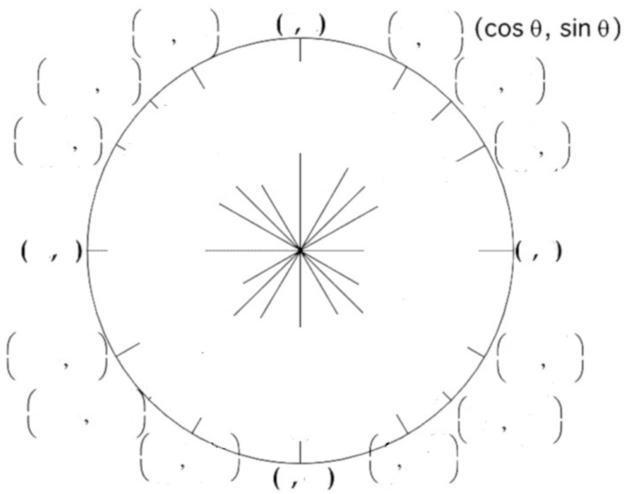
AP Calculus BC – Study Guide

Trigonometry…



Reciprocal identities:

$\sin x =$

$\cos x =$

$\tan x =$

$\csc x =$

$\sec x =$

$\cot x =$

Reciprocal identities:

$\tan x =$

$\cot x =$

Pythagorean identities:

$\sin^2 x + \cos^2 x =$

$1 + \tan^2 x =$

$1 + \cot^2 x =$

Power-reducing identities:

$\sin^2(x) =$

$\cos^2(x) =$

Double-angle identities:

$\sin(2x) =$

$\cos(2x) =$

Curve shapes (sketch)…

$f(x) = x^2$

$f(x) = x^3$

$f(x) = e^x$

$f(x) = \ln(x)$

$f(x) = \sqrt{x}$

$f(x) = \frac{1}{x}$

$f(x) = \sin(x)$

$f(x) = \cos(x)$

$f(x) = \tan(x)$

Conic sections…

Convert to standard form and sketch:

$$x^2 - 6x - 8y - 7 = 0$$

$$9x^2 + 4y^2 - 36x + 8y + 4 = 0$$

$$9x^2 - 4y^2 - 18x - 16y + 29 = 0$$

Limits and Continuity…

What must be true for $\lim_{x \to c} f(x)$ to exist?

What must be true for $f(x)$ to be continuous at *c*?

Evaluation tactics…(evaluate these limits):

$$\lim_{x \to 2} \frac{x-3}{x^2-7}$$

$$\lim_{x \to 5} \frac{x^2-25}{x-5}$$

$$\lim_{x \to 9} \frac{x^2-81}{\sqrt{x}-3}$$

What is L'Hopital's Rule?

Evaluate using L'Hopital's rule:

$$\lim_{x\to\infty}\frac{2x^2-x}{x^2+x}$$

$$\lim_{x\to\infty}e^{-x}\sqrt{x}$$

$$\lim_{x\to 0}\left(1+\frac{x}{2}\right)^{\cot x}$$

Special memorized limits:

$$\lim_{x\to 0}\frac{\sin x}{x}=$$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=$$

$$\lim_{x\to\infty}(1+x)^{1/x}=$$

Horizontal asymptotes occur when…

Vertical asymptotes occur when…

Derivatives…

Average rate of change of $f(x) =$
(from x = a to x = b)

Instantaneous rate of change of $f(x)$ at x is…

Limit definition of derivative, $f'(x) =$

Derivative shortcuts…

$$\frac{d}{dx}[c]=$$

$$\frac{d}{dx}\left[x^n\right]=$$

$$\frac{d}{dx}\left[e^x\right]=$$

$$\frac{d}{dx}\left[a^x\right]=$$

$$\frac{d}{dx}\left[\ln(x)\right]=$$

$$\frac{d}{dx}\left[\log_b(x)\right]=$$

$$\frac{d}{dx}\left[\sin(x)\right]=$$

$$\frac{d}{dx}\left[\cos(x)\right]=$$

$$\frac{d}{dx}\left[\tan(x)\right]=$$

$$\frac{d}{dx}\left[\tan(x)\right]=$$

$$\frac{d}{dx}\left[\sec(x)\right]=$$

$$\frac{d}{dx}\left[\csc(x)\right]=$$

$$\frac{d}{dx}\left[\cot(x)\right]=$$

$$\frac{d}{dx}\left[\sin^{-1}(x)\right]=$$

$$\frac{d}{dx}\left[\tan^{-1}(x)\right]=$$

$$\frac{d}{dx}\left[\sec^{-1}(x)\right]=$$

Antiderivative shortcuts…

$\int 0\ dx =$

$\int c\ dx =$

$\int x^n\ dx =$

$\int e^x\ dx =$

$\int e^{ax}\ dx =$

$\int a^x\ dx =$

$\int \frac{1}{x}\ dx =$

$\int \sin(x)\ dx =$

$\int \cos(x)\ dx =$

$\int \sec^2(x)\ dx =$

$\int \tan(x)\ dx =$

$\int \sec(x)\tan(x)\ dx =$

$\int \csc^2(x)\ dx =$

$\int \cot(x)\ dx =$

$\int \csc(x)\cot(x)\ dx =$

$\int \frac{1}{\sqrt{a^2 - x^2}} =$

$\int \frac{1}{a^2 + x^2} =$

$\int \frac{1}{x\sqrt{x^2 - a^2}} =$

Derivative properties/procedures…

$\frac{d}{dx}[cx]=$

$\frac{d}{dx}\left[f(x)\pm g(x)\right]=$

$\frac{d}{dx}\left[f(x)g(x)\right]=$ (product rule)

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=$ (quotient rule)

$\frac{d}{dx}\left[f\left(g(x)\right)\right]=$ (chain rule)

1) Implicit differentiation:

ex: Find $\frac{dy}{dx}$ *for* $xy^3+3x^2=4-y^5$

2) Logarithmic differentiation:

ex: Find $\frac{dy}{dx}$ *for* $y=x^{\left(5x^3+2x\right)}$

Integral properties/procedures…

$\int c\, f(x)\, dx =$

$\int \left[f(x) \pm g(x) \right] dx =$

$\int_b^a f(x)\, dx =$

1) <u>u-substitution</u> (integral version of chain rule)

ex: $\int x \cos\left(x^2\right) dx$

2) <u>by parts</u> (integral version of product rule)

ex: $\int x \ln(x)\, dx$

3) <u>trigonometric integrals</u>

ex: $\int \sin^3 x \cos^3 x\, dx$

4) <u>trigonometric substitution</u>

ex: $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$

5) <u>partial fraction expansion</u>

ex: $\int \frac{1}{x^2-5x+6}\,dx$

6) <u>complete the square to arctan form</u>

ex: $\int \frac{1}{x^2-4x+13}\,dx$

<u>Improper Integrals</u>:

$$\int_1^{\infty} \frac{1}{x^2}\,dx =$$

$$\int_1^{\infty} \frac{1}{x}\,dx =$$

Theorems…

What is the Intermediate Value Theorem?

What is the Mean Value Theorem?

What is the Squeeze Theorem?

Different representational forms of relationships…

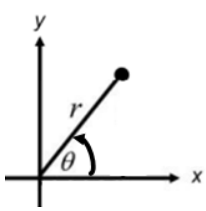
Rectangular: $y = f(x)$ *Parametric*: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

Convert to rectangular form and sketch:

$$\begin{cases} x = 2t \\ y = 4t + 3 \end{cases}$$

$$\begin{cases} x = 1 + 2\cos t \\ y = -2 + 3\sin t \end{cases}$$

Polar:



Formulas for converting polar - rectangular:

$x =$

$y =$

$x^2 + y^2 =$

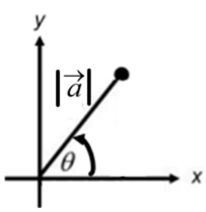
$\tan\theta =$

Convert to rectangular form and sketch:

$r = 8\sin\theta$

$\theta = \frac{5\pi}{6}$

<u>*Vectors*</u>:

Position: 

$\vec{a} = \langle a_x, a_y \rangle$, *a distance of* $|\vec{a}|$ *in direction* θ

Formulas for vector $\left(\vec{a} = \langle a_x, a_y \rangle\right)$:

$magnitude\ of\ a = |\vec{a}| =$

components :

$a_x =$

$a_y =$

Vectors are equal if… .

Find the vector from $(1,3)$ *to* $(9,4)$

Arithmetic operations and properties for different representations…

Multiplication by a constant…

$$3\langle 6,-3\rangle =$$

$$\lim_{x\to 2} 3f(x) =$$

$$\frac{d}{dx}\left[3f(x)\right] =$$

$$\int 3f(x)\,dx =$$

$$\sum_{n=1}^{\infty} 3a_n =$$

In general, multiplication of objects other than numbers is not straightforward (derivative of function multiplied requires product rule, integral requires integration by parts, multiplication of a vector by another vector not defined for this class, cannot multiply two series in summation form.)

Addition/subtraction…

$$\langle 8,1\rangle - \langle 2,5\rangle =$$

$$\lim_{x\to c}\left(x^3 - \frac{1}{x}\right) =$$

$$\frac{d}{dx}\left[x^3 - \sin(x)\right] =$$

$$\int \left(x^3 - \cos(x)\right) dx =$$

$$\sum_{n=1}^{\infty} \left(a_n \pm b_n\right) =$$

PEMDAS still applies…

$$2\langle 8,1\rangle - 3\langle 2,5\rangle =$$

For vectors things like limits, derivatives, or integrals apply separately to each term:

$$\lim_{t\to 4}\left(\left\langle t^2,\ t^3\right\rangle\right) =$$

$$\frac{d}{dx}\left[\left\langle t^2,\ t^3\right\rangle\right] =$$

$$\int \left\langle t^2,\ t^3\right\rangle dt =$$

For limits:

$$\lim_{x \to c}\left[f(x) \right]^n =$$

$$\lim_{x \to c}\left[\sqrt[n]{f(x)} \right] =$$

Derivatives in parametric form $\begin{cases} x = t^2 - 3t \\ y = \sin(t) \end{cases}$:

$$\frac{dy}{dx} =$$

$$\frac{d^2y}{dx^2} =$$

Derivatives in polar form $r = 4\sin(\theta)$:

$$\frac{dy}{dx} =$$

$\frac{dy}{dx}$ is the slope of the tangent line on the x-y plane.

Horizontal tangents occur when…

Vertical tangents occur when…

Intersections are always system solutions
(find the intersections):

$$\begin{cases} y = x^2 - 6 \\ y = -x \end{cases}$$

$$\begin{cases} r = 3(1 + \sin\theta) \\ r = 3(1 - \sin\theta) \end{cases}$$

$$\begin{cases} x_1 = 3\sin t \\ y_1 = 2\cos t \end{cases} \quad \begin{cases} x_2 = 3 + \cos t \\ y_2 = 1 + \sin t \end{cases} \quad 0 \le t < 2\pi$$

Applications of derivatives…

What do each of these tell us about f ?

$f(x)$ is

$f'(x)$ is

$f''(x)$ is

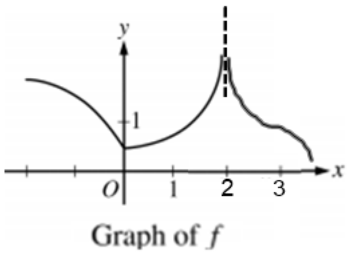
Critical points occur when…

Inflection points occur when…

Relative (local) max occurs when…

Relative (local) min occurs when…

Using a graph of the curve f



Where is f increasing? decreasing?

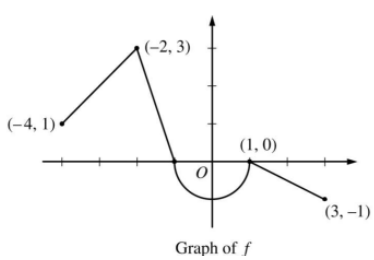
Where is f concave up? concave down?

Where is f continuous?

Where is f differentiable?

Where are the following for f?
- critical points
- relative maxima
-relative minima
- inflection points

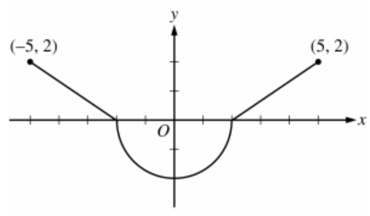
What are the absolute max/min over [-2,1]?



$$\int_{-4}^{3} f(x)\,dx =$$

$$\int_{3}^{-4} f(x)\,dx =$$

Using a graph of the derivative f'



Graph of f'

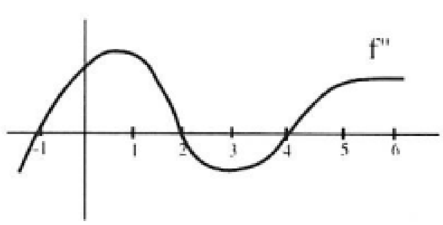
Where is f increasing? decreasing?

Where is f concave up? concave down?

Where are the following for f?
- critical points
- relative maxima
-relative minima
- inflection points

If $f(2)=1$, *then* $f(-5)=$

Using a graph of the concavity f''



Where is f concave up? concave down?

Where inflection points for f?

Tangent lines…

Rectangular:

$For\ (x-2)^2+(y+3)^2=4$

(a) *Write the equation of the tangent line at* $\left(1,-3+\sqrt{3}\right)$

(b) *Where does this curve have horizontal tangents?*

(c) *Where does this curve have vertical tangents?*

Parametric:

$$For\ \begin{cases} x=2t-\pi\sin t \\ y=2-\pi\cos t \end{cases}$$

(a) *Write the equation of the tangent line at* $t=\dfrac{2\pi}{3}$

(b) *Where does this curve have horizontal tangents?*

(c) *Where does this curve have vertical tangents?*

Polar:

For $r = 4\sin\theta$

(a) *Write the equation of the tangent line at* $\theta = \dfrac{\pi}{3}$

(b) *Where does this curve have horizontal tangents?*

(c) *Where does this curve have vertical tangents?*

Position, Velocity (speed), Acceleration…

In 1D:

An object moves in one direction with position x given by $x(t) = t^3 - 4t^2 + 3$.

(a) Find velocity as function of time.
(b) What acceleration as a function of time.
(c) What is the position of the particle at t = 2?
(d) What is the speed of the particle at t = 2?

An object is launched upward with an initial velocity of $30\ m/s$ from an initial height of $10\ m$ in gravity field with $a(t) = -9.8\ m/s^2$.

(a) Find velocity as a function of time.
(b) Find height as a function of time.
(c) At what time does the object reach maximum height and what is the max height?
(d) At what time does the object hit the ground?

In 2D (vector/parametric):

An object moves in the xy-plane with:

a velocity vector $\vec{v}(t) = \langle t^3 - 5t^2, \cos t \rangle$

…or could be given as parametric equations:

$$\begin{cases} x(t) = t^3 - 5t^2 \\ y(t) = \cos t \end{cases}$$

(a) Find the position vector if $\vec{x}(0) = \langle 3, 0 \rangle$.
(b) Find the acceleration vector.
(c) What is the position, velocity, and acceleration of the object at t = 2?
(d) What is the speed of the object at t = 2?

Related Rates Problems…

A 5-foot long ladder is leaning against a building. If the foot of the ladder is sliding away from the building at a rate of 2 ft/sec, how fast is the top of the ladder moving and in what direction when the foot of the ladder is 4 feet from the building?

Optimization Problems…

A cylindrical can (with circular base) is made with a material for the lateral side which costs \$3/cm^2, and a material for the top and bottom circular sides which costs \$5/cm^2. If the can must enclose a volume of $20\pi\ cm^3$ what should the radius and height be to minimize the material cost?

Applications of integrals…

Use Fundamental Theorem of Calculus PT1 to evaluate the definite integral:

$$\int_1^2 x^2\, dx =$$

Using the Fundamental Theorem of Calculus PT2 (net change theorem): The rate of change of the altitude of a hot-air balloon is given by $r(t) = t^3 - 4t^2 + 6 \quad (0 \le t \le 8)$. Find the change in altitude of the balloon during the time when the altitude is decreasing.

Using the Fundamental Theorem of Calculus PT2 to find a y-value from another given derivative: If $f'(x) = x^2 - 5x, \ and \ f(1) = 2$ find $f(4)$.

Integral as inverse operation of derivative: :

$$\frac{d}{dx}\left(\int_2^{3x^2} \left(t^3 - 4t\right) dt \right) =$$

$$\frac{d}{dx}\left(\int_{x^5}^{3x^2} \left(t^3 - 4t\right) dt \right) =$$

Average value of a function:
If $f(x) = x^2 - 5x$ find the average value of $f(x)\ over\ [2,6]$

Riemann Sums (approximation of definite integral):

Use a left-endpoint Riemann Sum with two subintervals of equal length to approximate $\int_{2}^{2.4} x^2\,dx$

Does this estimate under- or over-estimate the value?

Use a right-endpoint Riemann Sum with two subintervals of equal length to approximate $\int_{2}^{2.4} x^2\,dx$

Does this estimate under- or over-estimate the value?

Use the trapezoidal rule with two subintervals of equal length to approximate $\int_{2}^{2.4} x^2\,dx$

Area between curves (rectangular):

Find the area enclosed by $f(x)=x^2$ and $g(x)=\sqrt{x}$.

Area between curves (polar):

Find the area inside $r=3\sin\theta$ and outside $r=1+\sin\theta$.

Volumes:

Find the volume formed by rotating the area enclosed by $f(x)=x^2$ and $g(x)=\sqrt{x}$ around y-axis (disc method).

Find the volume formed by rotating the area enclosed

by $f(x)=x^2$ and $g(x)=\sqrt{x}$ around y-axis (shell method).

Find the volume formed by rotating the area enclosed by $f(x)=x^2$ and $g(x)=\sqrt{x}$ around the line $y=1$.

The region R enclosed by $f(x)=x^2$ and $g(x)=\sqrt{x}$ forms the base of a solid. For this solid, each cross section Perpendicular to the *x*-axis is a rectangle whose Height is 4 times the length of its base in region *R*. Find the volume of this solid.

Arclength (rectangular):

If $f(x)=\frac{x^3}{6}+\frac{1}{2x}$ find the length of this curve for $\frac{1}{2}\le x\le 1$.

Arclength (parametric):
Find the arclength of the curve $x=6t^2,\ y=2t^3$ over the interval $1\le t\le 4$.

Arclength (polar):
Find the arclength of one petal of $r=2\sin(3\theta)$.

Surface area of surface of revolution:

Find the area of the surface obtained by rotating
The curve $y = x^3,\ 0 \le x \le 2$ about the *x*-axis.

Displacement vs. total distance:

The velocity of a particle is given by
$\vec{v}(t) = \left\langle 3t^2 - 8t,\ 3t^2 - 12 \right\rangle$. Find:

(a) The displacement of the particle from $t = 1\ to\ t = 4$.

(b) The total distance traveled by the particle from $t = 1\ to\ t = 4$.

Differential Equations…

Slope fields:

Sketch a slope field for $\frac{dy}{dx} = \frac{1}{2}xy$

Which of the following could be a specific solution to the
Differential equation with the given slope field:

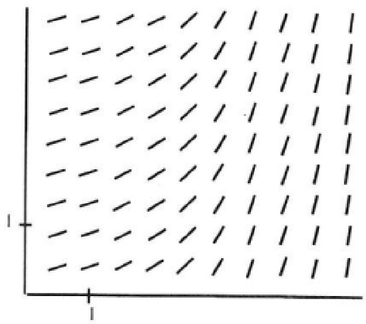
(A) $y = e^{-x}$

(B) $y = \sin x$

(C) $y = \sqrt{x}$

(D) $y = \ln x$

(E) $y = e^{0.5x}$



Solving separable differential equations:

Find the particular solution for $\frac{dy}{dx} = 3x^2 y, \quad y(2) = 1$.

Euler's method:

$f(x)$ is the solution to the differential equation

$\frac{dy}{dx} = x^2 y, \quad y(1) = 2$. Use Euler's method with a

step size of 0.1 to approximate $f(1.3)$.

(h can be negative)

Differential Equation Models:

Write the differential equation and solution equations:

Unrestricted population growth:

Radioactive Decay:

Logistic Model Growth:

Unrestricted population growth example:
A rabbit population with an initial size of 500 grows at
A rate proportional to its size. If there are1200 rabbits
At t = 10 days, when was the rabbit population 900?

Logistic growth example:

The number of moose in a national park is modeled by the function $M(t)$ that satisfies the logistical differential equation $\frac{dM}{dt}=\frac{3}{5}M-\frac{3}{1000}M^2 \text{ and } M(0)=50$.

(a) What is $\lim_{t\to\infty} M(t)$?

(b) What is the population of moose when the number of moose is growing most rapidly?

(c) At what time does max rate of growth occur?

Taylor Polynomials and Infinite Series…

Series convergence tests:
For each, state procedure (and conditions for use)
result for convergence, result for divergence…

nth-term test

Geometric series

p-series

Alternating series test

Alternating Series are absolutely convergent if…

Alternating Series are conditionally convergent if…

Integral test

Root test

Ratio test

Direct Comparison

Limit Comparison

Taylor Polynomials/Power Series…

<u>Taylor Polynomial form</u>:

Maclaurin means centered at…

Max Error (Lagrange Error)…

<u>Memorized Power Series</u>:

$e^x =$

$\sin x =$

$\cos x =$

Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \frac{(x-3)^n}{2^n}$$

The Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$.

Find a power series expansion for $\frac{x^2}{1-x^2}$

The function f is defined by the power series

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+...$$

Show that $1-\frac{1}{3!}$ approximates $f(1)$ with an error less than $\frac{1}{100}$

The Maclaurin series for the function f is given by $f(x)=\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n$. What is the value of $f(3)$?

Other things to know…

Notation forms for first derivatives:

$$y',\quad f'(x),\quad \frac{dy}{dx},\quad \frac{d}{dx}[y],\quad D_x(y)$$

Notation forms for higher-order derivatives:

$$y'',\quad f''(x),\quad \frac{d^2y}{dx^2}$$

$$y''',\quad f'''(x),\quad \frac{d^3y}{dx^3}$$

$$y^{(4)},\quad f^{(4)}(x),\quad \frac{d^{(4)}y}{dx^{(4)}}$$

... ...

$$y^{(n)},\quad f^{(n)}(x),\quad \frac{d^{(n)}y}{dx^{(4n)}}$$

Geometry Formulas:

$Circles:\quad area, A= \qquad circumference,\ C=$

$Triangles:\ A= \qquad (b\perp h)$

$Right-circular\ cylinders...$
$Surface\ Area=top/bottom+lateral \qquad Volume, V=$
$Surface\ Area=$