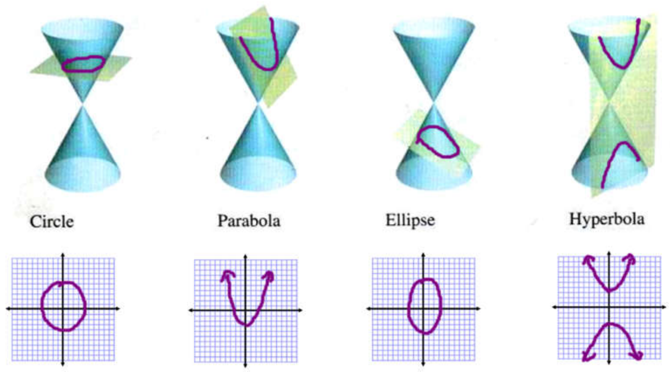
AP Calc BC – Lesson Notes – Unit 9: Conics, Vectors, Parametric Equations

Unit 9-1: Conic Sections
Larsen: 9.1

A conic section is a 2D curve which is the intersection of a plane with a cone...



...and all have equations of the general form:

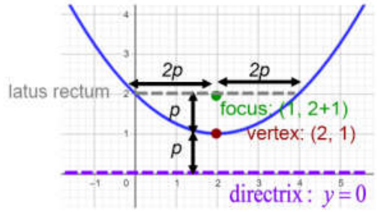
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

If the *xy* term is present, the conic section is not aligned with the x-y axes (is rotated)
We will not consider this case (it is solved with a rotational coordinate transformation)
- email me if you want a link to more detailed info about this case.

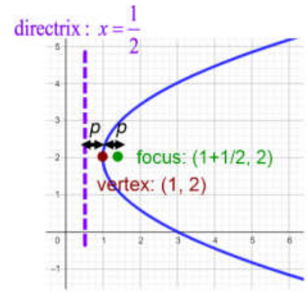
Quickly recognizing which conic section from the equation

$x^2 - 6x - 8y - 7 = 0$	One squared term = parabola
$9x^2 + 4y^2 - 36x + 8y + 4 = 0$	Two squared terms, same sign = ellipse
$9x^2 - 4y^2 - 18x - 16y + 29 = 0$	Two squared terms, different signs = hyperbola
$4x - y^2 - 2y - 9 = 0$	One squared term = parabola
$16x^2 - 4y^2 + 32x + 16y - 64 = 0$	Two squared terms, different signs = hyperbola
$2x^2 + 2y^2 + 12x - 16y + 40 = 0$	Two squared terms, same sign = ellipse, but coefficients of squared terms are same too, so circle

Parabolas



$$(x-2)^2 = 4(y-1)$$

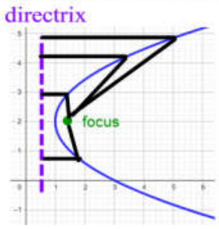


$$(y-2)^2 = 2(x-1)$$

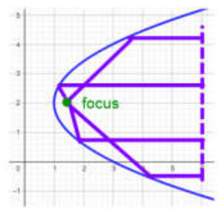
Standard form:	$(x-h)^2 = 4p(y-k)$	$p = \text{distance from vertex to focus}$
	$(y-k)^2 = 4p(x-h)$	$\text{and from vertex to directrix}$

x^2? (like $y=x^2$)

Geometrically, all points on a parabola are equidistant from focus and directrix...



...which makes this shape perfect for focusing energy (antenna dishes, flashlights, etc.)



all elements of a wavefront arrive at the same time at the focus

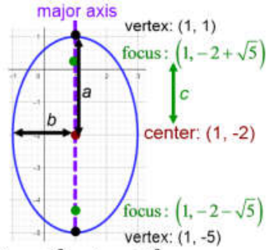
Parabola examples

$(x-h)^2 = 4p(y-k)$	$p = \text{distance from vertex to focus}$
$(y-k)^2 = 4p(x-h)$	$\text{and from vertex to directrix}$

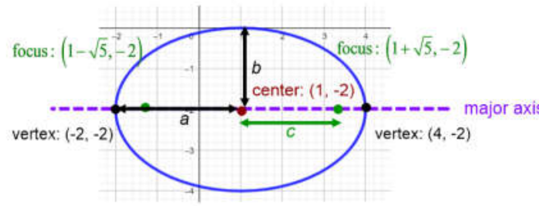
$$x^2 - 6x - 8y - 7 = 0$$

$$4x - y^2 - 2y - 9 = 0$$

Ellipses



$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$$

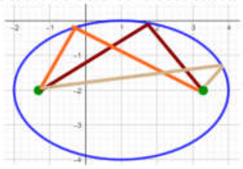


$$\frac{(x-1)^2}{9}+\frac{(y+2)^2}{4}=1$$

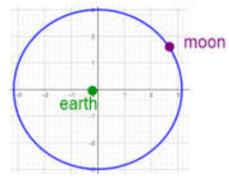
bigger denominator = longer direction

Standard form:			
$c^2=a^2-b^2$ $c=\text{distance from center to foci}$	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ *vertical major axis*	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ *horizontal major axis*	$\text{eccentricity}=e=\frac{c}{a}$ $e=1$ *is a circle,* *higher* $e \Rightarrow$ *more oval*

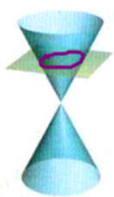
Geometrically, the sum of the distances from a point on an ellipse to each focus is constant:



In astronomical orbits, the object follows an elliptical path around the object being orbited which is at a focus (although the ellipse is usually close to circular):



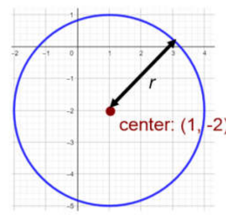
Circles



Circles are special cases of ellipses where *a* = *b*:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{a^2}=1$$

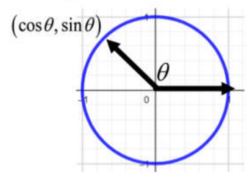
$$(x-h)^2+(y-k)^2=\cancel{a^2}\ r^2$$



$$(x-1)^2+(y+2)^2=9$$

Standard form: $(x-h)^2+(y-k)^2=r^2$

A circle with radius 1 (unit circle) is used to define the trigonometric functions:



Ellipse/Circle examples

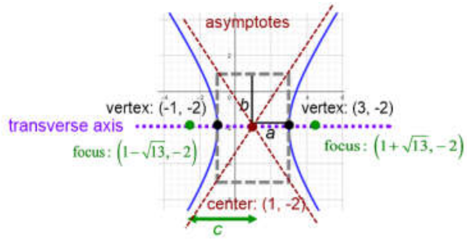
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ *horizontal major axis*	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ *vertical major axis*	$c^2=a^2-b^2$ $c=\text{distance from center to foci}$	$\text{eccentricity}=e=\frac{c}{a}$

$$9x^2+4y^2-36x+8y+4=0$$

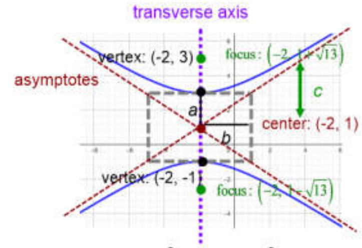
$$2x^2+2y^2+12x-16y+40=0$$

Hyperbolas

opens in direction of positive term



$$\frac{(x-1)^2}{4}-\frac{(y+2)^2}{9}=1$$



$$\frac{(y-1)^2}{4}-\frac{(x+2)^2}{9}=1$$

Standard form:

$c^2=a^2+b^2$

$c = distance\ from\ center\ to\ foci$

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

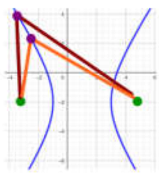
horizontal transverse axis

$$\frac{(y-1)^2}{4}-\frac{(x+2)^2}{9}=1$$

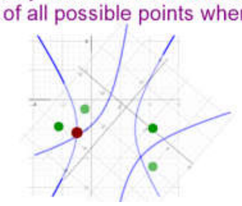
vertical transverse axis

$asymptotes: \ (y-k)=\pm\frac{b}{a}(x-h)$

Geometrically, the <u>difference</u> of the distances from a point on a hyperbola to each focus is constant...



...this is useful in location detection systems. If two points receive a signal and the delay between the times received is known, a hyperbola traces out the locus of all possible points where the emitting object might be.



If you have a 2nd set of two detectors, the location is at the intersection of the two hyperbolas from the two pairs of detectors.

Hyperbola examples

$c^2=a^2+b^2$

$c = distance\ from\ center\ to\ foci$

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

horizontal transverse axis

$$\frac{(y-1)^2}{4}-\frac{(x+2)^2}{9}=1$$

vertical transverse axis

$asymptotes: \ (y-k)=\pm\frac{b}{a}(x-h)$

$16x^2-4y^2+32x+16y-64=0$

$9x^2-4y^2-18x-16y+29=0$

Parabola	Ellipse	Hyperbola
$(x-h)^2 = 4p(y-k)$	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$
$(y-k)^2 = 4p(x-h)$	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$
x^2 like y= x^2 y^2 'other one'	a is always bigger, a under term of major axis	a not always bigger, a always under first term first term is transverse axis
p = dist. vertex to focus and dist. vertex to directrix	$c^2 = a^2 - b^2$	$c^2 = a^2 + b^2$

a = dist. center to vertex
c = dist. center to focus

Ellipse	Hyperbola
b = dist. center to point on minor axis	b = dist. to 'other side of box'

asymptotes from center through corners of box:

$$(y-k) = \pm\frac{b}{a}(x-h)$$

$$(y-k) = \pm\frac{a}{b}(x-h)$$

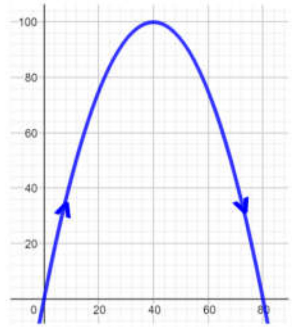
(look at box to see which)

eccentricity $e = \frac{c}{a}$

We know how to find a position equation from acceleration if we have motion in 1 dimension...

$$a(t) = -9.8$$
$$v(t) = \int a(t)\,dt = \int(-9.8)\,dt = -9.8t + v_0$$
$$x(t) = \int v(t)\,dt = \int(-9.8t + v_0)\,dt = -4.9t^2 + v_0 t + x_0$$

...but what if we wanted to model an object launched so that it follows a 2 dimensional path?



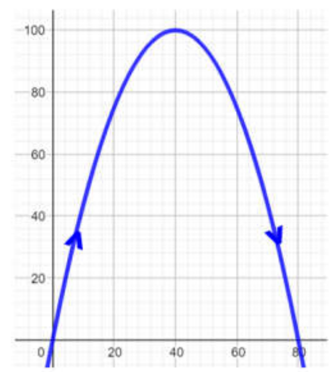
We can find a parabola which matches this...

Vertex: (40, 100)

$$(x-40)^2 = 4p(y-100)$$

Now make sure it goes through (0,0):

$$((0)-40)^2 = 4p((0)-100)$$
$$(-40)^2 = 4p(-100)$$
$$1600 = -400p$$
$$p = -4$$
$$(x-40)^2 = -16(y-100)$$
$$-16y + 1600 = x^2 - 80x + 1600$$
$$-16y = x^2 - 80x$$
$$\boxed{y = -\frac{1}{16}x^2 + 5x}$$



...and we can use a calculator graph to verify:

```
\Y1=-1/16X²+5X
```

```
WINDOW
 Xmin=0
 Xmax=80
 Xscl=1
 Ymin=0
 Ymax=100
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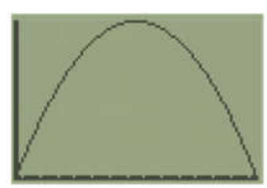
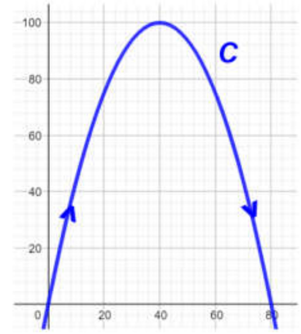


...but we don't have a good sense of where the object is at time t

Parametric Equations



This motion is in 2D, *x* and *y*, and both of these are varying with time, *t*.

We can instead represent this curve using separate equations for *x* and *y*, each written as functions of *t*. These are called **parametric equations**:

In this particular case, gravity is acting in the *y*-direction only. Once launched, the object continues to move steadily in the *x* direction. This suggests that we can just make the equation for x:

$$x = t$$

...and we can now substitute this expression for *x* into *y* to get the parametric equation for *y*:

$$y = -\frac{1}{16}x^2 - 5x$$

$$y = -\frac{1}{16}(t)^2 - 5(t)$$

We now call this path the **Plane Curve, C** with parametric equations:

$$\begin{cases} x = t \\ y = -\frac{1}{16}t^2 - 5t \end{cases}$$

We can now plug in any value of *t* and find the *x,y* location of the object at the time:

$$\boxed{\begin{cases} x = t \\ y = -\frac{1}{16}t^2 - 5t \end{cases}}$$

First, it is helpful to know what range of values *t* can take to describe the full object path. The object starts at (0,0) and ends at (80,0), where $y = 0$:

$$y = -\frac{1}{16}t^2 - 5t = 0$$

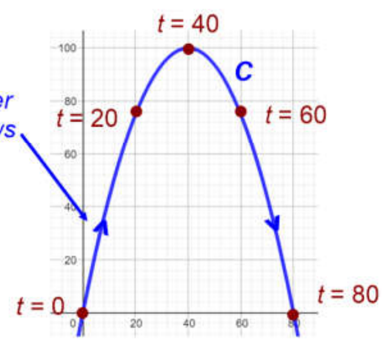
$$t\left(-\frac{1}{16}t - 5\right) = 0$$

$$t = 0,\ -\frac{1}{16}t - 5 = 0$$

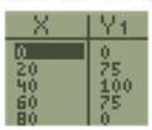
$$t = 80$$

$$\boxed{0 \le t \le 80}$$

a plane curve must always indicate direction as parameter increases positively with arrows



Now we can make a table with various values for *t* and find matching *x,y* (or just use a calculator's table features):



In the next section, we will use derivatives and integrals to find things like how fast an object's height is changing at a particular time, or the length of the arc the object travels over a time range (we can do this now even though the path is 2D because we have a single parameter variable to integrate over or differentiate with respect to).

Sketching plane curves from parametric equations - manually and using a calculator

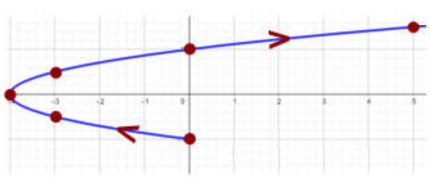
This idea applies to any curve, not just physics curves. Let's sketch the curve given by the parametric equations

$$\begin{cases} x = t^2 - 4 \\ y = \frac{1}{2}t \end{cases}$$

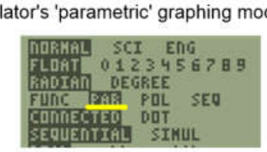
$$-2 \le t \le 3$$

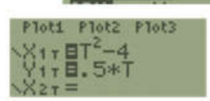
We could make a table and try various *t* values:

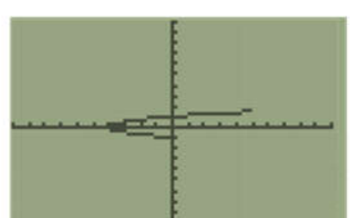
t	*x*	*y*
-2	0	-2
-1	-3	-1/2
0	-4	0
1	-3	1/2
2	0	2
3	5	3/2



Or if we are allowed to use calculators, we can use the calculator's 'parametric' graphing mode:







Eliminating the Parameter (converting from parametric to rectangular equations)

Another way which is sometimes helpful for graphing parametric equations is to eliminate the parameter to convert back to rectangular equations to see if it matches a graph shape you are familiar with.

$$\begin{cases} x = t^2 - 4 \\ y = \frac{1}{2}t \end{cases}$$

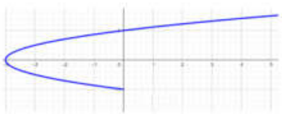
$t = 2y$

$x = (2y)^2 - 4$

$x + 4 = 4y^2$

$(y-0)^2 = \frac{1}{4}(x+4)$

...is a parabola with vertex (-4,0) opening to the right:



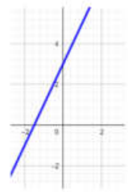
$$\begin{cases} x = 2t \\ y = 4t + 3 \end{cases}$$

$t = \frac{1}{2}x$

$y = 4\left(\frac{1}{2}x\right) + 3$

$y = 2x + 3$

...is a line:



$$\begin{cases} x = 1 + 2\cos t \\ y = -2 + 3\sin t \end{cases}$$

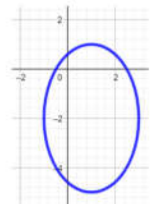
we know $\sin^2 t + \cos^2 t = 1$

$\cos t = \frac{x-1}{2}, \quad \sin t = \frac{y+2}{3}$

$\left(\frac{y+2}{3}\right)^2 + \left(\frac{x-1}{2}\right)^2 = 1$

$\frac{(y+2)^2}{9} + \frac{(x-1)^2}{4} = 1$

...is an ellipse with center (1,-2):



Range of Parameter, Limits of *x* and *y*

Graphing by eliminating the parameter may be quicker, but you don't have direction information or info about the limits of the parameter without plugging in a few values:

$$\begin{cases} x = 1 + 2\cos t \\ y = -2 + 3\sin t \end{cases}$$

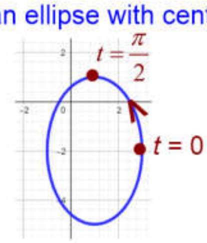
we know $\sin^2 t + \cos^2 t = 1$

$\cos t = \frac{x-1}{2}, \quad \sin t = \frac{y+2}{3}$

$\left(\frac{y+2}{3}\right)^2 + \left(\frac{x-1}{2}\right)^2 = 1$

$\frac{(y+2)^2}{9} + \frac{(x-1)^2}{4} = 1$

...is an ellipse with center (1,-2):



t	x	y
0	3	−2
$\frac{\pi}{2}$	1	1

Range of Parameter: (to go around once) $0 \le t \le 2\pi$

Limits of *x* and *y*: $-1 \le x \le 3$, $-5 \le y \le 1$ (we really shouldn't call these 'domain' and 'range' because this is not a function)

Parametrizations are not unique

If we are given a rectangular equation, coming up with a set of parametric equations using a parameter is called **parametrizing** the curve. Parametrizations are not unique: there are often many ways to parametrize a given curve.

$$y = \ln\left((2x+3)^2\right)$$

The most common way to parametrize is to just make the independent variable the parameter:

$$\begin{cases} x = t \\ y = \ln\left((2t+3)^2\right) \end{cases}$$

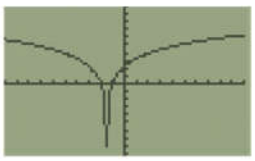
But we could do this instead:

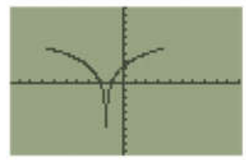
$$t = 2x+3$$

$$x = \frac{t-3}{2}$$

$$\begin{cases} x = \dfrac{t-3}{2} \\ y = \ln\left(t^2\right) \end{cases}$$

If you change parametrization, the range of the parameter also changes to cover a specified part of the curve. Here are graphs of the above parametrizations for $-10 \le t \le 10$





The Two Most Common Parametrizations

If you can solve an equation for one of the variables, the most common way to parametrize is to use the independent variable as the parameter:

$$y = \ln\left((2x+3)^2\right)$$

$$\begin{cases} x = t \\ y = \ln\left((2t+3)^2\right) \end{cases}$$

If you have cos, sin, try squaring and using the Pythagorean identity:

$$\begin{cases} x = 1+2\cos\theta \\ y = -2+3\sin\theta \end{cases}$$

(you can use any letter or symbol for the parameter)

$$\textit{we know } \sin^2\theta + \cos^2\theta = 1$$

$$\cos\theta = \frac{x-1}{2}, \quad \sin\theta = \frac{y+2}{3}$$

$$\left(\frac{y+2}{3}\right)^2 + \left(\frac{x-1}{2}\right)^2 = 1$$

$$\frac{(y+2)^2}{9} + \frac{(x-1)^2}{4} = 1$$

Examples

Graph the plane curve and write the corresponding rectangular equation:

$$\begin{cases} x = \cos\theta \\ y = 2\sin(2\theta) \end{cases}$$

Examples

Graph the plane curve and write the corresponding rectangular equation: $\begin{cases} x = -2 + 3\cos\theta \\ y = -5 + 3\sin(\theta) \end{cases}$

Graph the plane curve and write the corresponding rectangular equation: $\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}$

Find two different sets of parametric equations for $y = x^3$

Unit 9-3: Parametric Equations and Calculus
Larsen: 9.3

Derivative of a parametric function still means "slope"

Since we can ultimately express a curve given in parametric form on an x-y plane, the derivative dy/dx still means slope of the tangent line to the curve at a point, but the way we compute it is slightly different and relies on the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

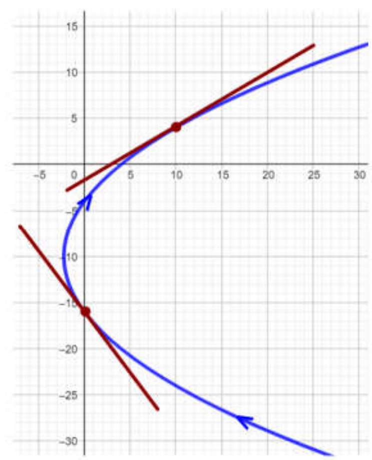
Solving for dy/dx...

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \quad \textit{as long as } \frac{dx}{dt} \neq 0$$

...and you can repeat for higher derivatives...

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\left(\frac{dx}{dt}\right)}$$

Graph the plane curve and find and interpret the meaning of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ $at\ t=-4, t=1$ for $\begin{cases} x = t^2+5t+4 \\ y = 4t \end{cases}$



	$at\ t=-4$	$at\ t=1$	
	$(0,-16)$	$(10,4)$	
$\frac{dx}{dt} = 2t+5$	$\frac{dx}{dt} = -3$	$\frac{dx}{dt} = 7$	How fast the point is increasing in *x* per unit of time, *t*.
$\frac{dy}{dt} = 4$	$\frac{dy}{dt} = 4$	$\frac{dy}{dt} = 4$	How fast the point is increasing in *y* per unit of time, *t*.
$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4}{2t+5}$	$\frac{dy}{dx} = -\frac{4}{3}$ **(decreasing)**	$\frac{dy}{dx} = \frac{4}{7}$ **(increasing)**	How fast the point is increasing in *y* per unit of *x*, the slope of the tangent line at this point.
$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{d}{dt}\left[t^2+5t+4\right]} = \frac{\frac{(2t+5)(0)-(4)(2)}{(2t+5)^2}}{2t+5} = \frac{-8}{(2t+5)^3}$	$\frac{d^2y}{dx^2} = 0.296$ **(concave up)**	$\frac{d^2y}{dx^2} = -0.023$ **(concave down)**	The concavity of the curve at this point.

We could also write equations for the tangent lines at these points:

	$at\ t=-4$	$at\ t=1$	
	$(0,-16)$	$(10,4)$	
$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4}{2t+5}$	$\frac{dy}{dx} = -\frac{4}{3}$	$\frac{dy}{dx} = \frac{4}{7}$	How fast the point is increasing in *y* per unit of *x*, the slope of the tangent line at this point.

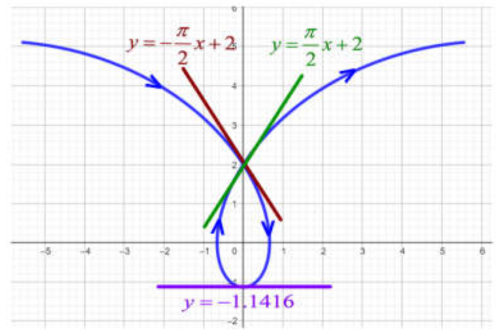
Tangent line equations

$$(y+16) = -\frac{4}{3}(x-0) \qquad (y-4) = \frac{4}{7}(x-10)$$

Need to be careful, graphs may loop and cross themselves

Graph the curve $\begin{cases} x = 2t - \pi \sin t \\ y = 2 - \pi \cos t \end{cases}$ $-3 \le t \le 3$ and find the parameters values where the graph crosses the y-axis.

Then write equations of tangent lines to the curve at these points.



Graph crosses the y-axis when x = 0, using calculator graph...

..this happens at 3 t values in the interval:

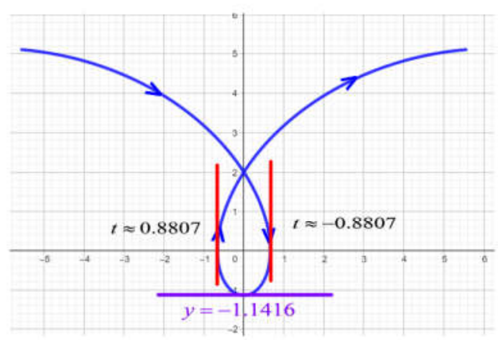
$t = -\frac{\pi}{2}$	$t = 0$	$t = \frac{\pi}{2}$
$(0, 2)$	$(0, -1.1416)$	$(0, 2)$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\pi \sin t}{2 - \pi \cos t}$$

$\frac{dy}{dx} = -\frac{\pi}{2}$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = \frac{\pi}{2}$
$(y-2) = -\frac{\pi}{2}(x-0)$	$(y+1.1416) = 0(x-0)$	$(y-2) = \frac{\pi}{2}(x-0)$
$y = -\frac{\pi}{2}x + 2$	$y = -1.1416$	$y = \frac{\pi}{2}x + 2$

We may be asked to find horizontal or vertical tangents

$\begin{cases} x = 2t - \pi \sin t \\ y = 2 - \pi \cos t \end{cases}$



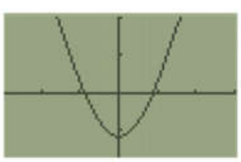
Some problems ask to find horizontal or vertical tangents...this mean on the x-y graph, so <u>horizontal tangents occur when</u>:

$$\frac{dy}{dx} = 0$$

<u>Vertical tangents would occur when this derivative is undefined</u> (typically, when the denominator goes to zero). For the derivative previously computed...

$$\frac{dy}{dx} = \frac{\pi \sin t}{2 - \pi \cos t}$$

would be undefined when $2 - \pi \cos t = 0$, solved by graphing in calculator:



at $t \approx -0.8807, \quad t \approx 0.8807$

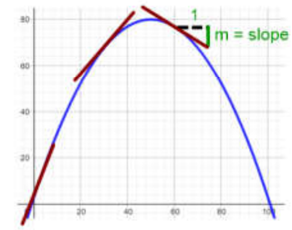
A real-world example...

A baseball is hit by a bat its trajectory is given by: $\begin{cases} x = t \\ y = 80 - \frac{77}{2500}(t-50)^2 \end{cases}$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \quad \textit{as long as } \frac{dx}{dt} \neq 0$$

What is the angle of elevation of the path of the ball at t = 0, t = 30, and t = 60?

We can graph this path:



The angle of elevation can be found using a tangent line at a given time: $\theta = \tan^{-1}(m)$

t	$\frac{dy}{dx}$	$\theta = \tan^{-1}(m)$ (use degrees)
0	3.08	72.013°
30	1.232	50.934°
80	-0.616	-31.633°

"Slope" at time t:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-\frac{77}{2500}2(t-50)^1(1)}{1} = -\frac{154}{2500}(t-50)$$

What do the derivatives $\frac{dx}{dt}, \frac{dy}{dt}$ mean?

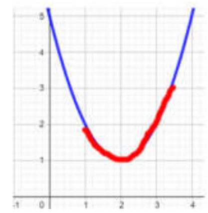
If we compute these at t = 0, t = 30, and t = 80...

	$\underline{t=0}$	$\underline{t=30}$	$\underline{t=80}$	
$\frac{dx}{dt} = 1$	$\frac{dx}{dt} = 1$	$\frac{dx}{dt} = 1$	$\frac{dx}{dt} = 1$	How fast the ball's height is changing per unit time.
$\frac{dy}{dt} = -\frac{154}{2500}(t-50)$	$\frac{dy}{dt} = 3.08$	$\frac{dy}{dt} = 1.232$	$\frac{dy}{dt} = -0.616$	How fast the ball is moving downrange per unit time.

We can now derive an equation for length of an arc along a 2D path

We have an equation for finding arc length using a derivative if a curve is expressed with y as a function of x:

$$arc\ length = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$$



But if we use the expression for dy/dx for a parametrically expressed curve...

$$f'(x) = \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

...then:

$$arc\ length = \int_a^b \sqrt{1+\left[\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}\right]^2}\,dx$$

$$= \int_a^b \sqrt{\frac{\left(\frac{dx}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2} + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\,dx$$

$$= \int_a^b \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)\frac{1}{\left(\frac{dx}{dt}\right)^2}}\,dx$$

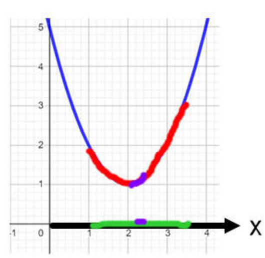
$$= \int_a^b \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)}\,\frac{1}{\left(\frac{dx}{dt}\right)}\,dx$$

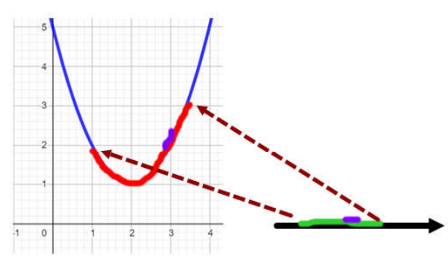
$$= \int_a^b \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)}\,\frac{dt}{dx}\,dx$$

$$\boxed{arc\ length = \int_a^b \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)}\,dt}$$

The difference between these is which variable we are 'integrating over':

$$arc\ length = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,\underline{dx} \qquad arc\ length = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\underline{dt}$$



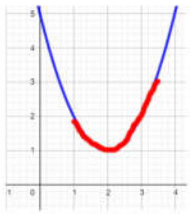


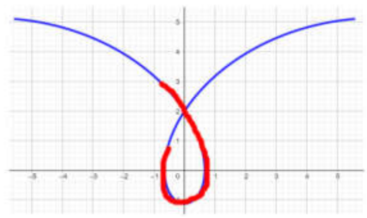
With the original formula, we are moving along the x-axis and integrating little sections of x (dx), and the integrand converts the small dx into a piece of arc length.

With the new formula, we are moving along the parameter (which might be something like time) a small amount dt and the integrand converts the small dt into a piece of arc length.

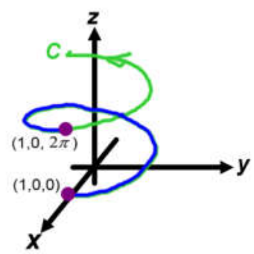
This new expression has two advantages over the previous version of arc length:

$$arc\ length = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\ dx \qquad arc\ length = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$





1) We can use it to find arc length for curves which are not functions.



2) In Calc 3 we can easily extend this to find arc length of curves in 3D space:

$$arc\ length = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

An arc length example...

Find the arc length of the curve on the given interval.

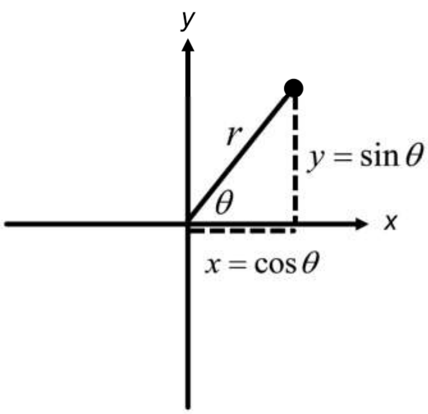
$$x = 6t^2,\ \ y = 2t^3 \qquad 1 \le t \le 4$$

Unit 9-4: Polar Coordinates and Graphs
Larsen: 9.4

Definition of Polar Coordinates

The coordinate system we've been using to graph points and equation curves is referred to as the **Cartesian** or **rectangular coordinate system**. But there is another way to locate points on the x-y plane that is sometimes advantageous called the **polar coordinate system.**

Instead of defining position distances in the x and y direction from axes, we define position using distance from the origin in a specified direction:



Converting...

<u>rectangular to polar</u>	<u>polar to rectangular</u>
$r=\sqrt{x^2+y^2}$	$x=r\cos\theta$
$\tan\theta=\dfrac{y}{x}$	$y=r\sin\theta$

The angle is specified in standard trigonometric position...0 along the positive x-axis and increasing positively counter-clockwise.

We can locate a point in the plane two ways, and can convert between methods:

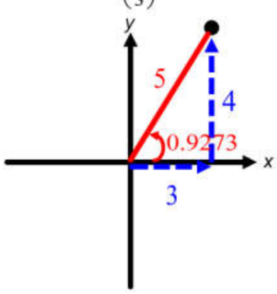
$x,y \quad\quad r,\theta$

$(3,4)\rightarrow(6,0.9273)$

$r=\sqrt{x^2+y^2}=\sqrt{3^2+4^2}=5$

$\tan\theta=\dfrac{y}{x}=\dfrac{4}{3}$

$\theta=\tan^{-1}\left(\dfrac{4}{3}\right)=0.9273$



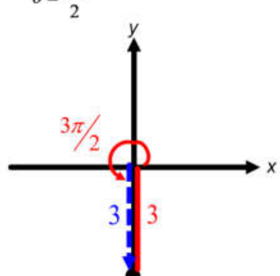
$x,y \quad\quad r,\theta$

$(0,-3)\rightarrow\left(3,{}^{3\pi}\!/_{2}\right)$

$r=\sqrt{x^2+y^2}=\sqrt{0^2+(-3)^2}=3$

$\tan\theta=\dfrac{y}{x}=\dfrac{3}{0}\ \textit{undefined}$

$\theta=\dfrac{3\pi}{2}$

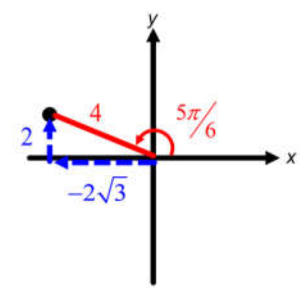


$r,\theta \quad\quad x,y$

$\left(4,\ {}^{5\pi}\!/_{6}\right)\rightarrow\left(-2\sqrt{3},\ 2\right)$

$x=r\cos\theta=4\cos\left({}^{5\pi}\!/_{6}\right)=4\left(-\dfrac{\sqrt{3}}{2}\right)=-2\sqrt{3}$

$y=r\sin\theta=4\sin\left({}^{5\pi}\!/_{6}\right)=4\left(\dfrac{1}{2}\right)=2$



Be careful about angles that arctan provides...

The arctan function can only provide a single angle as output and it is always between $-\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}$ but this may not match where the point is on the sketch:

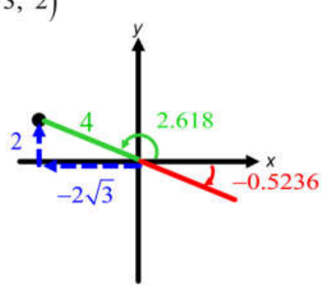
Example: Convert to polar coordinates $\left(-2\sqrt{3},\ 2\right)$

$r=\sqrt{x^2+y^2}=\sqrt{2^2+\left(-2\sqrt{3}\right)^2}=4$

$\tan\theta=\dfrac{y}{x}=\dfrac{2}{-2\sqrt{3}}$

$\theta=\tan^{-1}\left(\dfrac{2}{-2\sqrt{3}}\right)=-0.5236$

...gives an angle in quadrant IV



To get the correct angle, we need to add pi: $-0.5236+\pi=2.618$

Equations of curves can be written using either rectangular or polar coordinate

We can use our conversions to convert equations from one form to another. Sometimes, there are 'tricks' we need to use. It is best to see how this work by considering examples.

Convert from rectangular to polar form and sketch:

$x^2 + y^2 = 16$

$x^2 - y^2 = 9$

$x = 5$

$x^2 + y^2 - 4x = 0$

Convert from polar to rectangular form and sketch:

$r = 2$

$r = 8\sin\theta$

$\theta = \frac{5\pi}{6}$

$r = 2\csc\theta$

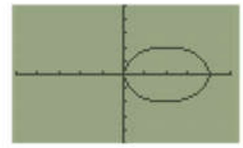
$r = \cot\theta\csc\theta$

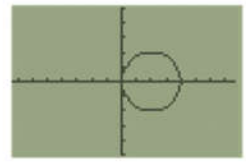
You can also use tables or calculator 'polar mode' to graph

$$r = 4\cos\theta$$

$$r^2 = 4r\cos\theta$$
$$x^2 + y^2 = 4x$$
$$\left(x^2 + 4x\right) + y^2 = 0$$
$$\left(x-2\right)^2 + y^2 = 16$$

If you can use a calculator, there is a polar mode:





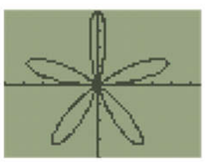
But you could also plug in angles to get radii:

θ	r
0	$4(1) = 4$
$\frac{\pi}{4}$	$4\left(\frac{\sqrt{2}}{2}\right) \approx 2.8$
$\frac{\pi}{2}$	$4(0) = 0$
$\frac{3\pi}{4}$	$4\left(\frac{-\sqrt{2}}{2}\right) \approx -2.8$
π	$4(-1) = -4$

(negative radii are graphed in opposite direction)

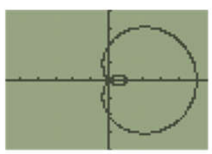
Sometimes the shapes get very complex, and some have names

$r = 4\sin(5\theta)$



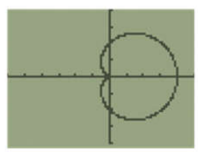
"rose"

$r = 2 + 3\cos\theta$



"limaçon"

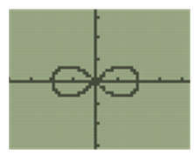
$r = 2 + 2\cos\theta$



"cardioid"

(microphone patterns)

$r^2 = 4\cos(2\theta)$



"lemniscate"

Derivatives in polar and tangent lines

The equations to convert from polar to rectangular: $x = r\cos\theta$, $y = r\sin\theta$

...are almost like parametric equations: they express x and y in terms of other variables (although in this case in terms of two variables, r and θ, not just one variable *t*. So we can pick one of these variables, θ and express *r* in terms of that, so that x and y are now in terms of just one variable θ.

$$r = f(\theta) \qquad x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta$$

Now, x and y form parametric equations, so we can find the derivative with the Chain Rule, but this will now require Product Rule as well:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{f(\theta)(\cos\theta) + \sin\theta\left(f'(\theta)\right)}{f(\theta)(-\sin\theta) + \cos\theta\left(f'(\theta)\right)}$$

(as long as the denominator is not zero)

<u>Horizontal tangents</u> occur when $\frac{dy}{dx} = 0$ which is when the numerator is zero, therefore when $\boxed{\frac{dy}{d\theta} = 0}$

<u>Vertical tangents</u> occur when $\frac{dy}{dx}$ is undefined, which is when the denominator is zero, therefore when $\boxed{\frac{dx}{d\theta} = 0}$

Derivatives in polar and tangent lines

Horizontal tangents occur when $\frac{dy}{dx}=0$ which is when the numerator is zero, therefore when $\frac{dy}{d\theta}=0$

Vertical tangents occur when $\frac{dy}{dx}$ is undefined, which is when the denominator is zero, therefore when $\frac{dx}{d\theta}=0$

We can use this result to quickly locate the angles where horizontal or vertical tangents occur.

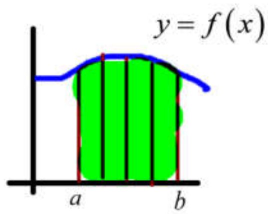
Example: find the points of horizontal and vertical tangency to the polar curve $r=4\sin\theta$

and find the equation of the tangent line at $\theta=\frac{\pi}{3}$

Unit 9-5: Area and Arc Length in Polar Coordinates
Larsen: 9.5

Area of a Polar Region

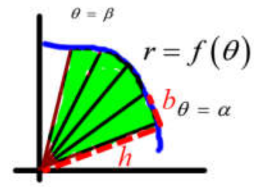
We can use a definite integral to find the area of a polar region. To derive the formula, we start with something like a Riemann Sum for rectangular area... except we will sum triangular segments...



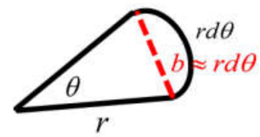
$$A = \int_{x=a}^{x=b} A_{rectangle}$$

$$A = \int_{x=a}^{x=b} h\, w$$

$$A = \int_{x=a}^{x=b} f(x)\, dx$$



The height of the rectangle is just the radius at that point.



For θ in radians $\theta = \frac{arc}{r}$

$arc = r\theta$

$d(arc) = rd\theta$

$$A = \int_{\theta=\alpha}^{\theta=\beta} A_{triangle}$$

$$A = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} bh$$

$$A = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2}(arclength) r$$

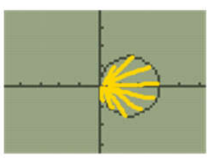
$$A = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r(r\, d\theta)$$

$$\boxed{A = \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} r^2\, d\theta}$$

Examples of area in polar

Find the area in the interior of $r = 3\cos\theta$

Always graph first, then imagine covering this area with radial 'slivers'



We can either integrate triangles from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$
or we can double the area integrated from $\theta = 0$ to $\theta = \frac{\pi}{2}$

Find the area in the inner loop of $r = 1 + 2\sin\theta$

Intersections of two curves in polar

Sometimes we need to find the area bounded by two curves and the start and end angles occur at intersections. To find intersections, first always graph to see what is going on, then treat the two curves as a system and solve simultaneously. Look also at the graph to see if the origin (0,0) is an intersection - because that corresponds to *r = 0* it can occur at any angle and often does not arise as a solution from the system.

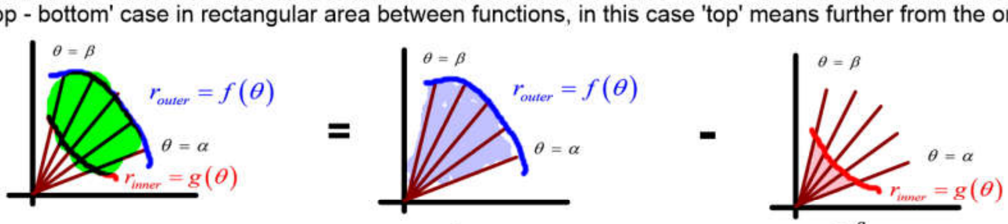
Examples...Find the intersection points of the curves:

$$r = 3(1+\sin\theta)$$
$$r = 3(1-\sin\theta)$$

$$r = 4\sin\theta$$
$$r = 2$$

Area between two curves in polar

We can find the area between two curves by finding the area of the outer region and subtracting the inner region (similar to the 'top - bottom' case in rectangular area between functions, in this case 'top' means further from the origin).



$$A_{between} = A_{outer} = \frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)\right)^2 \, d\theta - A_{inner} = \frac{1}{2}\int_{\alpha}^{\beta} \left(g(\theta)\right)^2 \, d\theta$$

$$A_{between} = \frac{1}{2}\int_{\alpha}^{\beta} \left[\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2\right] d\theta$$

(If you combine into one integral, make sure you square each radius function separately)

You need to always sketch first so you can see where intersections occur which usually define start and end angles. Also, be aware of the fact that in the middle of an area the 'outer' and 'inner' curves can switch.

Examples...

Find the area inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$

Find the common interior area of $r = 2\cos\theta$ *and* $r = 2\sin\theta$

Arc Length in Polar Coordinates

Starting with the arc length formula for parametrized functions: $arc\ length = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

and if we want to use θ as the parameter, we have $r = f(\theta)$

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

...then: $arc\ length = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

we can compute the two derivatives using the Product Rule:

$$\frac{dx}{d\theta} = f(\theta)(-\sin\theta) + \cos\theta\left(f'(\theta)\right) \qquad \frac{dy}{d\theta} = f(\theta)(\cos\theta) + \sin\theta\left(f'(\theta)\right)$$

the inside of the radical in the integral is then:

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= \left(f(\theta)(-\sin\theta) + \cos\theta\left(f'(\theta)\right)\right)^2$$

$$+ \left(f(\theta)(\cos\theta) + \sin\theta\left(f'(\theta)\right)\right)^2$$

$$= \left[f(\theta)\right]^2 \sin^2\theta - 2f(\theta)f'(\theta)\sin\theta\cos\theta + \left[f'(\theta)\right]^2\cos^2\theta$$

$$+ \left[f(\theta)\right]^2 \cos^2\theta + 2f(\theta)f'(\theta)\sin\theta\cos\theta + \left[f'(\theta)\right]^2\sin^2\theta$$

$$= \left[f(\theta)\right]^2\left(\sin^2\theta + \cos^2\theta\right) + \left[f'(\theta)\right]^2\left(\sin^2\theta + \cos^2\theta\right)$$

$$= \left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2$$

$$= r^2 + \left(\frac{dr}{d\theta}\right)^2$$

So arc length is:

$$arc\ length = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Examples of arc length in polar

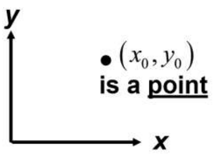
Finding arc length is then finding the portion of a single curve which you want to integrate over and determining the start and end angle values. One thing to be aware of it that some curves loop over themselves multiple times if the range of the parameter is large enough.

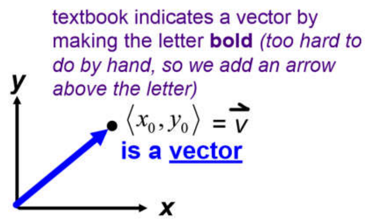
Find the arc length of the top half of the cardioid $r = 2 - 2\cos\theta$

Find the arc length of the curve $r = 2\sin(3\theta)$

Unit 9-6: Vectors (properties and applications)
Larsen: 9.6

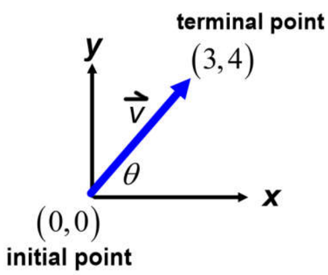
A vector is a directed line segment, and is characterized by its length and direction.





Initial, terminal point, length, direction

A vector can start from any initial point, but if you place its initial point at the origin, length and direction work the same way they do for polar coordinates:

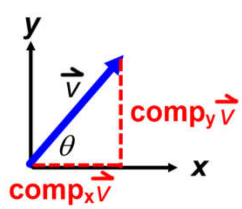


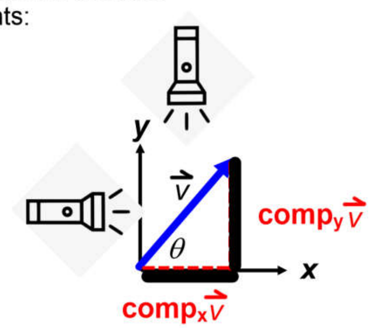
$$\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$$
$$\vec{v} = \langle 3, 4 \rangle$$

length: $|\vec{v}| = \sqrt{3^2 + 4^2} = 5$

direction: $\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.15°$

The length of the vector along each axis direction is called a **component.** Each vector has two components:

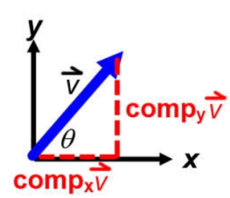




You can think of the components as being like the shadow of the vector on the x or y axis if you shined a flashlight on the vector.

This is called the **projection** of the vector onto the x-axis or y-axis.

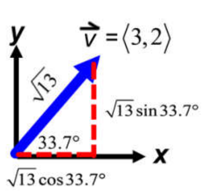
Converting between components and length, angle



$$\vec{v} = \langle comp_x, comp_y \rangle$$

$$|\vec{v}| = \sqrt{comp_x{}^2 + comp_y{}^2} \qquad comp_x = |\vec{v}|\cos\theta$$

$$\theta = \tan^{-1}\left(\frac{comp_y}{comp_x}\right) \qquad comp_y = |\vec{v}|\sin\theta$$

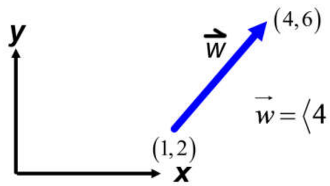


$$|\vec{v}| = \sqrt{3^2 + 2^2} = \sqrt{13} \qquad comp_x = \sqrt{13}\cos 33.7°$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right) = 33.7° \qquad comp_y = \sqrt{13}\sin 33.7°$$

Vector components from initial and terminal points

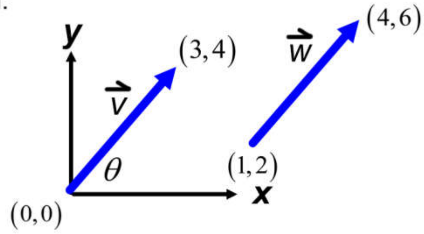
If a vector is defined by giving its initial and terminal points, you can find the components by subtracting end-start for each component:



$$\vec{w} = \langle 4-1, 6-2 \rangle = \langle 3,4 \rangle$$

Vector equality

Two vectors are considered 'equal' or 'the same vector' or 'equivalent' if their magnitudes and directions are the same, regardless of where the initial points are located:



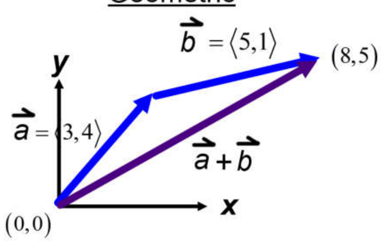
$$\vec{v} = \langle 3-0, 4-0 \rangle = \langle 3,4 \rangle \qquad \vec{w} = \langle 4-1, 6-2 \rangle = \langle 3,4 \rangle$$

$$\vec{v} = \vec{w}$$

Vector Addition

Adding two vectors is the equivalent of moving along the combined paths of both vectors to the new terminal point.

Geometric



Placing one vector's tail to the other's tip results in a new terminal point for the addition vector

the 'triangle law'

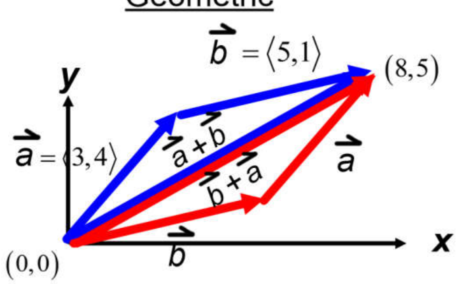
Algebraic

$$\vec{a} = \langle 3,4 \rangle$$
$$\vec{b} = \langle 5,1 \rangle$$
$$\vec{a}+\vec{b} = \langle 3+5, 4+1 \rangle$$
$$\vec{a}+\vec{b} = \langle 8,5 \rangle$$

Vector Addition is commutative

Reversing the order of the vectors being added gives the same result:

Geometric



the 'parallelogram law'

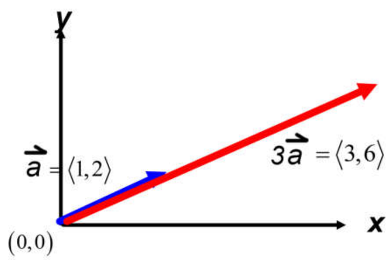
the sum is also the diagonal of the parallelogram

Algebraic

$$\vec{a} = \langle 3,4 \rangle$$
$$\vec{b} = \langle 5,1 \rangle$$
$$\vec{a}+\vec{b} = \langle 3+5, 4+1 \rangle$$
$$\vec{b}+\vec{a} = \langle 5+3, 1+4 \rangle$$

Multiplying a vector by a scalar

Multiplying a vector by a scalar (number) multiplies all components by that value, and scales the size of the vector...



$\vec{a} = \langle 1,2 \rangle$

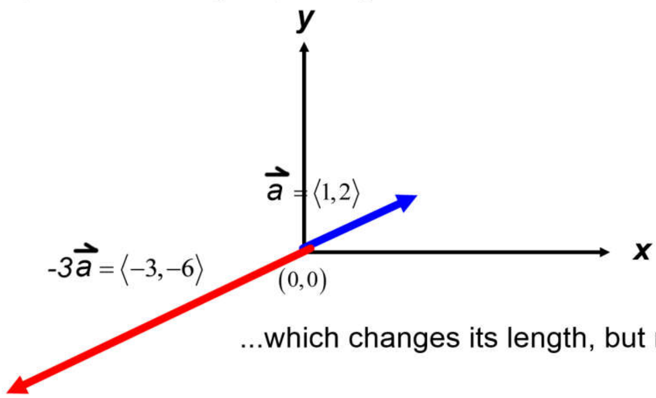
$3\vec{a} = 3\langle 1,2 \rangle$

$3\vec{a} = \langle 3,6 \rangle$

...which changes its length, but not its direction.

Negative vectors

However, if the scalar is negative, it changes the direction 180^0:



$\vec{a} = \langle 1,2 \rangle$

$-3\vec{a} = -3\langle 1,2 \rangle$

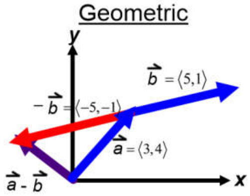
$-3\vec{a} = \langle -3,-6 \rangle$

...which changes its length, but not its 'direction'.

Vector Subtraction

Subtracting a vector is equivalent to adding a vector multiplied by -1:

<u>Geometric</u>



<u>Algebraic</u>

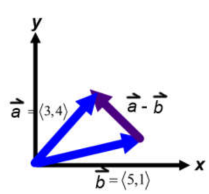
$\vec{a} = \langle 3,4 \rangle$

$\vec{b} = \langle 5,1 \rangle$

$\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right)$

$\vec{a} - \vec{b} = \langle 3,4 \rangle + \langle -5,-1 \rangle$

$\vec{a} - \vec{b} = \langle -2,3 \rangle$



subtraction is also geometrically equivalent to combining vector 'tail-to-tail' (drawn from b to a)

Properties of vectors

PROPERTIES OF VECTORS If **a**, **b**, and **c** are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Vector computational examples

1) Show vectors are equivalent:

u: $(-4, 0)$, $(1, 8)$

v: $(2, -1)$, $(7, 7)$

2) Find the magnitude of each vector and 3**v** - 2**u**

Sketch the original vectors and 3**v** - 2**u**

$$\vec{u} = \langle 3,\ 6 \rangle$$

$$\vec{v} = \langle 5, -2 \rangle$$

Applications of vectors

Things to know:

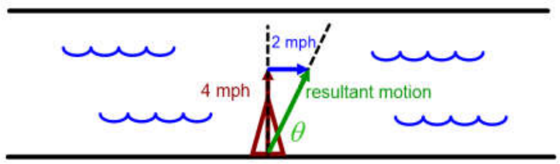
- If objects are not moving, then the sum of all force vectors = 0.

$$\sum H(\text{horiz components}) = 0$$
$$\sum V(\text{vert components}) = 0$$

- If objects are moving, then the overall motion is the sum of all individual motion vectors.

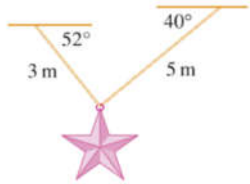
Vector Application Examples

A boat heads straight across a river at a speed of 4 mph, but the water in the river is flowing a 2 mph (as in the figure). What is the resultant and direction of the boat?



Vector Application Examples

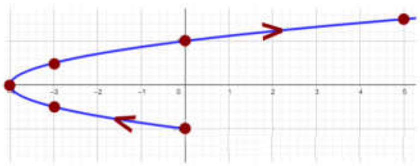
A star-shaped decoration is suspended, motionless, from two points as shown in the figure. If the decoration weighs 10 lbs, find the tension in each wire (the magnitude of each tension as well as the component forces)?



Unit 9-7: Vector-Valued Functions
Larsen: 9.7

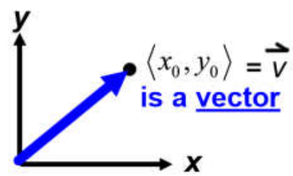
We've know how to represent a plane curve using parametric equations...

$$\begin{cases} x = t^2 - 4 \\ y = \frac{1}{2}t \end{cases}$$
$-2 \le t \le 3$



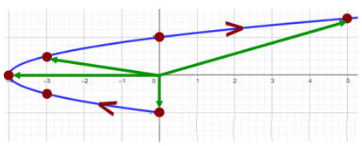
t	x	y
-2	0	-2
-1	-3	-1/2
0	-4	0
1	-3	1/2
2	0	2
3	5	3/2

...and we've defined how vectors work...



We can put these ideas together to define Vector-Valued Functions

$$\begin{cases} x = t^2 - 4 \\ y = \frac{1}{2}t \end{cases}$$
$-2 \le t \le 3$



t	x	y
-2	0	-2
-1	-3	-1/2
0	-4	0
1	-3	1/2
2	0	2
3	5	3/2

At each value of the parameter, *t*, we define a vector with initial point a the origin and terminal point at the point on the plane curve given by the parametric equations

$\vec{r}(2) = \langle 0, -2 \rangle$

$\vec{r}(0) = \langle -4, 0 \rangle$

$\vec{r}(1) = \langle -3, \frac{1}{2} \rangle$

$\vec{r}(3) = \langle 5, \frac{3}{2} \rangle$

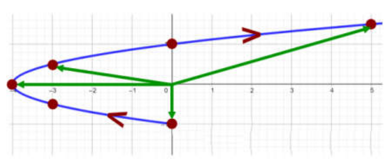
$$\boxed{\vec{r}(t) = \left\langle t^2 - 4, \frac{1}{2}t \right\rangle}$$

These are sometimes also called 'Vector Functions' or 'Position Vectors'

The domain of a vector-valued function is the allowed values of the parameter

$$\begin{cases} x = t^2 - 4 \\ y = \frac{1}{2}t \end{cases}$$
$-2 \le t \le 3$

t	x	y
-2	0	-2
-1	-3	-1/2
0	-4	0
1	-3	1/2
2	0	2
3	5	3/2



The domain is the possible list of parameter values

$\vec{r}(t) = \left\langle t^2 - 4, \frac{1}{2}t \right\rangle$

The output of the function are the resulting vectors (there really isn't a 'range' although you could think of the plane curve as representing the range of the vector-valued function.

The domain is defined by the allowable values for both of the parametric equations

If you are determining the domain of a vector-valued function, you start by assuming that the parameter can take any value, $-\infty < t < \infty$

Then remove from the domain any parameter values which either are not allowed:

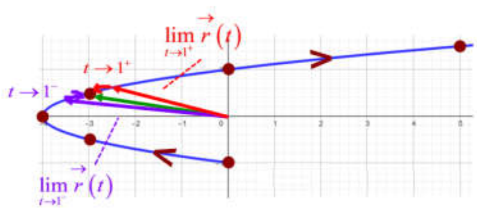
- The function definition itself may restrict the domain: $-2 \le t \le 3$
- In a real-world problem, the parameter *t* may represent time, in which case negative values would not make sense.
- Remove any parameter values which make either component parametric equation undefined:
  - Dividing by zero
  - Even roots of negatives
  - Logarithms of zero or negatives

Example: Find the domain of the vector-valued function: $\vec{r}(t) = \left\langle \sqrt{4-t^2}, \frac{3}{t-1} \right\rangle$

Limits and Continuity of Vector-Valued Functions

If you approach a particular value of the parameter from lower and higher values, a particular vector is being approached, so we can define the limit of a vector-valued function as being the vector approached...

$$\vec{r}(t) = \left\langle t^2 - 4, \frac{1}{2}t \right\rangle$$



t	x	y
-2	0	-2
-1	-3	-1/2
0	-4	0
1	-3	1/2
2	0	2
3	5	3/2

$$\lim_{t \to 1} \vec{r}(t)$$

$$\lim_{t \to 1} \left\langle t^2 - 4,\ \frac{1}{2}t \right\rangle$$

$$\left\langle \lim_{t \to 1} t^2 - 4,\ \lim_{t \to 1} \frac{1}{2}t \right\rangle$$

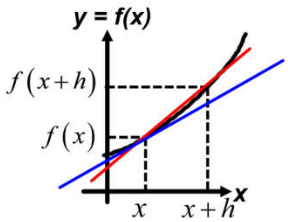
$$\left\langle (1)^2 - 4,\ \frac{1}{2}(1) \right\rangle$$

$$\left\langle -3,\ \frac{1}{2} \right\rangle$$

A vector-valued function is said to be **continuous on an interval** if the function is defined for all parameter values in the interval and the limit exists as the parameter approaches every value in the interval.

Derivative of a vector function

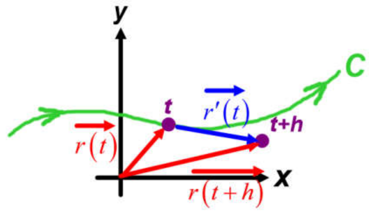
Recall from earlier:



$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

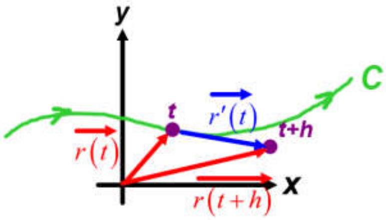
The derivative of a function (defined by the of the slope of the secant line) givens the slope of the tangent line to the curve at x.

For vector-valued functions:



$$\vec{r}'(t)=\lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

A similar limit structure defines the derivative of the vector function. Before taking the limit, this is a vector between two points on the plane curve which is roughly in the direction of the curve at the point where *t=t*.



$$\vec{r}'(t)=\lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

A similar limit structure defines the derivative of the vector function. Before taking the limit, this is a vector between two points on the plane curve which is roughly in the direction of the curve at the point where *t=t*.

The vector-valued function is a vector from the origin to the point on the plane curve.

The derivative of the vector-valued function is tangent to the place curve at this point and represents the current direction of travel as the parameter is increasing (so it is drawn with its initial point on the space curve).

Finding the derivative of a vector function at a parameter value

Although you could use limits to evaluate the derivative, typically we find the derivative by simply taking the derivative of each of the parametric equations of the vector-valued function using derivative shortcuts, then plug in parameter values:

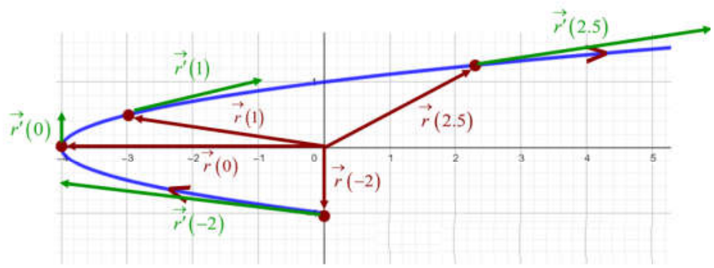
$$\vec{r}(t)=\left\langle t^2-4,\ \frac{1}{2}t\right\rangle \qquad \vec{r}'(t)=\left\langle 2t,\ \frac{1}{2}\right\rangle$$

$$\vec{r}(-2)=\left\langle (-2)^2-4,\ \tfrac{1}{2}(-2)\right\rangle=\langle 0,-1\rangle \qquad \vec{r}'(-2)=\left\langle 2(-2),\ \frac{1}{2}\right\rangle=\left\langle -4,\ \frac{1}{2}\right\rangle$$

$$\vec{r}(0)=\left\langle (0)^2-4,\ \frac{1}{2}(0)\right\rangle=\langle -4,0\rangle \qquad \vec{r}'(0)=\left\langle 2(0),\ \frac{1}{2}\right\rangle=\left\langle 0,\ \frac{1}{2}\right\rangle$$

$$\vec{r}(1)=\left\langle (1)^2-4,\ \frac{1}{2}(1)\right\rangle=\left\langle -3,\ \tfrac{1}{2}\right\rangle \qquad \vec{r}'(1)=\left\langle 2(1),\ \frac{1}{2}\right\rangle=\left\langle 2,\ \frac{1}{2}\right\rangle$$

$$\vec{r}(2.5)=\left\langle (2.5)^2-4,\ \frac{1}{2}(2.5)\right\rangle=\langle 2.25, 1.25\rangle \qquad \vec{r}'(2.5)=\left\langle 2(2.5),\ \frac{1}{2}\right\rangle=\left\langle 5,\ \frac{1}{2}\right\rangle$$



As you might imagine, if the parameter *t* represents time and the vector-valued function represents position, then the derivative of position gives velocity, but now as a vector we know not only the speed (magnitude of velocity) but also the direction at every value of time (this is the subject of the next section).

Properties of vector-valued derivatives

Things like product rule, and chain rule still apply...

THEOREM 9.16 Properties of the Derivative

Let **r** and **u** be differentiable vector-valued functions of t, let w be a differentiable real-valued function of t, and let c be a scalar.

1. $\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$ — Constant Multiple Rule
2. $\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$ — Sum and Difference Rules
3. $\frac{d}{dt}[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$ — Product Rule
4. $\frac{d}{dt}[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$ — Chain Rule

Example: Find the derivative of $\vec{r}(t) = \left\langle e^{(t^2+3t)},\ \frac{\ln(t^2-3t)}{t^3-4} \right\rangle$

Integrals with vector-valued functions

Because integrals are essentially anti-derivatives, we can also find integrals with vector-valued functions. As with derivatives, we just take the integral of each component's parametric equation separately:

Indefinite integral: $$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt,\ \int g(t)\,dt \right\rangle$$

Definite integral: $$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt \right\rangle$$

Example: Evaluate $\int \vec{r}(t)\,dt \ \ if\ \vec{r}(t) = \langle \cos t,\ t^3 - t \rangle$

Unit 9-8: Velocity and Acceleration as Vectors
Larsen: 9.8

Motion in one direction only

Earlier we learned that if a function represents distance (displacement) vs time, then the derivative is velocity and the 2nd derivative is acceleration:

distance (displacement): $s(t) = t^2 - 5t + 8 \;\; m$

average velocity: $v_{avg} = \dfrac{s_2 - s_1}{t_2 - t_1} = \dfrac{s(t+h) - s(t)}{h}$

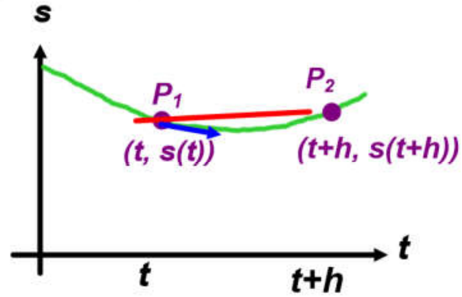
instantaneous velocity: $v(t) = \lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h} = s'(t)$

$v(t) = 2t - 5 \;\; m/s$

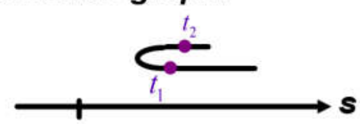
acceleration: $a(t) = v'(t)$

$a(t) = 2 \;\; m/s^2$

graph of displacement vs time:



1D motion graph:



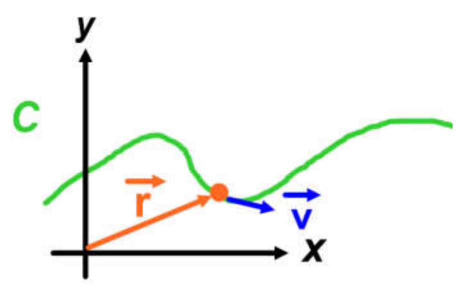
Motion in 2D

To analyze motion in 2 dimensions, we can represent the position on the 2D plane as a vector-valued function (a 'position vector' pointing from the origin to the location of the object at that time.

Then the velocity vector is the derivative of the position vector:

position: $\vec{r}(t) = \langle x(t),\ y(t) \rangle$

velocity: $\vec{v}(t) = \vec{r}'(t) = \langle x'(t),\ y'(t) \rangle$



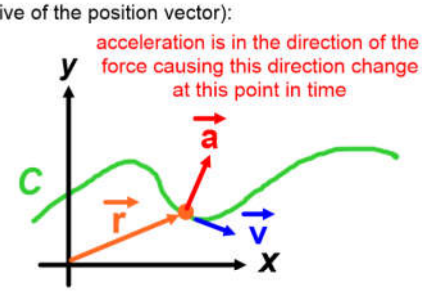
The **speed** is the magnitude of the velocity vector, but with a vector for velocity we know the direction the object is travelling at that time.

The acceleration vector is then the derivative of the velocity vector (and the 2nd derivative of the position vector):

position: $\vec{r}(t) = \langle x(t),\ y(t) \rangle$

velocity: $\vec{v}(t) = \vec{r}'(t) = \langle x'(t),\ y'(t) \rangle$

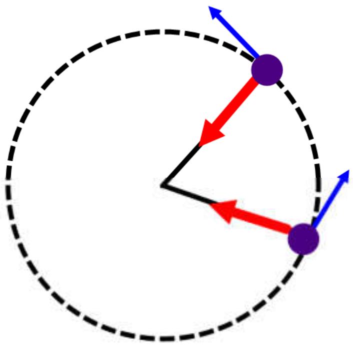
acceleration: $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = \langle x''(t),\ y''(t) \rangle$



The 2nd derivative of position is related to concavity but with a position function, what causes an object in motion to deviate from its course is a force. Newton's 2nd Law of Motion is $\vec{F} = m\vec{a}$ *or* $\vec{a} = \dfrac{\vec{F}}{m}$ which states that if a force in a particular direction is acting upon an object, there is an acceleration of the object in the direction of that force. (The mass is the property of matter which resists changing direction when force is applied: small mass = large acceleration for a given force, larger mass = small acceleration for a given force.)

Meaning/interpretation of acceleration

This allows us to explain things like circular motion. Imagine a ball connected to a string, and twirling the ball around in a circle at constant speed:



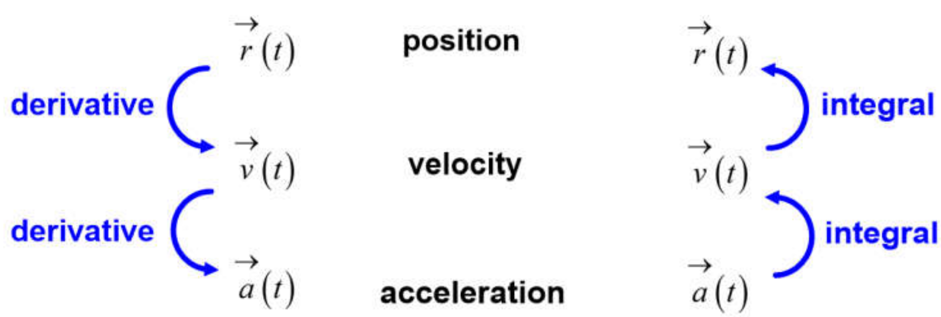
Although the speed (magnitude) of the velocity vector is constant, the direction of velocity is constantly changing for the ball to move around the circle.

The acceleration vector is always towards the center of the circle and is caused by the force the string exerts on the ball.

In physics, forces always cause accelerations - but not necessarily changes in speed. The change in velocity caused by the acceleration can be a change in the direction of the velocity vector (even when the speed is not changing).

$$\boxed{\vec{F} = m\vec{a}}$$

Given one function you can use derivatives or integrals to find the other functions



Remember, when you use an integral, you need to include an integration constant, which in this case would be a vector.

Examples: $\vec{r}(t) = \langle t^3 + 2\cos t,\ \sin t - t^2 \rangle$

Find $\vec{v}(t),\ \vec{a}(t)$

$\vec{a}(t) = \langle t,\ -10 \rangle,\ \ \vec{v}(2) = \langle 5, 2 \rangle,\ \ \vec{r}(1) = \langle 3, 4 \rangle$

Find $\vec{v}(t),\ \vec{r}(t),\ \vec{r}(3)$

Displacement vs. Total Distance

When you take the antiderivative of the velocity function as a vector-valued function, the velocity includes direction, so the resulting position represents the **displacement** of the object (the position of the object):

displacement: $$\int_a^b \vec{v}(t)\,dt = \left\langle \int_a^b x'(t)\,dt,\ \int_a^b y'(t)\,dt \right\rangle = \vec{r}(b) - \vec{r}(a)$$

If instead you need to find the **total distance traveled** then you can do a scalar integral of the speed:

total distance traveled: $$\int_a^b \left|\vec{v}(t)\right| dt = \int_a^b \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\ dt$$

Example: $\vec{a}(t) = \langle 2, 2t \rangle,\ \ \vec{v}(1) = \langle 3, 5 \rangle,\ \ \vec{r}(1) = \langle 2, 6 \rangle$

Find $\vec{v}(t),\ \vec{r}(t),\ \vec{r}(2)$ **and total distance traveled from** $t = 0$ **to** $t = 2$