# AP Calc BC – Lesson Notes – Unit 8: Intro to Differential Equations

Unit 8-1: Differential Equations: Slope Fields, Verifying Solutions, Solving by Integration, Euler's Method
Larsen: 5.1

**What is a differential equation?**

A differential equation is an equation (contains an equals sign which states the two sides are equal) but where at least one term contains a derivative.

$$\frac{dy}{dx}+5y=e^x \qquad \frac{d^2y}{dx^2}-3\frac{dy}{dx}+5y=6$$

A **solution** to a differential equation **is itself an equation** - an equation which completely satisfies the original differential equation.

To verify if an equation is a solution to given differential equation, you first take the required derivatives, and then plug everything in:

Ex) Verify that $y=100e^{.05t}+50$
is a solution to the differential equation $\frac{dy}{dt}=5e^{.05t}$

$y=100e^{.05t}+50$

so $y'=100(.05e^{.05t})=5e^{.05t} \rightarrow$

Substitute into the differential equation:

$\frac{dy}{dt}\stackrel{?}{=}5e^{.05t}$

$5e^{.05t}\stackrel{?}{=}5e^{.05t}$ ✓

(more complex with more complicated DEs, but this is the concept)

**Method for find a solution to a DE: Integration**

There are many methods for finding the solution functions to differential equations (which is why there is an entire course about this), but we will learn a few methods for solving DEs this year. The first is useful if we have a DE in which the highest order derivative is a first-derivative, and which can be solved for that derivative (called first-order differential equation). If this case, we can find the solution function by simply integrating the differential equation.

Ex) Find the solution(s) for $\frac{dy}{dt}=5e^{.05t}$

$y=\int 5e^{.05t}\,dt$

$y=\frac{5e^{.05t}}{.05}+c$

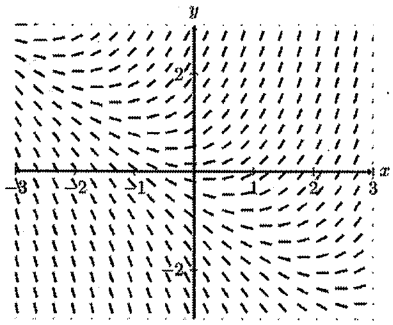
$\boxed{y=100e^{.05t}+c}$

## Slope Fields

For first-order differential equations, because they can be solved for the derivative on the left side, the expression on the right gives the 'slope' at any point *x,y* in the domain:

$$\frac{dy}{dx} = f(x,y)$$

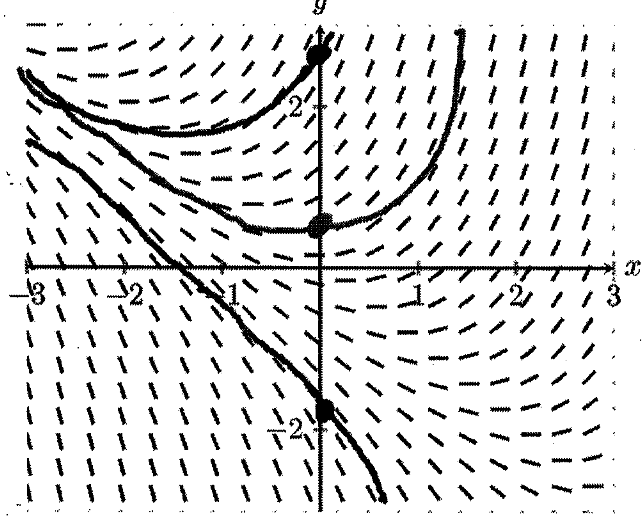
If you were to plug in various *x,y* points to the right side of the DE and evaluate, the result is the slope of the solution function at that point. This could be graphed by including slopes at many points, creating what is called a **slope field (or direction field)**:




...where the little line segments indicate the slope of the solution function curve at that *x,y* value.

## The solution curves follow the 'flow' of the slope field line segments

If you have found a solution by integrating and have not specified the value of the integration constant, the result is called the **general solution to the differential equation**, and repesents a family of function curves, each of which would follow the paths of the lineal elements in the slope field:




There are an infinite number of general solutions (corresponding to the infinite number of values we could set *C* to be).

But if we know the *y* value for a particular *x* value, this is called an **initial condition**, and it means we can establish one point on the solution function curve.

This solution which satisfies the DE but also passes through the initial condition point is called the **particular solution**.

## Finding a particular solution

To find a particular solution to a DE, we first find the general solution, then use the initial condition as a point that must be 'on the solution curve'...plugging in the *x* and *y* values and solving for the specific integration constant.

Ex: Find the particular solution:

$$y' = \frac{2}{x} \qquad y(1) = 8$$

$y = \int \frac{2}{x} dx$

$y = 2 \int \frac{1}{x} dx$

$y = 2\ln(x) + c$ (general solution)

$(8) = 2\ln(1) + c$

$8 = 0 + c$, so $C = 8$

$\boxed{y = 2\ln(x) + 8}$ (particular solution)

Ex: Verify the general solution, then find the particular solution:

$y = C_1 + C_2 \ln x$

*is a solution for* $xy'' + y' = 0$

$y(2) = 0$

$y'(2) = \frac{1}{2}$

$y' = C_2 \frac{1}{x} = C_2 x^{-1}$

$y'' = -C_2 x^{-2} = \frac{-C_2}{x^2}$

$xy'' + y' \overset{?}{=} 0$

$x\left(-\frac{C_2}{x^2}\right) + \left(C_2 \frac{1}{x}\right) \overset{?}{=} 0$

$-\frac{C_2}{x} + \frac{C_2}{x} \overset{?}{=} 0$ verified ✓

now: $y' = C_2 \frac{1}{x}$ & $y'(2) = \frac{1}{2}$

$\frac{1}{2} = C_2 \frac{1}{2}$, so $C_2 = 1$

then $y = C_1 + C_2 \ln x$

$y = C_1 + (1)\ln x$ and $y(2) = 0$

$0 = C_1 + (1)\ln(2)$ so $C_1 = -\ln(2)$

$\boxed{y = -\ln(2) + \ln x}$ (particular solution)

### We can't always find the solution function analytically

Not all differential equations will be in the forms we know how to solve (even after taking a full differential equations course). We may still need to know something about the solution, though - need to know the y for given x values in the solution.

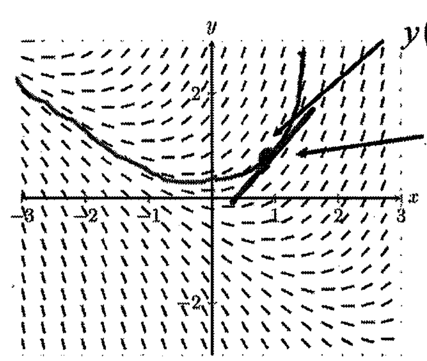
When we can't find a solution to a differential equation analytically, we use **numerical methods** to find the approximate solution - an approximate *y* value for any given value *x* for the solution curve, for a specific solution curve (that is, we need to be given an initial condition).

## Euler's Method

In this course we learn one such method, called **Euler's Method**. This method takes advantage of the fact that with a first-order differential equation of the form...

$$\frac{dy}{dx} = f(x, y) \qquad \text{...with an initial condition given:} \qquad y(x_0) = y_0$$

...we know the initial condition is on the solution curve, and we can plug in this $(x_0, y_0)$ into the differential equation to get the 'slope' of the tangent line to the solution curve at this point:



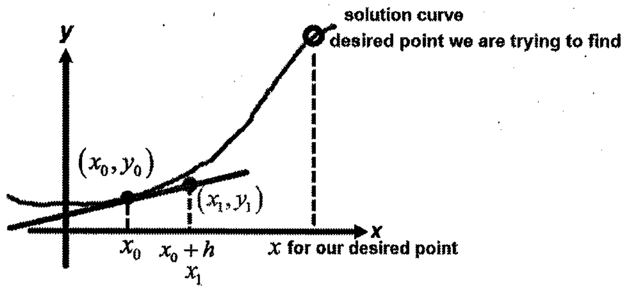

$y(x_0) = y_0$

$\frac{dy}{dx} = f(x_0, y_0) = \text{'slope'}$

## Euler's Method

Given: $\frac{dy}{dx} = f(x,y)$ ...with an initial condition given: $y(x_0) = y_0$

We can move a distance *h* away in x from the initial condition towards the x value we wish to know the solution curve *y* value...



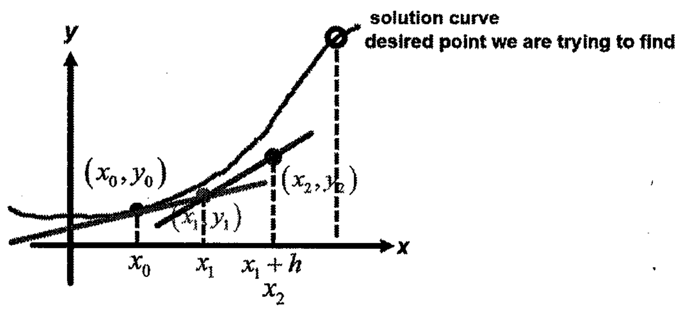

...and use the slope to compute an updated *y* value:

$$\frac{dy}{dx} = f(x_0, y_0) = \text{'slope'} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{h}$$

$$\text{so} \quad y_1 = y_0 + hf(x_0, y_0)$$

## Euler's Method

We then use this new point $(x_1, y_1)$ to establish the slope for another estimate moving closer to the desired point by plugging this point into the original DE *f(x,y)* function to establish the next slope:




This process continues iteratively until we are at the desired x value, and we then have an estimate of the *(x,y)* on the solution curve at the desired *x* value (and we trace out an approximation of the solution curve along the way).

### Easiest to understand this thorugh an example...

Use Euler's method to obtain a four-decimal approximation for $y(1.5)$ on the solution curve for $y' = 0.2xy \ \textit{with} \ y(1) = 1$

$$\frac{dy}{dx} = f(x_0, y_0) = \text{'slope'} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{h}$$

$$\text{so} \quad y_1 = y_0 + hf(x_0, y_0)$$

Let's set *h* = 0.1, taking 5 iterations to reach from x=1 to x=1.5:

$$(x_0, y_0) = (1,1)$$

$$y_1 = 1 + 0.1(0.2(1)(1)) = 1.02 \qquad (x_1, y_1) = (1.1, 1.02)$$

$$y_2 = 1.02 + 0.1(0.2(1.1)(1.02)) = 1.04244 \qquad (x_2, y_2) = (1.2, 1.04244)$$

$$y_3 = 1.04244 + 0.1(0.2(1.2)(1.04244)) = 1.0674... \qquad (x_3, y_3) = (1.3, 1.0674...)$$

$$y_4 = 1.0674.. + 0.1(0.2(1.3)(1.0674...)) = 1.0952... \qquad (x_4, y_4) = (1.4, 1.0952...)$$

$$y_5 = 1.0952.. + 0.1(0.2(1.4)(1.0952...)) = 1.125878.. \qquad (x_5, y_5) = (1.5, 1.125878...)$$

$$y(1.5) \approx 1.1259$$

Unit 8-2: Solving DEs using Separation of Variables
Larsen: 5.3

### Solving a Separable Differential Equation by Direct Integration

Solving a differential equation by direct integration requires that when you solve the DE for the derivative, the function on the RHS contains only the 'x' variable:

$$\frac{dy}{dx} = x^5$$

$$y = \int x^5 dx = \frac{1}{6}x^6 + C$$

### Solving a Separable Differential Equation by Separation of Variables

If you can solve the differential equation for the derivative, but the RHS function contains a mix of 'x' and 'y' variables, sometimes, you can still solve. First you must separate the variables - use algebra to rearrange the equation so that each side contains only one variable (including the differential)...

$$\frac{dy}{dx} = xy$$

$$\frac{1}{y}dy = x\,dx$$

$$\int \frac{1}{y}dy = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C_1$$

$$e^{\ln|y|} = e^{\frac{1}{2}x^2 + C_1}$$

$$y = e^{(\frac{1}{2}x^2 + C_1)}$$

$$y = e^{\frac{1}{2}x^2} e^{C_1}$$

$$\boxed{y = Ce^{\frac{1}{2}x^2}}$$

## Not all differential equations are separable

Separation of Variables works when the differential equation is first-order and when solved for the derivative, the RHS can be factored into two factors, each containing only one of the variables:

$$\frac{dy}{dx} = g(x)\,h(y)$$

...then it is possible to solve by integration.

| Separable DE | Non-Separable DE |
|---|---|
| $\frac{dy}{dx} = y^2 e^{3x+4y}$ | $\frac{dy}{dx} = y + \sin x$ |
| $\frac{dy}{dx} = y^2 e^{3x} e^{4y}$ | |
| $\frac{dy}{dx} = \left(e^{3x}\right)\left(y^2 e^{4y}\right)$ | |

Take Differential Equations - 2nd semester of next year's course - if you want to know more about how to solve other forms of differential equations :)

You may also be given an initial condition and asked to find the particular solution...

$2xy' - \ln(x^2) = 0 \qquad y(1) = 2$

$2x \frac{dy}{dx} = \ln(x^2)$

$dy = \frac{\ln(x^2)}{2x} dx$

$dy = \frac{2\ln(x)}{2x} dx$

$dy = \frac{\ln(x)}{x} dx$

$\int dy = \int \frac{1}{x} \ln(x) dx \longrightarrow$

$\int \frac{1}{x} \ln(x) dx$ by parts:

$u = \ln x \qquad dv = \frac{1}{x} dx$

$\frac{du}{dx} = \frac{1}{x} \qquad \int dv = \int \frac{1}{x} dx$

$du = \frac{1}{x} dv \qquad v = \ln x$

$uv - \int v\,du$

$\int \frac{1}{x} \ln x\,dx = (\ln x)^2 - \int \frac{1}{x} \ln x\,dx$ (original integral)

$2\int \frac{1}{x} \ln x\,dx = (\ln x)^2$

$\int \frac{1}{x} \ln x\,dx = \frac{(\ln x)^2}{2} + C$

$y = \frac{(\ln x)^2}{2} + C$

now $y(1) = 2$

$2 = \frac{(\ln 1)^2}{2} + C$

$2 = \frac{0}{2} + C$

$C = 2$

so $\boxed{y = \frac{(\ln x)^2}{2} + 2}$

**Try these...**

$\frac{dy}{dx} = \frac{3x^2}{y^2}$

$y^2 dy = 3x^2 dx$

$\int y^2 dy = \int 3x^2 dx$

$\frac{1}{3} y^3 = x^3 + C_1$

$y^3 = 3x^3 + 3C_1$

$\boxed{y^3 = 3x^3 + C}$

or

$\boxed{y = \sqrt[3]{3x^3 + C}}$

$\sqrt{x} + \sqrt{y}\,y' = 0 \qquad y(1) = 9$

$x^{1/2} + y^{1/2} \frac{dy}{dx} = 0$

$y^{1/2} dy = -x^{\frac{1}{2}} dx$

$\int y^{1/2} dy = -\int x^{1/2} dx$

$\frac{2}{3} y^{3/2} = -\frac{2}{3} x^{3/2} + C_1$

$y^{3/2} = -x^{3/2} + \frac{3}{2} C_1$

$y^{3/2} = -x^{3/2} + C$

now $y(1) = 9$

$9^{3/2} = -(1)^{3/2} + C$

$(\sqrt{9})^3 = -1 + C$

$27 = -1 + C$

$C = 28$

$\boxed{y^{3/2} = -x^{3/2} + 28}$

or

$\boxed{y = (-x^{3/2} + 28)^{2/3}}$

## Unit 8-3: DE Applications: Growth and Decay
## Larsen: 5.2

**Modeling = finding a (differential) equation to model a scenario**

The term 'modeling' is used to mean finding a function or equation (in this chapter, a differential equation) which applies to a given scenario, applying given conditions to solve for any constants and then using the differential equation and its solution to answer questions about the scenario.

**Growth/Decay proportion to amount**

There are many situations where the rate of change of a quantity (with respect to time) is proportional to the current amount of the quantity.

In problems like these, the appropriate differential equation to model this would be:

$$\frac{dy}{dt} = ky$$ ...where $k$ is a constant of proportionality.

**Some specific examples...**

**Unrestricted Population Growth:** $\frac{dP}{dt} = kP$

**Continuously Compounded Interest:** $\frac{dA}{dt} = kA$

**Radioactivity Decay:** $\frac{dQ}{dt} = kQ$

**Growth/Decay proportion to amount**

Let's take one of these, and solve for the form of the solution to the differential equation:

**Continuously Compounded Interest:** $\frac{dA}{dt} = kA$

This particular form can be solved using Separation of Variables...

$$\frac{dA}{dt} = kA$$

$$\frac{1}{A}dA = kdt$$

$$\int \frac{1}{A}dA = \int kdt$$

$$\ln|A| = kt + C_1$$

...and solving for A...

$$\ln|A| = kt + C_1$$

$$A = e^{kt+C_1}$$

$$A = e^{kt}e^{C_1}$$

$$A = Ce^{kt}$$

We usually have an initial condition: $A(0) = A_0$

...and can use it to solve for C... $A_0 = Ce^{k(0)}, \; C = A_0$

...which gives us the particular solution for the scenario: $\boxed{A = A_0e^{kt}}$

(for continuously compounded interest, the $k$ has a specific meaning: the annual interest rate)

## 'Proportional to' language

When problems state that a rate of change is proportional to something, we need to distinguish between directly proportional and inversely proportional:

"The rate of change of a population is proportional to the current population" $\frac{dP}{dt} = kP$

"The rate of change of the y is inversely proportional to the cube of time" $\frac{dy}{dt} = \frac{k}{t^3}$

"The rate of change of the population of y is proportional to the current value of y squared, and inversely proportional to the square root of time." $\frac{dy}{dt} = \frac{ky^2}{\sqrt{t}}$

## Using the differential equation model

Once we have found a differential equation to model a scenario, we can then find the solution function, and use initial conditions to establish the constants. Then we can use the completed model to answer other questions about the scenario.

Ex) The population of bacteria increases at a rate which is proportional to the amount of bacteria. A culture initially has 200 bacteria. At $t = 1$ hr, the population of bacteria has increased to 300 bacteria. If the rate of growth is proportional to the number of bacteria present, determine the time needed for the bacteria population to quadruple.

$\frac{dP}{dt} = kP$

$\int \frac{1}{P}\, dP = \int k\, dt$

$\ln|P| = kt + c_1$

$P = Ce^{kt}$

$P(0) = 200$

$200 = Ce^{k(0)}$

$C = 200$

$P = 200e^{kt}$

$P(1) = 300$

$300 = 200e^{k(1)}$

$e^k = \frac{300}{200} = \frac{3}{2}$

$k = \ln(\frac{3}{2})$

$P = 200e^{\ln(\frac{3}{2})t}$

$(4)(200) = 200e^{\ln(\frac{3}{2})t}$

$e^{\ln(\frac{3}{2})\cdot t} = 4$

$\ln(\frac{3}{2})\cdot t = \ln(4)$

$t = \frac{\ln(4)}{\ln(3/2)} = \boxed{3.419 \text{ hours}}$

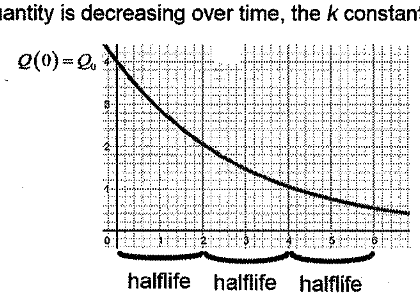
## Radioactive Decay

Radioactive substances spontaneously eject particles which results in the amount of material left decreasing over time. Initially, there is much material from which particles can eject so the rate of ejection is high, but as more and more material is removed, there is less material left from which particles can be ejected.

The rate of change of quantity of material with respect to time is proportional to the amount of material currently remaining, similar to population growth...

$\frac{dQ}{dt} = kQ$ ...which has the solution: $Q = Q_0 e^{kt}$

But because quantity is decreasing over time, the $k$ constant will be negative, producing radioactive decay.




"halflife" = the amount of time it takes for half of the quantity to be ejected.

## Radioactive Carbon Dating

An interesting application of this is using radioactive carbon dating to establish the age of previously living materials. In 1950, a chemist named Willard Libby found a way to use the ratio of the amount of radioactive carbon-14 to ordinary carbon in a living substance to establish the time when it died.

The action of cosmic radiation on nitrogen in the atmosphere turns some of the regular carbon into radioactive carbon-14, and the ratio of carbon-14 to regular carbon is constant and is absorbed by all living things, so the same ratio appears in the living tissues. But when an organism dies, the absorption of the carbon-14 by either breathing or eating stops. The regular carbon remains in the tissues, but the radioactive carbon-14 decays over time according to a radioactive decay model, and it is known that the 'half-life' of carbon-14 is 5,600 years (the time when only half of the initial carbon-14 remains).

Here is a specific example:

Ex) A fossilized bone is found to contain one-thousandth of the C-14 level found in living matter. Estimate the age of the fossil.

State an appropriate form DE: $\frac{dQ}{dt} = kQ$ Solve the DE: by separation of variables, solution is: $Q = Q_0 e^{kt}$

Use conditions to establish constants: For carbon-14, half-life=5600 yrs, so $Q(5600) = \frac{1}{2}Q_0$

$$\frac{1}{2}Q_0 = Q_0 e^{k(5600)}$$

$$e^{k(5600)} = \frac{1}{2}$$

$$5600k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5600} = -1.23776283 \cdot 10^{-4}$$

$$\boxed{Q = Q_0 e^{(-1.23776283 \cdot 10^{-4})t}}$$

Now answer the question: age when quantity is one-thousandth

$$Q(t_{age}) = \frac{1}{1000}Q_0 \qquad \frac{1}{1000}Q_0 = Q_0 e^{(-1.23776283 \cdot 10^{-4})t}$$

$$e^{(-1.23776283 \cdot 10^{-4})t} = \frac{1}{1000}$$

$$(-1.23776283 \cdot 10^{-4})t = \ln\left(\frac{1}{1000}\right)$$

$$t = \frac{\ln\left(\frac{1}{1000}\right)}{-1.23776283 \cdot 10^{-4}} = \boxed{55808 \text{ years}}$$

## Newton's Law of Cooling/Warming

A slightly different form DE is found in scenarios where objects starting at one temperature are immersed in a medium at a different temperature. Assuming the medium is large (the object isn't big enough to heat or cool the medium) than the rate of change (over time) of the temperature of the object is proportional to the difference between its temperature and the medium:

$$\frac{dT}{dt} = k(T - T_m) \quad \textit{where } T_m \textit{ is the temperature of the medium}$$

We can solve this using Separation of Variables:

$$\frac{dT}{T - T_m} = k\,dt$$

$$\int \frac{1}{T - T_m}\,dT = \int k\,dt$$

$$\ln|T - T_m| = kt + C_1$$

$$T - T_m = e^{kt + C_1}$$

$$T - T_m = Ce^{kt}$$

$$\boxed{T = T_m + Ce^{kt}}$$

## Newton's Law of Cooling/Warming example from Forensic Science

A murder victim's body is found by detectives who wish to establish the time of death. When the victim was alive, their body temperature was 98.6 °F, which as soon as death occurs, the body begins cooling towards the ambient temperature. This victim was found in a building with air conditioning which maintained the ambient temperature at a constant 78 °F. Detectives arrived on scene at 6:00am and found the core temperature of the body to be 84 °F. Core temperature was measured again at 6:30am and found to be 83 °F. What was the time of death?

We will use Newton's Law of Cooling which has solution: $T = T_m + Ce^{kt}$

Let's define 6:00am at t=0 and use hours, to 6:30am is t=0.5. Then time of death will be some negative time value.

$$T(0) = 84$$

$$T(0.5) = 83$$

$$T(t_{death}) = 98.6$$

$T(0) = 84$

$84 = 78 + Ce^{k(0)}$

$C = 84 - 78 = 6$

so $T = 78 + 6e^{kt}$

$T(0.5) = 83$

$83 = 78 + 6e^{k(0.5)}$

$6e^{.5k} = 5$

$e^{.5k} = \frac{5}{6}$

$.5k = \ln\left(\frac{5}{6}\right)$

$k = \frac{\ln(5/6)}{.5}$

so $T = 78 + 6e^{\frac{\ln(5/6)}{.5}t}$

Now when is $T = 98.6$?

$98.6 = 78 + 6e^{\frac{\ln(5/6)}{.5}t}$

$6e^{\frac{\ln(5/6)}{.5}t} = 20.6$

$e^{\frac{\ln(5/6)}{.5}t} = \frac{103}{30}$

$\frac{\ln(5/6)}{.5}t = \ln\left(\frac{103}{30}\right)$

$t = \frac{\ln\left(\frac{103}{30}\right)\cdot(.5)}{\ln(5/6)} = -3.3828$ hrs (before 6:00 AM)

.3828 hrs = 23 mins

3 hrs & 23 mins before 6:00 AM

time of death = $\boxed{2:37\text{ AM}}$

Unit 8-4: DE Applications: The Logistic Equation
Larsen: 5.4

**The Logistic Equation models population growth in an environment which can only support a defined maximum population**

We know that, in general, rate of population growth is proportional to the amount of population...

**Unrestricted Population Growth:** $\frac{dP}{dt} = kP$

...which produces this solution: $P = Ce^{kt}$

But more realistically, an environmental system has factors which limit the total population, things like availability of food and water. In these cases, the most common way to model population growth is with the following differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

The constant *L* is called the **carrying capacity** and represents the maximum population this system can support. By including this factor (which goes to zero as P approaches L) we are forcing the growth rate to zero as we approach this maximum population level.

**Solution: The Logistic Equation**

We can solve this using Separation of Variables...

$$\frac{dP}{dt} = kP\left(1-\frac{P}{L}\right)$$

$$\frac{1}{P\left(1-\frac{P}{L}\right)}dP = kdt$$

to proceed on the left, we employ Partial Fractions:

$$\frac{1}{P\left(1-\frac{P}{L}\right)} = \frac{A}{P} + \frac{B}{1-\frac{P}{L}}$$

$$A\left(1-\frac{P}{L}\right) + BP = 1$$

$$A - \frac{A}{L}P + BP = 1$$

$$\left(B-\frac{A}{L}\right)P + A = 0P + 1$$

*system*:

$$\begin{cases} B - \frac{A}{L} = 0 \\ A = 1 \end{cases}$$

$$B - \frac{1}{L} = 0, \ B = \frac{1}{L}$$

$$\frac{1}{P\left(1-\frac{P}{L}\right)} = \frac{A}{P} + \frac{B}{1-\frac{P}{L}} = \frac{1}{P} + \frac{\frac{1}{L}}{1-\frac{P}{L}}$$

$$\frac{1}{P\left(1-\frac{P}{L}\right)} = \frac{1}{P} + \frac{1}{L-P}$$

$$\int \frac{1}{P\left(1-\frac{P}{L}\right)}dP = \int kdt$$

$$\int \left[\frac{1}{P} + \frac{1}{L-P}\right]dP = \int kdt$$

$$\int \frac{1}{P}dP + \int \frac{1}{L-P}dP = \int kdt$$

$$u = L-P$$

$$du = -dP$$

$$\int \frac{1}{P}dP - \int \frac{1}{u}du = \int kdt$$

$$\ln|P| - \ln|u| = kt + C_1$$

$$\ln|P| - \ln|L-P| = kt + C_1$$

$$\ln\left|\frac{P}{L-P}\right| = kt + C_1$$

$$\frac{P}{L-P} = e^{kt+C_1} = C_2e^{kt}$$

$$P = C_2e^{kt}(L-P)$$

$$P = LC_2e^{kt} - PC_2e^{kt}$$

$$P + PC_2e^{kt} = LC_2e^{kt}$$

$$P\left(1+C_2e^{kt}\right) = LC_2e^{kt}$$

$$P = \frac{LC_2e^{kt}}{1+C_2e^{kt}}$$

$$P = \frac{L}{\frac{1}{C_2e^{kt}}+1}$$

$$P = \frac{L}{1+\frac{e^{-kt}}{C_2}}$$

$$\boxed{P = \frac{L}{1+Ce^{-kt}}}$$

## Solution: The Logistic Equation

This is the logistic equation: $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$

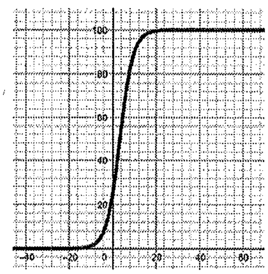
*L* is the carrying capacity (maximum population)

*C* is a constant which we need an initial condition to establish (population at time=0)

This is the solution function: $P = \frac{L}{1 + Ce^{-kt}}$

*k* is a 2nd constant which we need a second condition to establish

$$P = \frac{100}{1 + 3e^{-\frac{1}{3}t}}$$



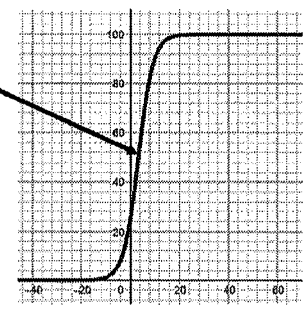
$$P(0) = \frac{100}{1 + 3e^{0}} = \frac{100}{1+3} = 25$$

$$\lim_{t \to \infty} \frac{100}{1 + 3e^{-\frac{1}{3}t}} = 100$$

*as time increases, the population approaches the carrying capacity, L*

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

At some value t, there is a point of inflection where concavity changes from up to down.



Let's find the point of inflection for this example: $P = \frac{100}{1 + 3e^{-\frac{1}{3}t}}$

Since this is the form of the solution... $P = \frac{L}{1 + Ce^{-kt}}$ $\quad k = \frac{1}{3},\ L = 100$

...so the original logistic equation DE would be: $\frac{dP}{dt} = \frac{1}{3}P\left(1 - \frac{P}{100}\right)$

To find inflection points, take 2nd derivative and set equal to zero: $\frac{dP}{dt} = \frac{1}{3}P\left(1 - \frac{P}{100}\right) = \frac{1}{3}P - \frac{1}{300}P^2$

(we must multiply by P' because of the Chain Rule) $\frac{d^2P}{dt^2} = \frac{1}{3}P' - \frac{1}{150}PP' = 0$

Now replace the P' factors with the original expression for $\frac{dP}{dt}$:

$$\frac{d^2P}{dt^2} = \frac{1}{3}\left(\frac{1}{3}P - \frac{1}{300}P^2\right) - \frac{1}{150}P\left(\frac{1}{3}P - \frac{1}{300}P^2\right) = 0$$

$$\frac{1}{9}P - \frac{1}{900}P^2 - \frac{1}{450}P^2 + \frac{1}{45000}P^3 = 0$$

$$\frac{1}{9}P - \frac{1}{300}P^2 + \frac{1}{45000}P^3 = 0$$

$$P^3 - 150P + 5000P = 0$$

$$P\left(P^2 - 150P + 5000\right) = 0$$

$$P = 0,\ P = \frac{150 \pm \sqrt{150^2 - 4(1)(5000)}}{2(1)} = \frac{150 \pm 50}{2} = 100,\ 50$$

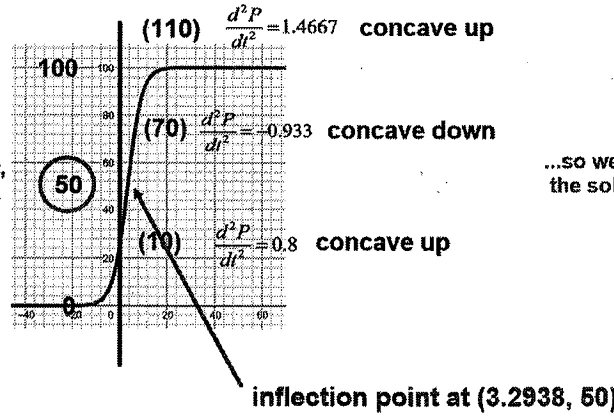
## The Logistic Equation

**Checking concavity around these values for *P*:**

$P = 0, 50, 100$

into $\frac{d^2P}{dt^2} = \frac{1}{3}\left(\frac{1}{3}P - \frac{1}{300}P^2\right) - \frac{1}{150}P\left(\frac{1}{3}P - \frac{1}{300}P^2\right)$

**This is the one we want, but we have a *P value...***




**...so we can plug this into the solution and find the matching *t*:**

$$P = \frac{100}{1+3e^{-\frac{1}{3}t}}$$

$$50 = \frac{100}{1+3e^{-\frac{1}{3}t}}$$

$$1+3e^{-\frac{1}{3}t} = \frac{100}{50} = 2$$

$$3e^{-\frac{1}{3}t} = 1$$

$$e^{-\frac{1}{3}t} = \frac{1}{3}$$

$$-\frac{1}{3}t = \ln\left(\frac{1}{3}\right)$$

$$t = \frac{\ln\left(\frac{1}{3}\right)}{\left(-\frac{1}{3}\right)} = 3.2958$$

## Slope Field for a Logistic Equation

So what does it mean to have concavity up for P values above 100?

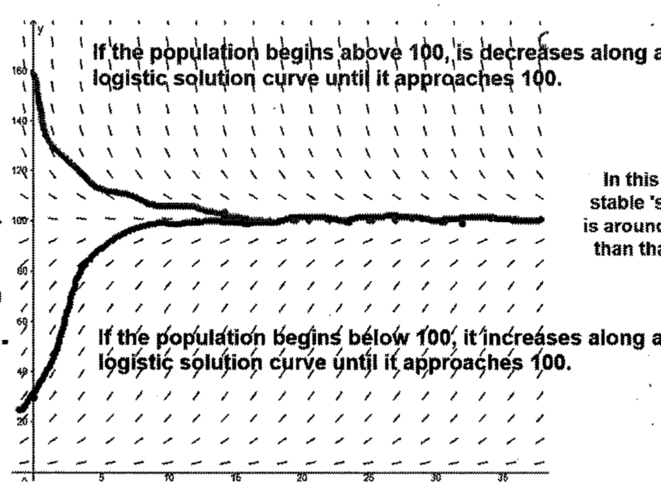
It doesn't really have a meaning in terms of the function for the solution curve for our initial condition, but it is interesting to consider the slope field for the original logistic differential equation. We have the differential equation which gives the slope for values in the domain, and it only depends upon the P value, so we can select P values and the lineal element slopes would be the same at that P value for all *t*:

| $P$ | $\frac{dP}{dt} = \frac{1}{3}P\left(1 - \frac{P}{100}\right)$ |
|---|---|
| 0 | 0 |
| 25 | 6.25 |
| 50 | 8.33 |
| 75 | 6.25 |
| 100 | 0 |
| 125 | -10.42 |
| 150 | -25 |

concave up

concave down

concave up




**In this system, there is only one stable 'state'...when the population is around 100. Any population other than that will approach this stable value over time.**

In our next (and last) section, we will look more closely at stability in more complex systems. But for this section, we are mainly asking you to get comfortable with the mechanics of working with logistics equations.

**One more example to work together...**

At time t = 0 a bacterial culture weighs 1 gram. Two hours later, the culture weights 4 grams. The maximum weight of the culture is 20 grams.

(a) Write a logistic differential equation that models the weight of the bacterial culture.

(b) Solve the differential equation.

(c) Find the culture's weight after 5 hours.

(d) When will the culture's weight reach 18 grams?

(e) After how many hours is the culture's weight increasing most rapidly?

(a) $L = 20$ $\quad \frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$ so $\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{20}\right)}$

(b) using the standard solution: $P = \frac{L}{1+Ce^{-kt}}$ $\quad P = \frac{20}{1+Ce^{-kt}}$

now, $P(0)=1$: $1 = \frac{20}{1+Ce^0}$

$1 = \frac{20}{1+C}$

$C+1 = \frac{20}{1}$, $C = 19$

So $P = \frac{20}{1+19e^{-kt}}$ $\quad$ then, $P(2)=4$: $4 = \frac{20}{1+19e^{-k(2)}}$

$1+19e^{-2k} = \frac{20}{4} = 5$

$19e^{-2k} = 4$

$e^{-2k} = \frac{4}{19}$, $-2k = \ln(4/19)$

$k = \frac{\ln(4/19)}{-2} = .77907$

so $\boxed{P = \frac{20}{1+19e^{-.77907t}}}$

(c) $P(5) = \frac{20}{1+19e^{-.77907(5)}} = \boxed{14.4256g}$

(d) $18 = \frac{20}{1+19e^{-.77907t}}$

$1+19e^{-.77907t} = \frac{20}{18} = \frac{10}{9}$

$19e^{-.77907t} = \frac{10}{9} - 1 = \frac{1}{9}$

$e^{-.77907t} = \frac{(1/9)}{19} = \frac{1}{171}$

$-.77907t = \ln\left(\frac{1}{171}\right)$

$t = \frac{\ln(\frac{1}{171})}{-.77907} = \boxed{6.60hrs}$

(e) weight increasing most rapidly when $\frac{dP}{dt}$ is a max

So need 2nd deriv and set = 0:

$\frac{dP}{dt} = (.77907)P\left(1 - \frac{P}{20}\right) = .77907P - .0389535P^2$

$\frac{d^2P}{dt^2} = .77907P' - .077907PP' \stackrel{set}{=} 0$ (chain rule)

replace $P'$s:

$.77907(.77907P - .0389535P^2) - .077907P(.77907P - .0389535P^2) = 0$

$.77907(.77907P - .0389535P^2) - .060695P^2 + .003035P^3 = 0$

$.60695P - .03034775P^2 - .060695P^2 + .003035P^3 = 0$

$.003035P^3 - .0910425P^2 + .60695P = 0$

$P^3 - 30P^2 + 200P = 0$

$P(P^2 - 30P + 200) = 0$

$P=0$ or $P = \frac{30 \pm \sqrt{30^2 - 4(1)(200)}}{2(1)} = \frac{30 \pm 10}{2} = 40$ or $10$

$\frac{dP}{dt}$ has critical values at $P = 0, 10, 40$, try each:

| $P$ | $\frac{dP}{dt} = (.77907)P\left(1 - \frac{P}{20}\right)$ |
|---|---|
| 0 | 0 |
| 10 | 3.89535 ← most rapidly (most positive 1st deriv) |
| 40 | −31.1628 |

at $P=10$, convert to $t$: $P = \frac{20}{1+19e^{-.77907t}}$

$10 = \frac{20}{1+19e^{-.77907t}}$

$1+19e^{-.77907t} = \frac{20}{10} = 2$

$19e^{-.77907t} = 1$

$e^{-.77907t} = 1/19$

$t = \frac{\ln(1/19)}{-.77907} = \boxed{3.78 \text{ hours}}$

# Unit 8-5: Predator-Prey Systems
# (not in Larsen - Stewart 9.6)

## Predator-Prey Systems of Differential Equations

If we look at how two variables are each changing with respect to time, that means there are now 3 variables: two y-variables both as functions of one x-variable. For example, we could consider how two different species interact in an environment: wolves and rabbits. One is a <u>predator</u> (meaning they eat the other species, here, the wolves) and the other is the <u>prey</u> (which serve as food for the predator, here the rabbits).

If there were no predators, and ample plant food for the rabbits, we could assume the rabbit population would follow unrestricted population growth:

$$\frac{dR}{dt} = aR$$

If wolves eat only rabbits, then wolves are fighting over a limited supply of food so the wolf population will decline at a rate proportional to the size of the wolf population (too many wolves, wolf population declines faster):

$$\frac{dW}{dt} = -bW$$

Now bring these populations together in the same ecosystem. We could assume that the species encounter each other at a rate proportional to the size of both populations - proportional to the product *RW.*

Since an encounter is beneficial for wolves, but detrimental to rabbits, we could combine all this behavior in the following system:

$$\begin{cases} \dfrac{dR}{dt} = aR - cRW \\ \dfrac{dW}{dt} = -bW + dRW \end{cases} \qquad \text{where } a, b, c, d \text{ are constants}$$

This particular model is known as the **Lotka-Volterra** equations.

The populations of rabbits and wolves may fluctuate, but the populations must change in a way which always makes both of these differential equations true.

What can we learn about the behavior of the populations, given this system? Let's look at a specific example with the constants filled in for a particular ecosystem:

$$\begin{cases} \dfrac{dR}{dt} = 0.08R - 0.001RW \\ \dfrac{dW}{dt} = -0.02W + 0.00002RW \end{cases}$$

We could look at the *R* and *W* values which make both derivatives zero. These are called the **equilibrium solutions.**

$$\begin{array}{ll} 0.08R - 0.001RW = 0 & -0.02W + 0.00002RW = 0 \\ 0.08R(1 - 0.0125W) = 0 & -0.02W(1 - 0.001R) = 0 \\ R = 0 \ \ or & W = 0 \ \ or \\ 1 - 0.0125W = 0 & 1 - 0.001R = 0 \\ W = 80 & R = 1000 \end{array}$$

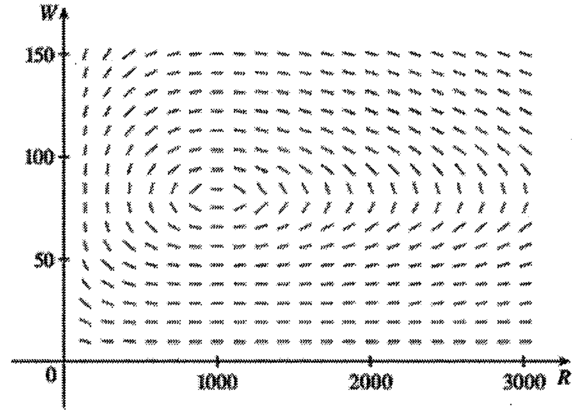
R=0, W=0 is the trivial solution: if there are no rabbits or wolves, then there are no population changes.

R=1000, W=80 is more interesting...this suggests that with these values there is a 'balance' between the populations, enough rabbits to keep the wolves alive but not too many wolves so that the rabbits get depleted. This is called an **equilibrium solution.**

But it isn't necessarily true that the populations just quickly approach and stay at these values. To see what actually happens, we need to produce something called a **Phase Diagram** which has on the x and y axes the population values of the rabbits and wolves.

First, we will need to combine the two derivatives in a way which eliminates the time variable. We can use the Chain Rule:

$$\frac{dW}{dt} = \frac{dW}{dR}\frac{dR}{dt} \quad so \quad \frac{dW}{dR} = \frac{\frac{dW}{dt}}{\frac{dR}{dt}}$$

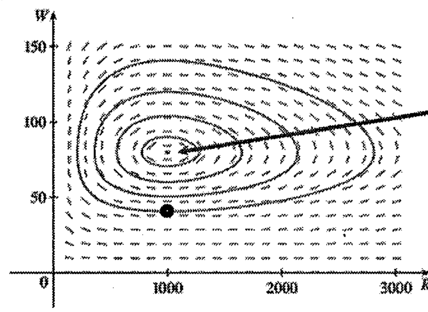



$$\frac{dW}{dR} = \frac{\frac{dW}{dt}}{\frac{dR}{dt}} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

We can then plug in various values of R and W to create a slope field.

If we start with some initial population values for the rabbits and wolves, a particular solution curve will follow the elements in this slope field. Different starting values will lead to different curves:

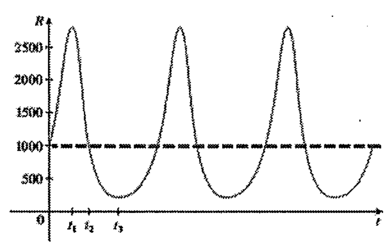
$$\frac{dW}{dR} = \frac{\frac{dW}{dt}}{\frac{dR}{dt}} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

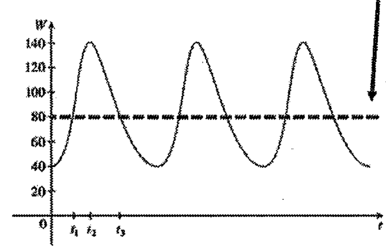



Here is the equilibrium point on the phase diagram. This is value around which each population is oscillating over time.

We could then look at how each population actually varies with respect to time. Consider the outer most curve...perhaps this started at the initial condition indicated by the dot: R=1000, W = 40.

Here is how the rabbit and wolf populations would change over time (following the curve in the phase diagram).



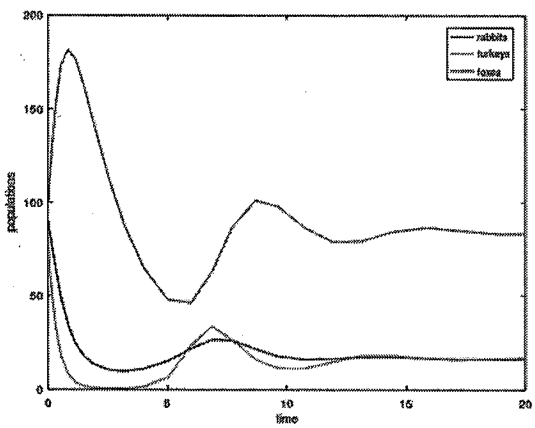
Although this is beyond the scope of this course (take next year's Calc III/Differential Equations :), we could consider the interaction of 3 species: one predator (foxes) and two different prey (rabbits and turkeys). Here is a possible system of differential equations:

$$\begin{cases} \dfrac{dR}{dt} = 0.01(100-R)R - 0.01RF \\ \dfrac{dT}{dt} = 0.04(80-T)T - 0.03TF \\ \dfrac{dF}{dt} = -0.5F + 0.01RF + 0.02TF \end{cases} \qquad R(0)=90 \quad T(0)=80 \quad F(0)=100$$

This assumes that rabbits and turkeys don't grow without bound, but uses a logistical growth model for rabbits and turkeys which is saying there are limits in the ecosystem other than foxes, assumes that interactions between turkeys and rabbits don't matter to population (no TR terms), and assumes that foxes are twice as likely to eat a turkey when encountered as a rabbit.

(Note: we could have used a logistical model for our rabbits two, in our 2 species system.)

In this particular scenario, the timeplots of the 3 species over time look like this...




**84 foxes**

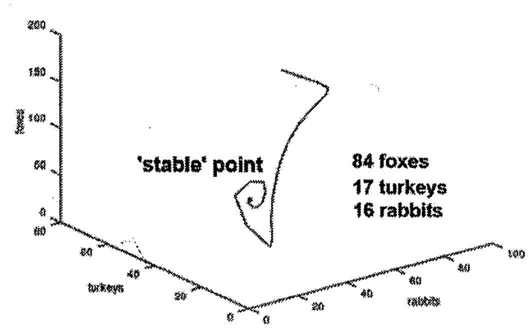
**all populations approach 'stable' values**

**17 turkeys**

**16 rabbits**

...and don't oscillate around the equilibrium value but actually approach and stay at this value. (This can also happen in our 2 species scenarios).

Because there are now 3 population variables, the phase diagram becomes a 3D plot:




**84 foxes**
**17 turkeys**
**16 rabbits**

In Differential Equations, we learn techniques for solving 3 and higher-dimensional systems of equations manually, but often need to resort to software (so we include a short unit at the end of that course where we show you how to use OCTAVE (an open-source, free, equivalent to MATLAB) to solve systems of differential equations and produce these plots.

There are lot of different ways to ask questions about Predator-Prey systems, so best to start working through the homework problems.