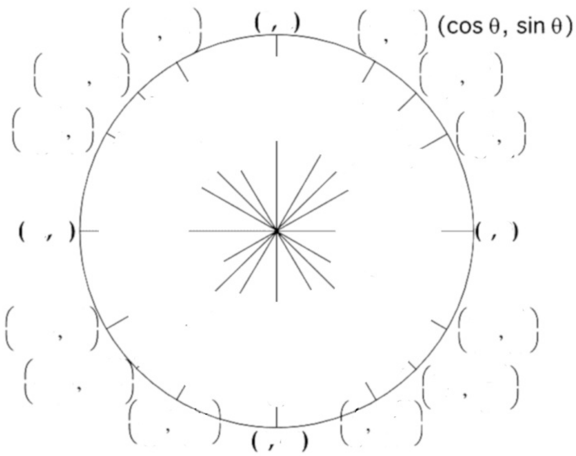
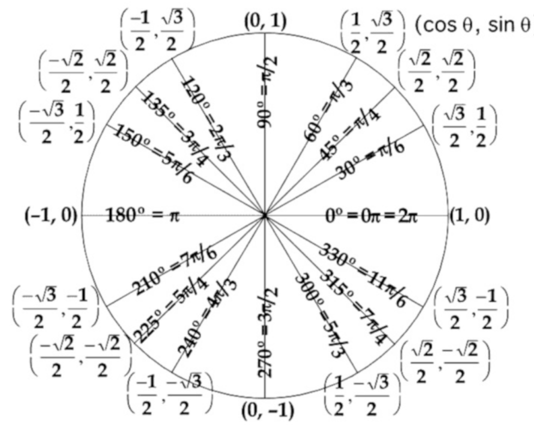
AP Calculus BC – Study Guide

Trigonometry…





Important Trig Identities:

$\sin x =$

$\cos x =$

$\tan x =$

$\cot x =$

$\sec x =$

$\csc x =$

$\sin^2 x + \cos^2 x =$
$(\textit{the other two forms})$

$\sin^2(x) =$
$\cos^2(x) =$

$\sin(2x) =$
$\cos(2x) =$

$$\sin x = \frac{1}{\csc x}$$

$$\cos x = \frac{1}{\sec x}$$

$$\tan x = \frac{1}{\cot x} = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

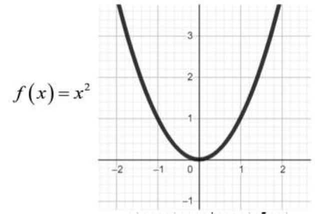
$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$
$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

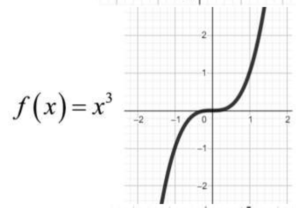
$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

Curve shapes (sketch)…

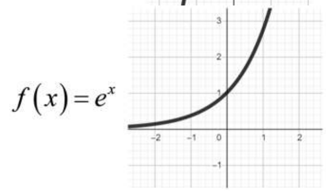
$f(x) = x^2$



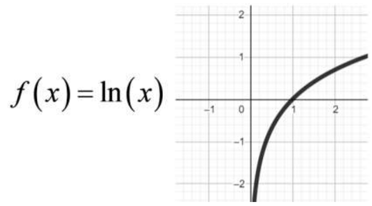
$f(x) = x^3$



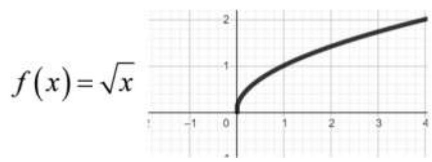
$f(x) = e^x$



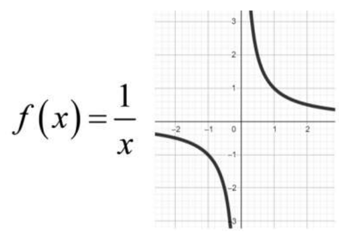
$f(x) = \ln(x)$



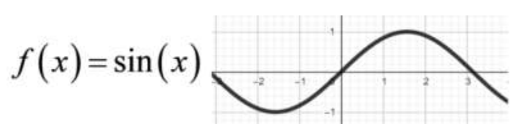
$f(x) = \sqrt{x}$



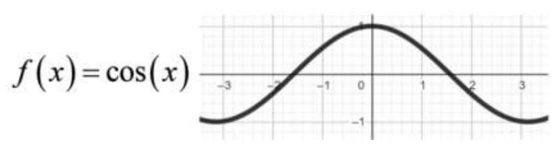
$f(x) = \frac{1}{x}$



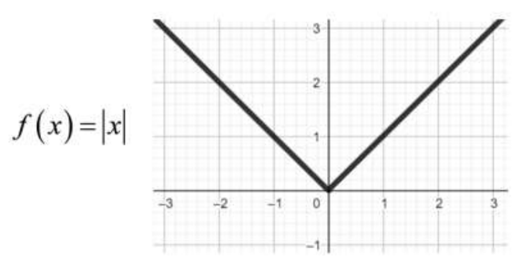
$f(x) = \sin(x)$



$f(x) = \cos(x)$



$f(x) = |x|$



Geometry Formulas:

$Circles:$

$area,\ A =$

$circumference,\ C =$

$Triangles:$

$A = \qquad (b \perp h)$

$Right-circular\ cylinders...$

$Surface\ Area = top/bottom + lateral$

$Surface\ Area =$

$Volume, V =$

Geometry Formulas:

$Circles:$

$area,\ A = \pi r^2$

$circumference,\ C = 2\pi r$

$Triangles:$

$A = \frac{1}{2}bh \qquad (b \perp h)$

$Right-circular\ cylinders...$

$Surface\ Area = top/bottom + lateral$

$Surface\ Area = 2\left(\pi r^2\right) + 2\pi rh$

$Volume, V = \pi r^2 h$

Limits and Continuity…

What must be true for $\lim_{x \to c} f(x)$ to exist?

$$\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$$
$$\textit{where } L \textit{ is a finite number}$$

What must be true for $f(x)$ to be continuous at *c*?

1) $f(c)$ *must exist*

2) $\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$
limit must exist

3) $f(c) = L$

Evaluation tactics…(evaluate these limits):

$\lim_{x \to 2} \frac{x-3}{x^2-7}$

Plug in:

$$\lim_{x \to 2} \frac{x-3}{x^2-7} = \frac{(2)-3}{(2)^2-7} = \frac{-1}{-3} = \frac{1}{3}$$

$\lim_{x \to 5} \frac{x^2-25}{x-5}$

Factor and cancel:

$$\lim_{x \to 5} \frac{x^2-25}{x-5} \left(\frac{0}{0}\right) = \lim_{x \to 5} \frac{(x-5)(x+5)}{x-5} = \lim_{x \to 5} (x+5) = 10$$

$\lim_{x \to 9} \frac{x^2-81}{\sqrt{x}-3}$

Rationalize:

$$\lim_{x \to 9} \frac{x^2-81}{\sqrt{x}-3} = \lim_{x \to 9} \frac{(x^2-81)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$$
$$= \lim_{x \to 9} \frac{(x-9)(x+9)(\sqrt{x}+3)}{x-9} = \lim_{x \to 9} (x+9)(\sqrt{x}+3) = (18)(6)$$

What is L'Hopital's Rule?

$$\textit{If } \lim_{x \to c} \frac{f(x)}{g(x)} \textit{ is indeterminant form } \frac{0}{0} \textit{ or } \frac{\pm\infty}{\pm\infty}$$
$$\textit{then } \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Evaluate using L'Hopital's rule:

$$\lim_{x\to\infty}\frac{2x^2-x}{x^2+x}$$

$$\lim_{x\to\infty} e^{-x}\sqrt{x}$$

$$\lim_{x\to 0}\left(1+\frac{x}{2}\right)^{\cot x}$$

$$\lim_{x\to\infty}\frac{2x^2-x}{x^2+x}\ \left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to\infty}\frac{4x-1}{2x+1}=\lim_{x\to\infty}\frac{4}{2}=2$$

$$\lim_{x\to\infty} e^{-x}\sqrt{x}\ \ (0\bullet\infty)$$

$$=\lim_{x\to\infty}\frac{\sqrt{x}}{e^x}\left(\frac{\infty}{\infty}\right)\ =\lim_{x\to\infty}\frac{\frac{1}{2}x^{-\frac{1}{2}}}{e^x}=\lim_{x\to\infty}\frac{1}{2\sqrt{x}e^x}=\frac{1}{\infty}=0$$

function raised to a function? Ln of both sides…

$$y=\lim_{x\to 0}\left(1+\frac{x}{2}\right)^{\cot x}$$

$$\ln(y)=\ln\left(\lim_{x\to 0}\left(1+\frac{x}{2}\right)^{\cot x}\right)=\lim_{x\to 0}\left[\ln\left(\left(1+\frac{x}{2}\right)^{\cot x}\right)\right]$$

$$\ln(y)=\lim_{x\to 0}\left[\cot(x)\ln\left(\left(1+\frac{x}{2}\right)\right)\right]\ \left(\frac{\cos 0}{\sin 0}\ln 1=\infty\bullet 0\right)$$

$$\ln(y)=\lim_{x\to 0}\left[\frac{\ln\left(\left(1+\frac{x}{2}\right)\right)}{\tan(x)}\right]\ \left(\frac{0}{0}\right)\ l'Hopital's\ rule...$$

$$\ln(y)=\lim_{x\to 0}\left[\frac{\frac{1}{1+\frac{x}{2}}\left(\frac{1}{2}\right)}{\sec^2(x)}\right]=\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{(\cos 0)^2}\right)}=\frac{1}{2}$$

$$y=e^{\frac{1}{2}}=\sqrt{e}$$

Special memorized limits:

$$\lim_{x\to 0}\frac{\sin x}{x}=$$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0$$

Horizontal asymptotes occur when…

Vertical asymptotes occur when…

Horizontal asymptotes occur when…
$\lim_{x\to\pm\infty} f(x)=any\ constant$

Vertical asymptotes occur when…
$\lim_{x\to c} f(x)=\pm\infty$

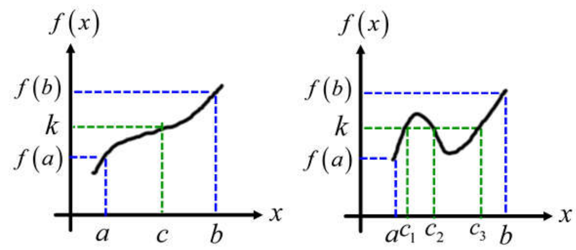
(whenever the function's y value is approaching infinity as x approaches a number – usually at uncancelled zeros in the denominator of rational functions)

Important Theorems…

What is the Intermediate Value Theorem?

Intermediate Value Theorem

If f is continuous on $[a,b]$, $f(a) \neq f(b)$, and k is any numberbetween $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c) = k$.

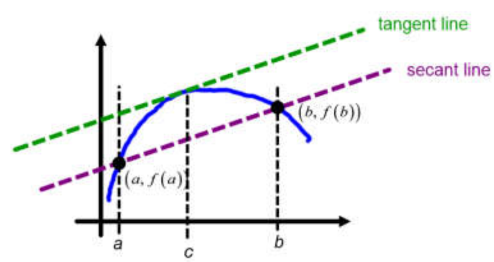


Note: This theorem doesn't provide a method for finding the value(s) *c*, and doesn't indicate the number of *c* values which map to *k*, it only guarantees the existence of at least one number *c* such that f(c) = k.

What is the Mean Value Theorem?

Mean Value Theorem

Let f be continuous on $[a,b]$, and differentiable on (a,b), then there exists a number c in (a,b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



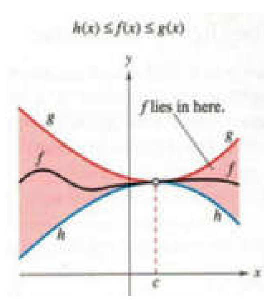
In other words, you can find a mean (average) rate of change across and interval, and there is some input value where the instantaneous rate of change equals the mean rate of change.

(Special case when slope = 0 is called 'Rolle's Theorem')

What is the Squeeze Theorem?

Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$ then $\lim_{x \to c} f(x)$ exists and is equal to L.



Derivatives…

Average rate of change of $f(x) =$
(from x = a to x = b)

Average rate of change of $f(x) = \dfrac{f(b) - f(a)}{b - a}$

Instantaneous rate of change of $f(x)$ at x is…

Instantaneous rate of change of $f(x)$ at x $f'(x)$

Limit definition of derivative, $f'(x) =$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$-or-$$

$$f'(a) = \lim_{x \to a} \frac{f(x+a) - f(a)}{x - a}$$

Notation forms for first derivatives:

Notation forms for first derivatives:

$$y', \quad f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}[y], \quad D_x(y)$$

Notation forms for higher-order derivatives:

Notation forms for higher-order derivatives:

$$y'', \quad f''(x), \quad \frac{d^2 y}{dx^2}$$

$$y''', \quad f'''(x), \quad \frac{d^3 y}{dx^3}$$

$$y^{(4)}, \quad f^{(4)}(x), \quad \frac{d^{(4)} y}{dx^{(4)}}$$

$$\ldots \quad \ldots \quad \ldots$$

$$y^{(n)}, \quad f^{(n)}(x), \quad \frac{d^{(n)} y}{dx^{(4n)}}$$

Derivative shortcuts…

$$\frac{d}{dx}[c]=$$ $$\frac{d}{dx}[c]=0$$

$$\frac{d}{dx}\left[x^n\right]=$$ $$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$$

$$\frac{d}{dx}\left[e^x\right]=$$ $$\frac{d}{dx}\left[e^x\right]=e^x$$

$$\frac{d}{dx}\left[a^x\right]=$$ $$\frac{d}{dx}\left[a^x\right]=a^x\ln(a)$$

$$\frac{d}{dx}\left[\ln(x)\right]=$$ $$\frac{d}{dx}\left[\ln(x)\right]=\frac{1}{x}$$

$$\frac{d}{dx}\left[\log_b(x)\right]=$$ $$\frac{d}{dx}\left[\log_b(x)\right]=\frac{1}{x\ln(b)}$$

$$\frac{d}{dx}\left[\sin(x)\right]=$$ $$\frac{d}{dx}\left[\sin(x)\right]=\cos(x)$$

$$\frac{d}{dx}\left[\cos(x)\right]=$$ $$\frac{d}{dx}\left[\cos(x)\right]=-\sin(x)$$

$$\frac{d}{dx}\left[\tan(x)\right]=$$ $$\frac{d}{dx}\left[\tan(x)\right]=\sec^2(x)$$

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$$\frac{d}{dx}\left[\sec(x)\right]=$$ $$\frac{d}{dx}\left[\sec(x)\right]=\sec(x)\tan(x)$$

$$\frac{d}{dx}\left[\csc(x)\right]=$$ $$\frac{d}{dx}\left[\csc(x)\right]=-\csc(x)\cot(x)$$

$$\frac{d}{dx}\left[\cot(x)\right]=$$ $$\frac{d}{dx}\left[\cot(x)\right]=-\csc^2(x)$$

$$\frac{d}{dx}\left[\sin^{-1}(x)\right]=$$ $$\frac{d}{dx}\left[\sin^{-1}(x)\right]=\frac{1}{\sqrt{1-x^2}}\ \left(\frac{d}{dx}\left[\cos^{-1}(x)\right]=\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\frac{d}{dx}\left[\tan^{-1}(x)\right]=$$ $$\frac{d}{dx}\left[\tan^{-1}(x)\right]=\frac{1}{1+x^2}\ \left(\frac{d}{dx}\left[\cot^{-1}(x)\right]=\frac{-1}{1+x^2}\right)$$

$$\frac{d}{dx}\left[\sec^{-1}(x)\right]=$$ $$\frac{d}{dx}\left[\sec^{-1}(x)\right]=\frac{1}{|x|\sqrt{x^2-1}}\ \left(\frac{d}{dx}\left[\csc^{-1}(x)\right]=\frac{-1}{|x|\sqrt{x^2-1}}\right)$$

Antiderivative shortcuts…

$\int 0\ dx =$	$\int 0\ dx = C$
$\int c\ dx =$	$\int c\ dx = cx + C$
$\int x^n\ dx =$	$\int x^n\ dx =$
$\int e^x\ dx =$	$\int e^x\ dx = e^x + C$
$\int e^{ax}\ dx =$	$\int e^{ax}\ dx = \frac{e^{ax}}{a} + C$
$\int a^x\ dx =$	$\int a^x\ dx = \frac{a^x}{\ln(a)} + C$
$\int \frac{1}{x}\ dx =$	$\int \frac{1}{x}\ dx = \ln\lvert x\rvert + C$
$\int \sin(x)\ dx =$	$\int \sin(x)\ dx = -\cos(x) + C$
$\int \cos(x)\ dx =$	$\int \cos(x)\ dx = \sin(x) + C$
$\int \sec^2(x)\ dx =$	$\int \sec^2(x)\ dx = \tan(x) + C$
$\int \csc^2(x)\ dx =$	$\int \csc^2(x)\ dx = -\cot(x) + C$
$\int \tan(x)\ dx =$	$\int \tan(x)\ dx = \ln\lvert\sec(x)\rvert + C = -\ln\lvert\cos(x)\rvert + C$
$\int \cot(x)\ dx =$	$\int \cot(x)\ dx = -\ln\lvert\csc(x)\rvert + C = \ln\lvert\sin(x)\rvert + C$
$\int \sec(x)\tan(x)\ dx =$	$\int \sec(x)\tan(x)\ dx = \sec(x) + C$
$\int \csc(x)\cot(x)\ dx =$	$\int \csc(x)\cot(x)\ dx = -\csc(x) + C$
$\int \sec(x)\ dx =$	$\int \sec(x)\ dx = \ln\lvert\sec(x) + \tan(x)\rvert + C$
$\int \csc(x)\ dx =$	$\int \csc(x)\ dx = \ln\lvert\csc(x) - \cot(x)\rvert + C$
$\int \frac{1}{\sqrt{a^2 - u^2}}\ du =$	$\int \frac{1}{\sqrt{a^2 - u^2}}\ du = \sin^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{1}{a^2 + u^2}\ du =$	$\int \frac{1}{a^2 + u^2}\ du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{1}{\lvert u\rvert\sqrt{u^2 - a^2}}\ du =$	$\int \frac{1}{\lvert u\rvert\sqrt{u^2 - a^2}}\ du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$

Derivative properties/procedures…

$\frac{d}{dx}[cx]=$

$\frac{d}{dx}[cx]=c\frac{d}{dx}[x]$ (constants can be moved out)

$\frac{d}{dx}\left[f(x)\pm g(x)\right]=$

$\frac{d}{dx}\left[f(x)\pm g(x)\right]=\frac{d}{dx}\left[f(x)\right]\pm\frac{d}{dx}\left[g(x)\right]$

(derivative of each term separately)

$\frac{d}{dx}\left[f(x)g(x)\right]=$ (product rule)

$\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+g(x)f'(x)$

(1^{st} times deriv. of 2^{nd} plus 2^{nd} times deriv. of 1st)

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=$ (quotient rule)

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}$

(low-dhigh minus high-dlow over low squared)

$\frac{d}{dx}\left[f\left(g(x)\right)\right]=$ (chain rule)

$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)\bullet g'(x)$

(deriv. of outside (with same inside) times deriv. of inside)

1) <u>Implicit differentiation</u>:

ex: Find $\frac{dy}{dx}$ $for\ xy^3+3x^2=4-y^5$

1) <u>Implicit differentiation</u>:

$$x\frac{d}{dx}\left[y^3\right]+y^3\frac{d}{dx}[x]+\frac{d}{dx}\left[3x^2\right]=\frac{d}{dx}[4]-\frac{d}{dx}\left[y^5\right]$$

$$x\left(3y^2\frac{dy}{dx}\right)+y^3(1)+6x=0-5y^4\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(3xy^2+5y^4\right)=-6x-y^3$$

$$\frac{dy}{dx}=\frac{-6x-y^3}{3xy^2+5y^4}$$

2) <u>Logarithmic differentiation</u>:

ex: Find $\frac{dy}{dx}$ $for\ \ y=x^{\left(5x^3+2x\right)}$

2) <u>Logarithmic differentiation</u>:

$$\ln(y)=\ln\left(x^{5x^3+2x}\right)$$

$$\ln(y)=\left(5x^3+2x\right)\ln(x)$$

$$\frac{d}{dx}\left[\ln(y)\right]=\frac{d}{dx}\left[\left(5x^3+2x\right)\ln(x)\right]$$

$$\frac{d}{dx}\left[\ln(y)\right]=\left(5x^3+2x\right)\frac{d}{dx}\left[\ln(x)\right]+\ln(x)\frac{d}{dx}\left[\left(5x^3+2x\right)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\left(5x^3+2x\right)\frac{1}{x}+\ln(x)\left(15x^2+2\right)$$

$$\frac{dy}{dx}=\left[\left(5x^3+2x\right)\frac{1}{x}+\ln(x)\left(15x^2+2\right)\right]y$$

$$\frac{dy}{dx}=\left[\left(5x^3+2x\right)\frac{1}{x}+\ln(x)\left(15x^2+2\right)\right]x^{\left(5x^3+2x\right)}$$

Integral properties/procedures…

$\int c\ f(x)\,dx =$

$\int c\ f(x)\,dx = c\int f(x)\,dx$ (constants can be moved out)

$\int [f(x) \pm g(x)]\,dx =$

$\int [f(x) \pm g(x)]\,dx = \int [f(x)]\,dx \pm \int [g(x)]\,dx$
(can split into separate integrals for each term)

$\int_b^a f(x)\,dx =$

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

1) <u>u-substitution</u> (integral version of chain rule)

ex: $\int x\cos(x^2)\,dx$

1) <u>u-substitution</u> (integral version of chain rule)

$\int x\cos(x^2)\,dx \quad u = x^2$

$\frac{du}{dx} = 2x, \quad du = 2xdx, \quad xdx = \frac{1}{2}du$

substitute into original integral :

$\int \cos(u)\frac{1}{2}du = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u) = \frac{1}{2}\sin(x^2) + C$

2) <u>by parts</u> (integral version of product rule)

ex: $\int x\ln(x)\,dx$

2) <u>by parts</u> (integral version of product rule)

$\int x\ln(x)\,dx \quad u = \ln(x) \qquad dv = x\,dx$

$\frac{du}{dx} = \frac{1}{x} \qquad \int dv = \int x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

substitute into pattern :

$uv - \int v\,du = (\ln(x))\left(\frac{1}{2}x^2\right) - \int \frac{1}{2}x^2\frac{1}{x}dx$

$= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int xdx$

$= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$

3) <u>trigonometric integrals</u>

ex: $\int \sin^3 x\cos^3 x\,dx$

3) <u>trigonometric integrals</u>

$\int \sin^3 x\cos^3\,dx \quad$ (*split off something to form du*)

$\int \sin^3 x\cos^2 x\cos xdx$

$\int \sin^3 x(1-\sin^2 x)\cos xdx$

$\int (\sin^3 x - \sin^5 x)\cos xdx$

$\int \sin^3 x\cos xdx - \int \sin^5 x\cos xdx$

$u = \sin x, \frac{du}{dx} = \cos x, \quad \cos xdx = du$

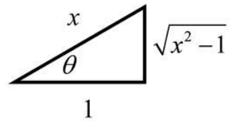
$\int u^3du - \int u^5du$

$\frac{1}{4}u^4 - \frac{1}{6}u^6 + C = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

4) <u>trigonometric substitution</u>

ex: $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$

4) <u>trigonometric substitution</u>



$$\cos\theta = \frac{1}{x} \qquad \tan\theta = \frac{\sqrt{x^2-1}}{1}$$

$$x = \frac{1}{\cos\theta} = \sec\theta \qquad \sqrt{x^2-1} = \tan\theta$$

$$\frac{dx}{d\theta} = \sec\theta\tan\theta \qquad dx = \sec\theta\tan\theta d\theta$$

5) <u>partial fraction expansion</u>

ex: $\int \frac{1}{x^2-5x+6}\,dx$

5) <u>partial fraction expansion</u>

$$\int \frac{1}{x^2-5x+6}\,dx \qquad \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$\int \frac{1}{(x-3)(x-2)}\,dx \qquad A(x-2)+B(x-3)=1$$

$$Ax-2A+Bx-3B=1$$

$$(A+B)x+(-2A-3B)=(0)x+(1)$$

$$system: \begin{cases} A+B=0 \\ -2A-3B=1 \end{cases} \qquad A=1,\ B=-1$$

$$1\int \frac{1}{x-3}\,dx - 1\int \frac{1}{x-2}\,dx$$

$$\ln|x-3| - \ln|x-2| + C = \ln\left|\frac{x-3}{x-2}\right| + C$$

6) <u>complete the square to arctan form</u>

ex: $\int \frac{1}{x^2-4x+13}\,dx$

6) <u>complete the square to arctan form</u>

$$\int \frac{1}{x^2-4x+13}\,dx \qquad x^2-4x+\underline{\underline{4}}+13-\underline{\underline{4}}$$

$$(x-2)^2+9$$

$$\int \frac{1}{(x-2)^2+9}\,dx \quad now\, u-sub:\ u=x-2,\ \frac{du}{dx}=1,\ du=dx$$

$$\int \frac{1}{u^2+3^2}\,dx$$

$$\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)+C = \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+C$$

<u>Improper Integrals</u>:

$$\int_1^\infty \frac{1}{x^2}\,dx =$$

$$\int_1^\infty \frac{1}{x}\,dx =$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{b}-\left(-\frac{1}{1}\right)\right] = -\frac{1}{\infty}+1 = 0+1 = 1$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|b| - (\ln|1|)\right] = \infty - 0 = \infty$$

(integral may converge to a number, or diverge)

Applications of derivatives…

<u>What do each of these tell us about</u> f ?

$f(x)$ is

$f'(x)$ is

$f''(x)$ is

Critical points occur when…

Inflection points occur when…

Relative (local) max occurs when…

Relative (local) min occurs when…

<u>What do each of these tell us about</u> f ?

$f(x)$ *is the* $y-$*value at* x

$f'(x)$ *is the instantaneous rate of change ('slope') at* x

$f'(x)>0$ f *is increasing*

$f'(x)<0$ f *is decreasing*

$f''(x)$ *is the concavity ('curvature') at* x

$f''(x)>0$ f *is concave up*

$f''(x)<0$ f *is concave down*

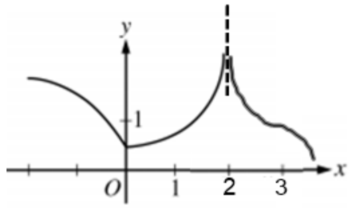
Critical points occur when $f'(x)=0$ *or DNE*
and the sign of $f'(x)$ changes.

Inflection points occur when $f''(x)=0$ *or DNE*
and the sign of $f''(x)$ changes.

Relative (local) max occurs when $f'(x)=0$ *or DNE*
and the sign of $f'(x)$ goes from + to -

Relative (local) min occurs when $f'(x)=0$ *or DNE*
and the sign of $f'(x)$ goes from - to +

Using a graph of the curve f



Graph of f

Where is f increasing? decreasing?

Where is f concave up? concave down?

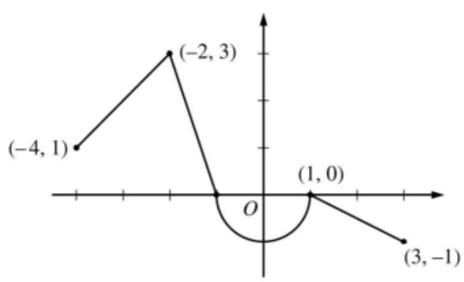
Where is f continuous?

Where is f differentiable?

Where are the following for f ?

- critical points
- relative maxima
-relative minima
- inflection points

What are the absolute max/min over [-2,1]?

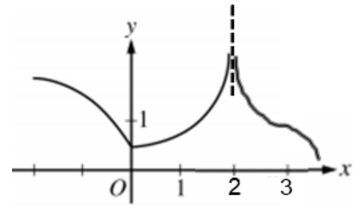


Graph of f

$$\int_{-4}^{3} f(x)\,dx =$$

$$\int_{3}^{-4} f(x)\,dx =$$

Using a graph of the curve f



Graph of f

f *is increasing over* $(0,2)$ [f *going up*]
decreasing over $(-2,0)\cup(2,4)$ [f *going down*]

f *is concave up over* $(0,2)\cup(2,3)$
concave down over $(-2,0)\cup(3,4)$

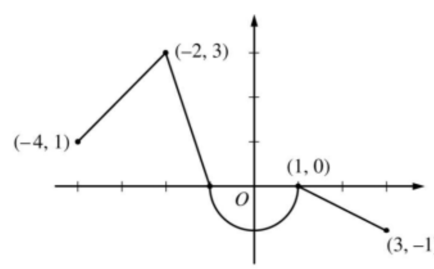
f *is continuous over* $(-2,2)\cup(2,4)$

f *is differentiable over* $(-2,0)\cup(0,2)\cup(2,4)$

Where are the following for f ?

critical points at (-2,2), (0,0.5), (2.5,1)
no relative maxima
relative minima at (0,0.5)
inflection points at (0,0.5), (3,1)

What are the absolute max/min over [-2,1]?
Absolute min at (0,0.5), absolute max at (-2,2)

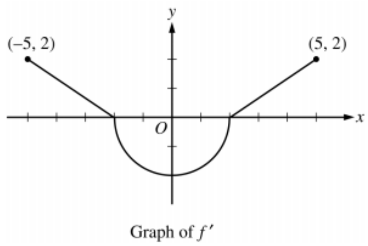


Graph of f

$$\int_{-4}^{3} f(x)\,dx = areas = 4+2-\frac{\pi}{2}-1 = 5-\frac{\pi}{2}$$

$$\int_{3}^{-4} f(x)\,dx = -\int_{-4}^{3} f(x)\,dx = -5+\frac{\pi}{2}$$

Using a graph of the derivative f'



Where is f increasing? decreasing?

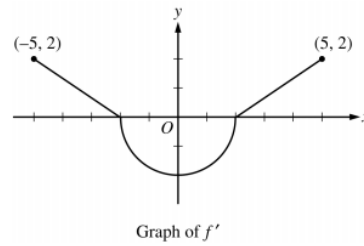
Where is f concave up? concave down?

Where are the following for f ?

- critical points
- relative maxima
-relative minima
- inflection points

$If\ f(2)=1,\ then\ f(-5)=$

Using a graph of the derivative f'



f *is increasing over* $(-5,-2)\cup(2,5)\ [f'>0]$

decreasing over $(-2,2)\ [f'<0]$

f *is concave up over* $(0,5)\quad [f'\ going\ up]$

concave down over $(-5,0)\ [f'\ going\ down]$

Where are the following for f ?

critical points at x=-2, x=2 [f' = 0]
relative maxima at x = 2 [f' from – to +]
relative minima at x = -2 [f' from + to -]
inflection point at x =0 [f' graph changing direction]

We can use the Net Change Theorem

(part of the Fundamental Theorem of Calculus):

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

evaluate definite integral by plugging limits into antiderivative

This also means an integral of a derivative of something is equal to the accumulation (net change) in the value this is a derivative of :

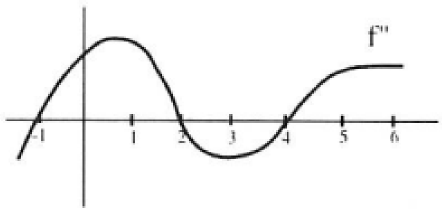
$$\int_a^b f'(x)\,dx = f(b)-f(a)$$

Pick one limit to be what you have and the other what you need:

$$\int_{-5}^{2} f'(x)\,dx = f(2)-f(-5) \text{ and evaluate integral using areas}$$

$$3-\frac{1}{2}\pi(2)^2 = 1-f(-5), \qquad f(-5)=1-3+2\pi = 2\pi-2$$

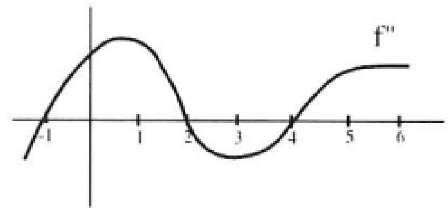
Using a graph of the concavity f''



Where is f concave up? concave down?

Where inflection points for f ?

Using a graph of the concavity f''



f *is concave up over* $(-1,2)\cup(4,6)\quad[f''>0]$

concave down over $(-2,-1)\cup(2,4)\,[f''<0]$

Where inflection points for f ?

at x =-1, x=2, x=4 [f'' = 0 and sign is changing]

Tangent lines…

<u>Rectangular</u>:

For $(x-2)^2+(y+3)^2=4$

(a) *Write the equation of the tangent line at* $\left(1,-3+\sqrt{3}\right)$

(b) *Where does this curve have horizontal tangents?*

(c) *Where does this curve have vertical tangents?*

For $(x-2)^2+(y+3)^2=4$

(a) *Write the equation of the* tan *gent line at* $\left(1,-3+\sqrt{3}\right)$

$m=\dfrac{dy}{dx}$ [*use implicit differentiation if needed*]:

$$2(x-2)(1)+2(y+3)\left(\frac{dy}{dx}\right)=0,\quad \frac{dy}{dx}=\frac{-x+2}{y+3}=\frac{-(1)+2}{\left(-3+\sqrt{3}\right)+3}=\frac{1}{\sqrt{3}}$$

$$\left(y-\left(-3+\sqrt{3}\right)\right)=\frac{1}{\sqrt{3}}(x-1)$$

(b) *Where does this curve have horizontal tangents?*

where $\dfrac{dy}{dx}=0$ $(numerator=0)$, $-x+2=0$, *at* $x=2$ $(2\ points)$

(c) *Where does this curve have vertical tangents?*

where $\dfrac{dy}{dx}=DNE$ $(denominator=0)$, $y+3=0$, *at* $y=-3$ $(2\ points)$

Position, Velocity (speed), Acceleration…

In 1D:

An object moves in one direction with position x given by $x(t)=t^3-4t^2+3$.

(a) Find velocity as function of time.
(b) What acceleration as a function of time.
(c) What is the position of the particle at t = 2?
(d) What is the speed of the particle at t = 2?

In 1D:

$x(t)=t^3-4t^2+3$

$(a)\quad v(t)=x'(t)=3t^2-8t$

$(b)\quad a(t)=v'(t)=6t-8$

$(c)\quad x(2)=(2)^3-4(2)^2+3=-5\quad (\textit{include units if given in problem})$

$(d)\quad speed=|v(2)|=|3(2)^2-8(2)|=|-4|=4$

An object is launched upward with an initial velocity of $30\ m/s$ from an initial height of $10\ m$ in gravity field with $a(t)=-9.8\,m/s^2$.

(a) Find velocity as a function of time.
(b) Find height as a function of time.
(c) At what time does the object reach maximum height and what is the max height?
(d) At what time does the object hit the ground?

$a(t)=-9.8$

$(a)\quad v(t)=\int a(t)\,dt=\int(-9.8)\,dt=-9.8t+C_1$

$v(0)=30,\ so\ 30=-9.8(0)+C_1,\ C_1=30$

$v(t)=-9.8t+30$

$(b)\quad x(t)=\int v(t)\,dt=\int(-9.8t+30)\,dt=-4.9t^2+30t+C_2$

$x(0)=10,\ so\ 10=-4.9(0)^2+30(0)+C_2,\ C_2=10$

$x(t)=-4.9t^2+30t+10$

$(c)\quad Max\ height\ when\ v=0:\ -9.8t+30=0,\ t=3.06122\text{ sec}$

$x(3.06122)=55.91837\ m$

$(d)\quad On\ ground\ when\ x=0:\ -4.9t^2+30t+10=0$

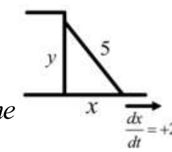
$$at\ t=\frac{-30\pm\sqrt{(30)^2-4(-4.9)(10)}}{2(-4.9)}=\cancel{-0.3169},\ 6.439\text{ sec}$$

Related Rates Problems…

A 5-foot long ladder is leaning against a building. If the foot of the ladder is sliding away from the building at a rate of 2 ft/sec, how fast is the top of the ladder moving and in what direction when the foot of the ladder is 4 feet from the building?

Draw a picture and assign variables to things which vary then find equations which relate the variables:

$x^2 + y^2 = 5^2$



Anything changing is a derivative with respect to time (+ if value is increasing)

$\frac{dx}{dt} = +2$

At this snapshot in time, variables have 'snaphot' values:

$(4)^2 + y^2 = 5^2, \quad y = 3$

Differentiate implicitly WRT time, plug in values, and solve:

$x^2 + y^2 = 25$

$\frac{d}{dt}\left[x^2\right] + \frac{d}{dt}\left[y^2\right] = \frac{d}{dt}[25]$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2(4)(2) + 2(4)\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{16}{8} = -2 \ ft/\sec$

negative b/c top of ladder is moving so y is decreasing (downward)

Optimization Problems…

A cylindrical can (with circular base) is made with a material for the lateral side which costs \$3/cm^2, and a material for the top and bottom circular sides which costs \$5/cm^2. If the can must enclose a volume of $20\pi\, cm^3$ what should the radius and height be to minimize the material cost?

Need functions for the objective function (what is being optimized) and any constraints.

Objective Function	Constraint
$Cost, C = \left(A_{lateral}\right)\left(\frac{\$3}{cm^3}\right) + \left(A_{top/bottom}\right)\left(\frac{\$5}{cm^3}\right)$	$V = 20\pi\ cm^3$
$C = (2\pi rh)(3) + (2)\left(\pi r^2\right)(5)$	$\pi r^2 h = 20\pi$

$C = 6\pi rh + 10\pi r^2$ *cost in terms of r and h*

Now solve constraint for one variables, substitute into objective function:

$h = \frac{20\pi}{\pi r^2} = \frac{20}{r^2} \quad so \quad C = 6\pi r\left(\frac{20}{r^2}\right) + 10\pi r^2 = 120\pi r^{-1} + 10\pi r^2$

Now find min *by taking derivative and finding where* $C'(r) = 0$

$C'(r) = -120\pi r^{-2} + 20\pi r = 0$

$20\pi r = \frac{120\pi}{r^2}, \quad r^3 = \frac{120}{20} = 6, \quad r = \sqrt[3]{6} = (6)^{\frac{1}{3}} = 1.81712\ cm$

Use constraint equation to find other dimension:

$h = \frac{20}{r^2} = \frac{20}{(1.81712)^2} = 6.057\ cm$

Should use 2nd − derivative to verify this is a min *not a* max:

$C''(r) = 240\pi r^{-3} + 20\pi$ *is + for + r, so concave up, so this is a* min.

Applications of integrals…

Use Fundamental Theorem of Calculus PT1 to evaluate the definite integral:

$$\int_1^2 x^2\,dx =$$

Use Fundamental Theorem of Calculus PT1 to evaluate the definite integral:

$$\int_1^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^2 = \left(\frac{1}{3}(2)^3\right) - \left(\frac{1}{3}(1)^3\right) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Using the Fundamental Theorem of Calculus PT2 (net change theorem): The rate of change of the altitude of a hot-air balloon is given by $r(t) = t^3 - 4t^2 + 6 \quad (0 \le t \le 8)$. Find the change in altitude of the balloon during the time when the altitude is decreasing.

First graph $r(t)$ *in calculator and find that this rate is negative for* $1.572 < t < 3.514$

Then, since $r(t)$ *is the derivative of altitude:*

$$\int_{1.572}^{3.514} a'(t)\,dt = \int_{1.572}^{3.514} \left(t^3 - 4t^2 + 6\right)dt = a(3.514) - a(1.572)$$

is the change in altitude $= -4.431\ (Math\ 9)$

Using the Fundamental Theorem of Calculus PT2 to find a y-value from another given derivative: If $f'(x) = x^2 - 5x,\ and\ f(1) = 2$ find $f(4)$.

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\int_1^4 \left(3x^2 - \frac{5}{2}x\right)dx = f(4) - f(1)$$

$$\left[x^3 - 5x^2\right]_1^4 = f(4) - 2$$

$$\left((4)^3 - 5(4)^2\right) - \left((1)^3 - 5(1)^2\right) = f(4) - 2$$

$$-12 = f(4) - 2, \quad f(4) = -10$$

Integral as inverse operation of derivative:

$$\frac{d}{dx}\left(\int_2^{3x^2} \left(t^3 - 4t\right)dt\right) =$$

$$\frac{d}{dx}\left(\int_{x^5}^{3x^2} \left(t^3 - 4t\right)dt\right) =$$

Integral as inverse operation of derivative:

$$\frac{d}{dx}\left(\int_a^{b(x)} f(t)\,dt\right) = f(b(x)) \bullet b'(x)\ \ [chain\ rule]$$

$$\frac{d}{dx}\left(\int_2^{3x^2} \left(t^3 - 4t\right)dt\right) = \left(\left(3x^2\right)^3 - 4\left(3x^2\right)\right) \bullet (6x)$$

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\,dt\right) = f(b(x)) \bullet b'(x) - f(a(x)) \bullet a'(x)$$

$$\frac{d}{dx}\left(\int_{x^5}^{3x^2} \left(t^3 - 4t\right)dt\right) = \left(\left(3x^2\right)^3 - 4\left(3x^2\right)\right) \bullet (6x) - \left(\left(x^5\right)^3 - 4\left(x^5\right)\right) \bullet \left(5x^4\right)$$

Average value of a function:

If $f(x) = x^2 - 5x$ find the average value of $f(x)\ over\ [2,6]$

Average value of a function:

$$average\ value\ of\ f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\frac{1}{6-2}\int_2^6 \left(x^2 - 5x\right)dx - \frac{1}{4}\left[\frac{1}{3}x^3 - \frac{5}{2}x^2\right]_2^6 = \frac{8}{3}$$

NOTE: This is different than 'average rate of change of $f(x)$*'*

which would instead be: $\dfrac{f(6) - f(2)}{6-2}$