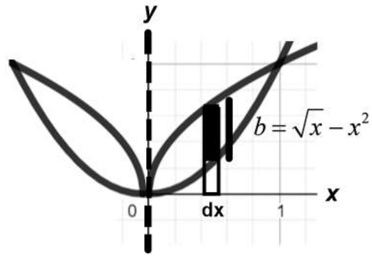
$$V = \int_a^b \pi R^2 \, dh - \int_a^b \pi r^2 \, dh$$

$$= \int_a^b \pi \left(1 - x^2\right)^2 dx - \int_a^b \pi \left(1 - \sqrt{x}\right)^2 dx$$

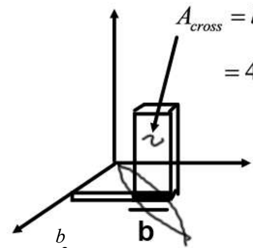
$$= \pi \int_0^1 \left(\left(1 - x^2\right)^2 - \left(1 - \sqrt{x}\right)^2\right) dx = 1.152$$



The region R enclosed by $f(x)=x^2$ and $g(x)=\sqrt{x}$ forms the base of a solid. For this solid, each cross section Perpendicular to the *x*-axis is a rectangle whose Height is 4 times the length of its base in region *R*. Find the volume of this solid.



$$A_{cross}=b(4b)=4b^2$$
$$=4\left(\sqrt{x}-x^2\right)^2$$



$$V=\int_a^b A_{cross}\,dx$$
$$=\int_0^1 4\left(\sqrt{x}-x^2\right)^2\,dx=0.514$$

<u>Arclength (rectangular)</u>:

If $f(x)=\frac{x^3}{6}+\frac{1}{2x}$ find the length of this curve for $\frac{1}{2}\le x\le 1.$

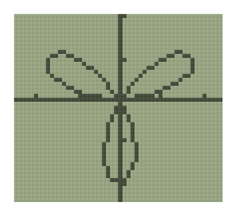
$$arclength=\int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$$
$$f(x)=\frac{1}{6}x^3+\frac{1}{2}x^{-1},\quad f'(x)=\frac{1}{2}x^2-\frac{1}{2}x^{-2}$$
$$=\int_{1/2}^{1}\sqrt{1+\left[\frac{1}{2}x^2-\frac{1}{2}x^{-2}\right]^2}\,dx=0.646$$

<u>Arclength (parametric)</u>:
Find the arclength of the curve $x=6t^2,\ y=2t^3$ over the interval $1\le t\le 4$.

$$arclength=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
$$x=6t^2,\quad \frac{dx}{dt}=12t,\quad y=2t^3,\quad \frac{dy}{dt}=6t^2$$
$$arclength=\int_1^4\sqrt{(12t)^2+\left(6t^2\right)^2}\,dt=156.525$$

<u>Arclength (polar)</u>:
Find the arclength of one petal of $r=2\sin(3\theta)$.

$$arclength=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$



one petal when $r=0$:

$2\sin(3\theta)=0\quad substitute:\phi=3\theta$

$2\sin\phi=0,\quad \sin\phi=0,\quad \phi=0,\pi$

back substitute: $\quad 3\theta=0\quad 3\theta=\pi$

$\theta=0,\quad \theta=\frac{\pi}{3}$

$$r=2\sin(3\theta),\quad \frac{dr}{d\theta}=6\cos(3\theta)$$
$$arclength=\int_0^{\pi/3}\sqrt{\left(2\sin(3\theta)\right)^2+\left(6\cos(3\theta)\right)^2}\,d\theta=4.455$$

Surface area of surface of revolution:

Find the area of the surface obtained by rotating The curve $y=x^3,\ 0\le x\le 2$ about the *x*-axis.

Surface area of surface of revolution:

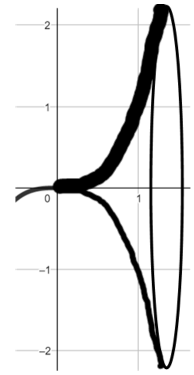
$$arclength=\int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$$

concept is: if you rotate each 'piece' of arc around the axis, this piece forms a 'strip' with surface area $= 2\pi(d\,'arclength')$, *so*:

$$surface\ area=\int_a^b 2\pi r\sqrt{1+\left[f'(x)\right]^2}\,dx \quad and\ r=f(x)$$

$$f(x)=x^3,\quad f'(x)=3x^2$$

$$surface\ area=\int_0^2 2\pi x^3\sqrt{1+\left[3x^2\right]^2}\,dx=203.044$$



Displacement vs. total distance:

The velocity of a particle is given by $\vec{v}(t)=\langle 3t^2-8t,\ 3t^2-12\rangle$. Find:

(a) The displacement of the particle from $t=1$ *to* $t=4$.

(b) The total distance traveled by the particle from $t=1$ *to* $t=4$.

Displacement vs. total distance:

$$(a)\ total\ displacement=\vec{r}(4)-\vec{r}(1)$$

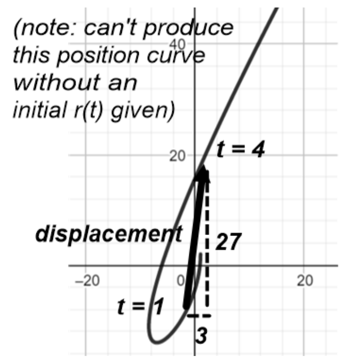
$$\int_1^4 \vec{r}'(t)\,dt=\int_1^4 \vec{v}(t)\,dt=\vec{r}(4)-\vec{r}(1)$$

$$\int_1^4 \langle 3t^2-8t,\ 3t^2-12\rangle\,dt=\vec{r}(4)-\vec{r}(1)$$

$$\left\langle \int_1^4 3t^2-8t\,dt,\ \int_1^4 3t^2-12\,dt\right\rangle=\vec{r}(4)-\vec{r}(1)$$

$$\left\langle \left[t^3-4t^2\right]_1^4,\ \left[t^3-12t\right]_1^4\right\rangle=\vec{r}(4)-\vec{r}(1)$$

$$\langle 3,\ 27\rangle=\vec{r}(4)-\vec{r}(1)$$



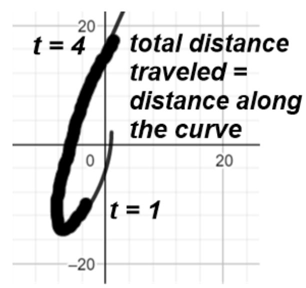
$$(b)\ total\ distance\ traveled=\int_a^b \left|\vec{v}(t)\right|dt$$

$$\left|\vec{v}(t)\right|=\sqrt{\left(3t^2-8t\right)^2+\left(3t^2-12\right)^2}$$

total distance traveled

$$=\int_1^4 \sqrt{\left(3t^2-8t\right)^2+\left(3t^2-12\right)^2}\ dt$$

41.655



Differential Equations…

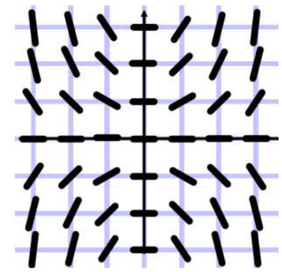
Slope fields:

Sketch a slope field for $\frac{dy}{dx} = \frac{1}{2}xy$

Slope fields:

plug various (x,y) into

$\frac{dy}{dx}$ to get slopes at pts



Which of the following could be a specific solution to the Differential equation with the given slope field:

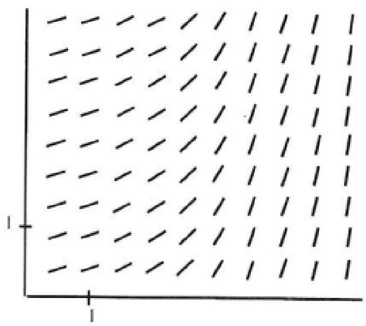
(A) $y = e^{-x}$

(B) $y = \sin x$

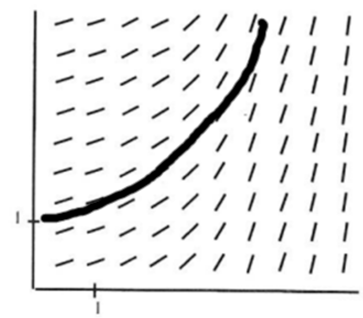
(C) $y = \sqrt{x}$

(D) $y = \ln x$

(E) $y = e^{0.5x}$



solution curves follow the direction in slope field (E) [this is an exponential curve shape]



Solving separable differential equations:

Find the particular solution for $\frac{dy}{dx} = 3x^2 y, \quad y(2) = 1$.

Solving separable differential equations:

$\textit{Separate the variables: } \frac{1}{y}dy = 3x^2 dx$

$\textit{integrate both sides: } \int \frac{1}{y}dy = \int 3x^2 dx$

$(\textit{general, implicit solution}): \ln|y| = x^3 + C$

$\textit{plug in initial condition: } \ln(1) = (2)^3 + C$

$\textit{solve for } C: \quad 0 = 8 + C, \quad C = -8$

$\textit{write the particular, implicit solution:}$

$\ln|y| = x^3 - 8$

$\textit{if needed, solve for } y\ (\textit{explicit solution}):$

$e^{\ln|y|} = e^{(x^3-8)}, \quad y = e^{(x^3-8)} = e^{-8}e^{x^3}$

Euler's method:

$f(x)$ is the solution to the differential equation $\frac{dy}{dx} = x^2 y, \quad y(1) = 2$. Use Euler's method with a step size of 0.1 to approximate $f(1.3)$.

(h can be negative)

Euler's method:

(x,y)	$y_{n+1} = y_n + h\left(\frac{dy}{dx}\right)$
$(1,2)$	$y = 2 + (0.1)\left((1)^2(2)\right) = 2.2$
$(1.1, 2.2)$	$y = 2.2 + (0.1)\left((1.1)^2(2.2)\right) = 2.4662$
$(1.2, 2.4662)$	$y = 2.4662 + (0.1)\left((1.2)^2(2.4662)\right) = 2.8213$
$(1.3, 2.8213)$	$f(1.3) \approx 2.8213$

Differential Equation Models:

Write the differential equation and solution equations:

Unrestricted population growth:

Radioactive Decay:

Logistic Model Growth:

Unrestricted population growth example:
A rabbit population with an initial size of 500 grows at
A rate proportional to its size. If there are1200 rabbits
At *t* = 10 days, when was the rabbit population 900?

Differential Equation Models:

Write the differential equation and solution equations:

Unrestricted population growth:

$$DE: \frac{dP}{dt} = kP \qquad solution: \ P = P_0 e^{kt}$$

Continuously compounded interest is same form:

$$\frac{dA}{dt} = kA \qquad A = Pe^{rt}$$

Radioactive Decay:

$$DE: \frac{dQ}{dt} = kQ \qquad solution: \ Q = Q_0 e^{kt}$$

$$(halflife = time\ for\ quantity\ to\ reduce\ by\ half)$$

Logistic Model Growth:

growth limited by environment

maximum population = carrying capacity, L

$$DE: \frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right) \qquad solution: \ P = \frac{L}{1 + Ce^{-kt}}$$

$$\left(population\ grows\ fastest\ when\ P = \frac{1}{2}L\right)$$

$$\frac{dP}{dt} = kP, \ solution: \ P = P_0 e^{kt}$$

$$P = 500e^{kt}, \ use\ P(10) = 1200:$$

$$1200 = 500e^{k(10)}, e^{10k} = \frac{1200}{500}, \ 10k = \ln\left(\frac{1200}{500}\right)$$

$$k = 0.087547, so\ P = 500e^{0.087547t}$$

$$Now, solve\ for\ t\ when\ P = 900:$$

$$900 = 500e^{0.087547t}, e^{0.087547t} = \frac{900}{500},$$

$$0.087547t = \ln\left(\frac{900}{500}\right), \ t = 6.714\ days$$

Logistic growth example:

The number of moose in a national park is modeled by the function $M(t)$ that satisfies the logistical differential equation $\frac{dM}{dt}=\frac{3}{5}M-\frac{3}{1000}M^2$ and $M(0)=50$.

(a) What is $\lim_{t\to\infty} M(t)$?

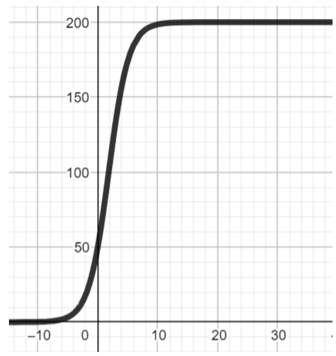
(b) What is the population of moose when the number of moose is growing most rapidly?

(c) At what time does max rate of growth occur?

(a) *logistic DE form*: $\frac{dP}{dt}=kP\left(1-\frac{P}{L}\right)$

factoring to get the 1: $\frac{dM}{dt}=\frac{3}{5}M\left(1-\frac{1}{200}M\right)$

$\frac{dM}{dt}=\frac{3}{5}M\left(1-\frac{M}{200}\right)$ *so carrying capacity* $=200$



From curve shape,

$\lim_{t\to\infty} M(t)=$ *carrying capacity*

$=200$

(b) *Fastest growth for logistic occurs when population is half the carrying capacity*:

When there are 100 *moose.*

(c) *DE form*: $\frac{dM}{dt}=\frac{3}{5}M\left(1-\frac{M}{200}\right)=kP\left(1-\frac{P}{L}\right)$

solution form: $M=\frac{L}{1+Ce^{-kt}}=\frac{200}{1+Ce^{-\frac{3}{5}t}}$

using initial condition: $M(0)=50$:

$50=\frac{200}{1+Ce^{-\frac{3}{5}(0)}}=\frac{200}{1+C},\ 1+C=\frac{200}{50},\ C=3$

final solution equation: $M=\frac{200}{1+3e^{-\frac{3}{5}t}}$

Now, solve for time when $M=100$:

$100=\frac{200}{1+3e^{-\frac{3}{5}t}},\ 1+3e^{-\frac{3}{5}t}=\frac{200}{100},\ 3e^{-\frac{3}{5}t}=1,$

$-\frac{3}{5}t=\ln\left(\frac{1}{3}\right),\quad t=1.831$ *years.*

Taylor Polynomials and Infinite Series…

Series convergence tests:
For each, state procedure (and conditions for use)
result for convergence, result for divergence…

nth-term test

Geometric series

p-series

Alternating series test

Alternating Series are absolutely convergent if…

Alternating Series are conditionally convergent if…

Series convergence tests:

nth term test

$\lim_{n\to\infty} a_n \neq 0 \; [\textit{diverges}]$

$(\textit{cannot be used to show convergence})$

Geometric series

$\textit{form}: \sum_{n=0}^{\infty} ar^n$

$|r| < 1 \quad [\textit{converges}]$

$|r| \geq 1 \quad [\textit{diverges}]$

p-series

$\textit{form}: \sum_{n=1}^{\infty} \frac{1}{n^p}$

$p > 1 \quad [\textit{converges}]$

$0 < p \leq 1 \quad [\textit{diverges}]$

Alternating series test

$\textit{form}: \sum_{n=1}^{\infty} (-1)^{n-1} a_n$

$1) \lim_{n\to\infty} a_n = 0 \quad \textit{and}$

$2) \; a_{n+1} \leq a_n \quad [\textit{converges}]$

$\textit{if either not met, inconclusive}$

$\sum_{n=1}^{\infty} \left|(-1)^{n-1} a_n\right| \textit{ converges}$

$\left(\textit{by theorem, } \sum_{n=1}^{\infty} (-1)^{n-1} a_n \textit{ also converges}\right)$

$\sum_{n=1}^{\infty} \left|(-1)^{n-1} a_n\right| \textit{ diverges but } \sum_{n=1}^{\infty} (-1)^{n-1} a_n \textit{ converges}$

Integral test

Root test

Ratio test

Direct Comparison

Limit Comparison

Integral test

$$form: \sum_{n=1}^{\infty} a_n \quad a_n = f(n) \quad f(n)\ terms\ positive\ and\ decreasing$$

$$evaluate \int_1^{\infty} f(x)\,dx$$

$$if\ integral\ converges, series\ converges$$

$$if\ integral\ diverges, series\ diverges$$

Root test

$$\sum_{n=1}^{\infty} \sqrt[n]{|a_n|} < 1 \qquad [converges]$$

$$\sum_{n=1}^{\infty} \sqrt[n]{|a_n|} > 1\ or\ \infty \qquad [diverges]$$

Ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \qquad [converges]$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1\ or\ \infty \qquad [diverges]$$

Direct Comparison

$$if\ 0 < a_{orig} \le a_{new}\ and\ \sum_{n=1}^{\infty} a_{new}\ converges, then\ \sum_{n=1}^{\infty} a_{orig}\ converges$$

$$f\ 0 < a_{new} \le a_{orig}\ and\ \sum_{n=1}^{\infty} a_{new}\ diverges, then\ \sum_{n=1}^{\infty} a_{orig}\ diverges$$

Limit Comparison

$$If\ \lim_{n\to\infty} \frac{a_{orig}}{a_{new}} > 0\ (a\ finite, positive\ number)$$

$$then\ series\ are\ 'linked'\ so...$$

$$If\ \sum_{n=1}^{\infty} a_{new}\ converges, then\ \sum_{n=1}^{\infty} a_{orig}\ converges$$

$$If\ \sum_{n=1}^{\infty} a_{new}\ diverges, then\ \sum_{n=1}^{\infty} a_{orig}\ diverges$$

Taylor Polynomials/Power Series…

Taylor Polynomial form:

Maclaurin means centered at…

Max Error (Lagrange Error)…

Memorized Power Series:

$e^x =$

$\sin x =$

$\cos x =$

Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \frac{(x-3)^n}{2^n}$$

The Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$.

Find a power series expansion for $\frac{x^2}{1-x^2}$

Taylor Polynomial form:

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin means centered at $x = 0$.

Max Error (Lagrange Error)…

$$max\ error = \frac{f^{(n)}(z)}{(n+1)!}(x-c)^{n+1}$$

$$where\ f^{(n)}(z)\ is\ \max\ value\ of\ derivative$$

Memorized Power Series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

Find the radius of convergence of

$$ratio\ test: \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+2}{2n+3}\frac{(x-3)^{n+1}}{2^{n+1}}\frac{2n+1}{n+1}\frac{2^n}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+2}{2n+3}\frac{(x-3)\cancel{(x-3)^n}}{2\cancel{2^n}}\frac{2n+1}{n+1}\frac{\cancel{2^n}}{\cancel{(x-3)^n}}\right|$$

$$= \lim_{n\to\infty}\frac{(n+2)(2n+1)}{2(2n+3)(n+1)}|x-3| = \lim_{n\to\infty}\frac{2n^2+\ldots\ldots}{4n^2+\ldots.}|x-3|$$

$$\frac{1}{2}|x-3| < 1,\quad |x-3| = 2,\quad radius\ of\ convergence = 2$$

$$You\ can\ do\ algebraic\ operations\ with\ power\ series:$$

$$let\ u = x^2$$

$$\frac{x^2}{1-x^2} = x^2\frac{1}{1-u} = x^2\sum_{n=0}^{\infty}u^n = x^2\sum_{n=0}^{\infty}(x^2)^n = x^2\sum_{n=0}^{\infty}x^{2n} = \sum_{n=0}^{\infty}x^2x^{2n} = \sum_{n=0}^{\infty}x^{2n+2}$$

$$= x^2 + x^4 + x^6 + x^8 + \ldots$$

The function f is defined by the power series

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots$$

Show that $1-\frac{1}{3!}$ approximates $f(1)$ with an error less than $\frac{1}{100}$

$$\max\ error \le \left|first\ neglected\ term\right|$$

$$\max\ error \le \left|\frac{x^4}{5!}\right|$$

$$\max\ error \le \left|\frac{(1)^4}{5!}\right| = 0.0083$$

$$0.0083<\frac{1}{100}$$

The Maclaurin series for the function f is given by $f(x)=\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n$. What is the value of $f(3)$?

$$f(3)=\sum_{n=0}^{\infty}\left(-\frac{3}{4}\right)^n \textit{ is a Geometric series with } r=-\frac{3}{4}$$

$$\textit{which converges to a sum of } \frac{a}{1-r}=\frac{1}{1-\left(-\frac{3}{4}\right)}=\frac{4}{7}$$

$$So\ f(3)=\frac{4}{7}.$$

Other things to know…

Notation forms for first derivatives:

$$y',\quad f'(x),\quad \frac{dy}{dx},\quad \frac{d}{dx}[y],\quad D_x(y)$$

Notation forms for higher-order derivatives:

$$y'',\quad f''(x),\quad \frac{d^2y}{dx^2}$$

$$y''',\quad f'''(x),\quad \frac{d^3y}{dx^3}$$

$$y^{(4)},\quad f^{(4)}(x),\quad \frac{d^{(4)}y}{dx^{(4)}}$$

… … …

$$y^{(n)},\quad f^{(n)}(x),\quad \frac{d^{(n)}y}{dx^{(4n)}}$$

Geometry Formulas:

$Circles:\quad area,\ A=\qquad circumference,\ C=$

$Triangles:\ A=\qquad (b\perp h)$

$Right-circular\ cylinders...$

$Surface\ Area=top/bottom+lateral\qquad Volume, V=$

$Surface\ Area=$

Notation forms for first derivatives:

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… … …

$$y^{(n)},\quad f^{(n)}(x),\quad \frac{d^{(n)}y}{dx^{(4n)}}$$

Geometry Formulas:

$Circles:\quad area,\ A=\pi r^2\qquad circumference,\ C=2\pi r$

$Triangles:\ A=\frac{1}{2}bh\qquad (b\perp h)$

$Right-circular\ cylinders...$

$Surface\ Area=top/bottom+lateral\qquad Volume, V=\pi r^2 h$

$Surface\ Area=2\left(\pi r^2\right)+2\pi rh$