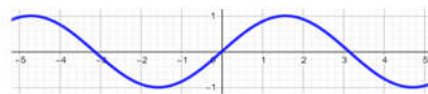# AP Calc BC – Lesson Notes – Unit 7: Infinite Series

Unit 7 is mainly about infinite series

It isn't intuitive, but sometimes you can represent a function with an infinite series: $f(x) = \sin x$
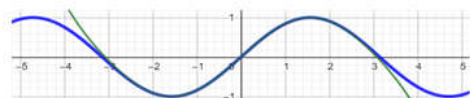
$$f(x) \approx x$$

$$f(x) \approx x - \frac{1}{3!}x^3$$

$$f(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

Most of this unit is about finding an infinite series to represent a given function, and to answer question about it, like over what interval of x does the series 'converge' to an accurate representation and how much error will the approximation allow for a given number of included terms.

$$f(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

$$f(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9$$

Why represent functions as series?

- Sometimes, it is easier to work with the series (especially if using only a few terms) as an approximation to the actual function.

- Series are also sometimes useful in proofs in more advanced mathematics. One example of a theorem which can be proved using infinite series is Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This formula is used frequently in physics and engineering to represent information with two orthogonal components as a single entity called a 'phasor'.

In AP Calculus BC, we are going to learn the definitions and procedures for finding and analyzing infinite series, but know that you will need these later if you take advanced science or engineering courses in college.

Unit 7-1: Sequences

Sequences are functions with integers as the domain

A **sequence** is just a list of numbers in a particular order, but it formally defined as a function whose domain is the set of positive integers (usually starting at n=1, although sometimes starting at n=0):

$$n: \qquad 1 \quad 2 \quad 3 \quad 4 \quad 5 \; ...$$
$$sequence: \quad 5 \quad 10 \quad 17 \quad 26 \quad 37 \; ...$$

For some sequences, it is possible to define the pattern by writing the output as a function of the value of n:

$$f(n) = a_n = (n+1)^2 + 1 \qquad \text{(this is the expression for the } n\text{th term of the above sequence)}$$

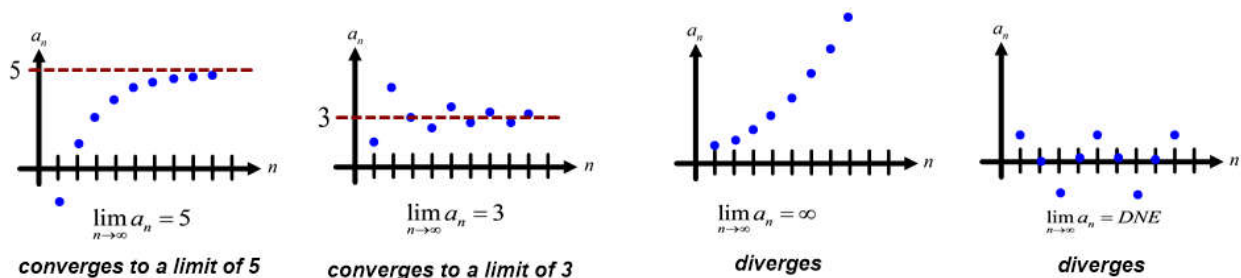n is called the **index** which locates the position of a particular number in the sequence

a_n is the expression defining the **n^{th} term** of the sequence

Given an expression for the nth term, you may be asked to write out some terms of the sequence:

#1. Write out the first 4 terms of the sequence defined by $\quad a_n = (-1)^{n-1} \dfrac{n}{n+1}$

Limit of a sequence

In calculus, we are mainly concerned with whether or not a sequence approaches a limiting value as n approaches infinity. Some sequences approach a single numerical value and these are said to **converge**. Other sequences go off to positive or negative infinity or oscillate between values and these are said to **diverge**.

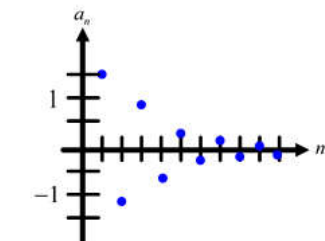| $\lim\limits_{n\to\infty} a_n = 5$	$\lim\limits_{n\to\infty} a_n = 3$	$\lim\limits_{n\to\infty} a_n = \infty$	$\lim\limits_{n\to\infty} a_n = DNE$
converges to a limit of 5	**converges to a limit of 3**	**diverges**	**diverges**

If a sequence converges, there are various methods we may need to employ to determine the limit of the sequence.

If you are allowed to use a calculator, you can enter the expression for a_n in as a function and use table features to quickly see if a sequence converges or diverges:

$$a_n = (-1)^{n-1} \dfrac{3n}{n^2+1}$$

...suggests this sequence is converging to 0

But to know for sure we need to find the limit analytically.

Finding the limit of a sequence analytically

$$a_n = (-1)^{n-1} \frac{3n}{n^2+1}$$

We first try evaluating the limit. After graphing, we can see that the sequence is converging to a number and the sign is oscillating, but the value around which it is converting is determined by: $\frac{3n}{n^2+1}$

We try evaluating this limit as n approaches infinity, initially by plugging in a large value:

$$\lim_{n \to \infty} \frac{3n}{n^2+1} = \frac{\infty}{\infty}$$

This is an indeterminant form, which allows us to use L'Hopital's Rule:

$$\lim_{n \to \infty} \frac{3n}{n^2+1} = \lim_{n \to \infty} \frac{3}{2n} = \frac{3}{\infty} = 0$$

So we verify that this sequence does indeed converge specifically to zero.

Two more examples:

Does the sequence defined by the expression for the general term converge, and if so, to what value?

#2. $a_n = \dfrac{5n^2+3n}{2n^2-n+4}$
#3. $a_n = \dfrac{n!}{(n+1)!}$

Properties of Limits of Sequences

A number of properties and theorems for functions and limits also apply to limits of sequences:

$Let \ \lim_{n \to \infty} a_n = L \ and \ \lim_{n \to \infty} b_n = K$

1) $Scalar \ multiple: \ \lim_{n \to \infty}(ca_n) = cL$

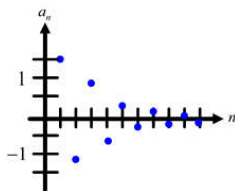2) $Sum \ or \ difference: \ \lim_{n \to \infty}(a_n \pm b_n) = L \pm K$

3) $Product: \ \lim_{n \to \infty}(a_n b_n) = LK$

4) $Quotient: \ \lim_{n \to \infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{L}{K} \ (for \ b_n \neq 0, K \neq 0)$

A number of properties and theorems for functions and limits also apply to limits of sequences:

5) $Absolute \ Value:$ If $\lim_{n \to \infty}|a_n| = 0$ then $\lim_{n \to \infty} a_n = 0$

6) $Squeeze \ Theorem:$ If $\lim_{n \to \infty} a_n = L = \lim_{n \to \infty} c_n$ and there exists an integer N such that $a_n \leq b_n \leq c_n$ for all $n > N$, then $\lim_{n \to \infty} b_n = L$.

(If a series is oscillating around zero, then the sequence converges to zero)
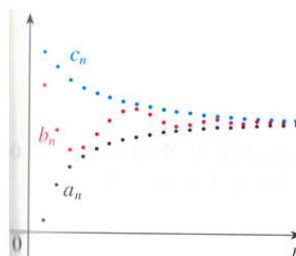
(If you can't evaluate the limit of the sequence directly but can find functions which bound the sequence above and below which both approach the same value, then the original sequence converges to this value - rarely used)

Monotonic and Bounded Sequences

If we just want to know if a sequence converges but do not necessarily need the limiting value we can use the following definitions and theorem:

> A sequence a_n is **monotonic** when its terms are nondecreasing: $a_1 \le a_2 \le a_3 \le \ldots a_n \le \ldots$
> or nonincreasing: $a_1 \ge a_2 \ge a_3 \ge \ldots a_n \ge \ldots$

> A sequence a_n is **bounded above** when there is a real number M such that $a_n \le M$ for all n. The number M is called an **upper bound** of the sequence.
>
> A sequence a_n is **bounded below** when there is a real number N such that $N \le a_n$ for all n. The number N is called a **lower bound** of the sequence.
>
> A sequence a_n is **bounded** when it is bounded above and bounded below. $N \le a_n$

> If a sequence a_n is **bounded and monotonic**, then it **converges**.

This example shows how these definitions are used...

Determine whether the sequence is monotonic and whether it is bounded: $a_n = \dfrac{3n}{n+2}$

Monotonic? Use the derivative to determine where increasing/decreasing:

$$f(n) = \frac{3n}{n+2} \qquad f'(n) = \frac{(n+2)3 - 3n(1)}{(n+2)^2} = \frac{2}{(n+2)^2} \qquad f'(n) = 0 \text{ or } DNE: \textbf{\textit{nowhere and testing }} n = 0, f'(0) > 0 \text{ (increasing)}$$

$-\infty$ —————————————— ∞
increasing everywhere

...so the sequence is **monotonic** (specifically, nondecreasing)

<u>Bounded?</u> Since it is increasing, the lower bound value would be at $n = 1$: $\qquad a_1 = \dfrac{3(1)}{(1)+2} = 1$ lower bound

The upper bound would be as n approaches infinity: $\qquad a_n = \lim\limits_{n \to \infty} \dfrac{3n}{n+2} = \dfrac{\infty}{\infty}$

$$\text{by } L'Hopital: \; = \lim\limits_{n \to \infty} \frac{3}{1} = 3 \qquad \text{upper bound}$$

...so the sequence is **bounded**

Therefore the sequence **converges**.

Finding an expression for the nth term

In order to analytically determine convergence, we must have an expression for the nth term, and some problems just directly ask us to find this for a given sequence.

First, you can determine if the sequence is <u>arithmetic</u> or <u>geometric</u> which each have defined formulas:

<u>arithmetic:</u>   10   7   4   1 -2 $a_n = a_1 + (n-1)d$

            -3   -3   -3   -3   the common difference, d $a_n = 10 + (n-1)(-3) = 10 - 3n + 3 = 13 - 3n$

<u>geometric:</u>   100   50   25 $\dfrac{25}{2}$ $\dfrac{25}{4}$

     $*\frac{1}{2}$ $*\frac{1}{2}$ $*\frac{1}{2}$ $*\frac{1}{2}$ the common ratio, r $a_n = a_1(r)^{n-1}$

$$a_n = 100\left(\frac{1}{2}\right)^{n-1}$$

If the signs alternate, include multiplying by $(-1)^n$ or $(-1)^{n-1}$

$$-100 \quad 50 \quad -25 \quad \frac{25}{2} \quad -\frac{25}{4} \qquad a_n = (-1)^n 100\left(\frac{1}{2}\right)^{n-1}$$

Finding an expression for the nth term

You can also try writing the values of n below each term, then try different operations on n:

sequence:	⑤	⑩	⑰	㉖	37	← these...
n:	1	2	3	4	5	
n^2:	1	④	⑨	⑯	㉕	← ...are one larger than these

$$a_n = (n+1)^2 + 1$$

More examples...

#4. List the first 4 terms of the sequence:

$$a_n = 1 - (0.2)^n$$

[calculator function seq under list, OPS]

#5. Find an expression for the nth term:

$$\left\{ 1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27} .... \right\}$$

#6. Determine if the sequence converges and if so find the limiting value:

$$a_n = \frac{3 + 5n^2}{n + n^2}$$

$$a_n = \frac{n!}{2^n}$$

#7. Determine if the sequence is monotonic, bounded, and if it converges:

$$a_n = \frac{n}{n^2 + 1}$$

Unit 7-2: Series and Convergence

Series vs. Sequences

A **sequence** is a set of numbers in a particular order but a **series** is the summation of the terms of a sequence. We typically use sigma notation along with the expression for the nth term of the series. Comparing a sequence and corresponding series:

Sequence: $\quad a_n = \dfrac{1}{2^n} \qquad \dfrac{1}{2} \quad \dfrac{1}{4} \quad \dfrac{1}{8} \quad \dfrac{1}{16} \quad \dots$

Series: $\quad \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{2^n} = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} \quad \dots$

Finite Series: If there are a fixed number of terms in the series, we say the series is a finite series:

$$\sum_{n=1}^{4} \dfrac{1}{2^n} = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} = \dfrac{15}{16}$$

Finite series always converge to a computed sum value.

Infinite Series: If there are an infinite number of terms, we say the series is an infinite series:

$$\sum_{n=1}^{\infty} \dfrac{1}{2^n} = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} \quad \dots = ??$$

Infinite series may or may not converge to a sum value (much of this unit is about how to determine if infinite series converge and if so to what sum).

Convergence of Series vs. Sequences

We say a **sequence** converges if the values of the terms approach a numerical value as n approaches infinity...

Sequence: $\quad a_n = \dfrac{1}{2^n} \qquad \dfrac{1}{2} \quad \dfrac{1}{4} \quad \dfrac{1}{8} \quad \dfrac{1}{16} \quad \dots$

...and our main way to determine convergence is to compute the limit of the expression for the nth term as n approaches infinity:

$$\lim_{n \to \infty} \dfrac{1}{2^n} = \dfrac{1}{\infty} = 0 \quad \text{...so this sequence converges to a value of 0.}$$

We say a **series** converges if the sum of the terms approaches a numerical value as n approaches infinity.

$$\sum_{n=1}^{\infty} \dfrac{1}{2^n} = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} \quad \dots = ??$$

NOTE: The value to which the series converges is different than the value to which the sequence converges.

One way to determine convergence of a series is to consider the **sequence of partial sums** of a series. A **partial sum** is the sum of the terms up to a specified value of n:

Partial sums:

$$S_1 = \dfrac{1}{2}$$

$$S_2 = \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{3}{4}$$

$$S_3 = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} = \dfrac{7}{8}$$

$$S_4 = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} = \dfrac{15}{16}$$

$$S_n = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \dots + \dfrac{1}{2^n} = \dfrac{2^n - 1}{2^n}$$

To determine if a series converges, instead of taking the limit of the expression for the nth term, we take the limit of **an expression for the nth partial sum** as n increases...

$$\lim_{n \to \infty} \dfrac{2^n - 1}{2^n} = 1$$

...since this limit approaches the number 1, this infinite series **converges** and its sum is 1.

Convergence of Series vs. Sequences

This particular series has an interesting geometrical interpretation as well...

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} \; ... = ??$$

...each term is a fraction which adds up to the total area of a 1 x 1 square:
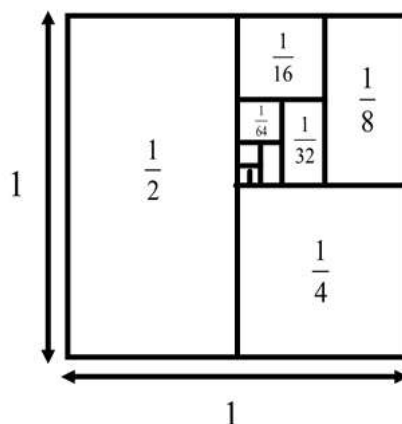
$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

$$S_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$$

$$S_4 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}$$

$$S_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...+\frac{1}{2^n}=\frac{2^n-1}{2^n}$$

$$\lim_{n\to\infty}\frac{2^n-1}{2^n}=1$$

Example: Determine if the series converges and if so, to what sum.

$$\sum_{n=1}^{\infty}1$$

$$\sum_{n=1}^{\infty}1=1+1+1+1 \; ...$$

The nth partial sum would be: $S_n = n$

So we can determine convergence using the limit: $\lim_{n\to\infty}(n)=\infty$

So this series **diverges**.

Determining convergence of infinite series when it is difficult to find S_n, Properties

In the examples we've seen so far, we have been able to determine convergence by writing out some terms of the series, and then finding a formula for the nth partial sum, then taking the limit of this partial sum expression as n approaches infinity.

But sometimes it is difficult to determine a formula for the nth partial sum, so we need to resort to other techniques to determine convergence and/or to find the sum:

- **Recognizing Known Series**: Sometimes, we can recognize that a given series is a particular known type which has conditions which determine whether or not it converges and what sum it converges to.

- **Theorems and Tests**: Sometimes, we can employ certain theorems or 'tests' to determine whether a series converges or diverges

For series which converge, the following properties also apply:

Let $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ be convergent series, and let A, B, and c be real numbers.

If $\sum_{n=1}^{\infty}a_n=A$ and $\sum_{n=1}^{\infty}b_n=B$, then the following series converge to the indicated sums:

$$\sum_{n=1}^{\infty}ca_n=cA$$

$$\sum_{n=1}^{\infty}(a_n\pm b_n)=A\pm B$$

Geometric Series

One frequently encountered infinite series pattern we can recognize is the **Geometric Series**:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots + ar^4 + \ldots \quad (for\ a \neq 0)$$

We were introduced to this back in honors algebra 2, but can now prove some things about geometric series.

The expression for the nth partial sum would be: $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

if we multiply this by r and then subtract this result from S_n: $\quad S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

$$-rS_n = \quad ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$$

$$(1-r)S_n = a - ar^n$$

$$(1-r)S_n = a(1-r^n)$$

$$S_n = \frac{a}{1-r}(1-r^n)$$

Taking the limit: $\lim_{n \to \infty} \frac{a}{1-r}(1-r^n)$ since we are subtracting the term r^n as n increases, this will go to zero only if $|r| < 1$

Therefore, a geometric series will converge if $|r| < 1$ and will converge to the sum $\frac{a}{1-r}$

Examples: Determine if the series converges and if so to what sum.

#1. $\sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n$

$a = 3\left(\frac{1}{4}\right)^0 = 3$

$r = \frac{1}{4}$ converges

$to: \frac{a}{1-r} = \frac{3}{1-\frac{1}{4}} = \frac{3}{\left(\frac{3}{4}\right)} = 4$

#2. $\sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^n$

$a = 3\left(\frac{1}{4}\right)^1 = \frac{3}{4}$

$r = \frac{1}{4}$ converges

$to: \frac{a}{1-r} = \frac{\left(\frac{3}{4}\right)}{1-\frac{1}{4}} = \frac{\left(\frac{3}{4}\right)}{\left(\frac{3}{4}\right)} = 1$

#3. $\sum_{n=1}^{\infty} 3(1.1)^n$

$a = 3(1.1)^1 = 3.3$

$r = 1.1 \quad diverges$

You can use geometric series to investigate repeating decimals...

Example: Write the repeating decimal as a geometric series and as a ratio of two integers: $0.\overline{36}$

#4. $0.\overline{36} = 0.36363636363636\ldots$

$= 0.36 + 0.0036 + 0.000036 + 0.00000036 + \ldots$

$= \sum_{n=0}^{\infty} 0.36\left(\frac{1}{100}\right)^n$

$a = 0.36$

$r = \frac{1}{100}$ converges

$to: \frac{a}{1-r} = \frac{0.36}{1-\frac{1}{100}} = \frac{0.36}{\frac{99}{100}} = \frac{100(0.36)}{99} = \frac{36}{99}$

The nth-term test for divergence

If we can't find an expression for the nth partial sum, and the series isn't geometric, we can at least determine if the series diverges using the following theorem and test.

Assume that $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L$ (even though we can't determine the S_n)

It would be true that $S_n = S_{n-1} + a_n$

If we take $L = \lim_{n\to\infty} S_n$

$$= \lim_{n\to\infty}(S_{n-1} + a_n)$$

$$= \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n$$

We know $\lim_{n\to\infty} S_n = L$

but since n is approaching infinity it is also true that $\lim_{n\to\infty} S_{n-1} = L$

so it must be true that $L = L + \lim_{n\to\infty} a_n$

which implies that the sequence $\{a_n\}$ converges to zero

Therefore: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

This is a theorem

This theorem isn't super useful directly:

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

But remember from geometry that you can either negate the sides of a statement or invert them to form the inverse, converse, and contrapositive? One of these, the contrapositive is always true if the statement is true, so the contrapositive of this theorem is also always true:

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

This is called the '**nth-term test for Divergence**'

The contrapositive of a true statement is always true, but the inverse or converse are not necessarily true or false.

We can therefore only use this test to demonstrate that a series diverges.

'nth-term test for Divergence'

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

1) Compute the limit of the sequence for the series $\lim_{n\to\infty} a_n$

2) If the limit is non-zero: you can state that the infinite series diverges.

If the limit is zero, you can make no conclusions: the infinite series might converge or might not converge (and if it does converge, you do not know that sum to which it converges).

Examples

Determine if the series converges and if so to what sum.

#5. $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$

#6. $\sum_{n=1}^{\infty} \frac{1}{n}$

Examples

Determine if the series converges and if so to what sum.

#7. $-2 + \dfrac{5}{2} - \dfrac{25}{8} + \dfrac{125}{32} - \ldots$

#8. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\dfrac{n}{2n+5}\right)$

Determine if the series converges and if so to what sum.

#9. $\displaystyle\sum_{n=1}^{\infty} \dfrac{n}{\sqrt{1+n^2}}$

Find the values of x for which the series converges, and find the sum of the series for those values of x.

#10. $\displaystyle\sum_{n=1}^{\infty} \dfrac{x^n}{3^n}$

Unit 7-3: The Integral Test and *p*-series

The Integral Test

Much of this unit is about whether or not a series converges and if so to what sum. We've seen that sometimes we can recognize a given series as a particular form for which there are specific rules (like telescoping or geometric series) and we can sometimes use theorems (like the n-th term test) for at least determining convergence. In this section, we expand our list of recognizable series and theorems about convergence with **The Integral Test**:

If f is positive, continuous, and <u>decreasing</u> for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\, dx$$

either both converge or both diverge.

This theorem applies when you have a series whose terms are always positive and always decreasing...in this case, we can use an integral of the a_n expressed as a function and if the integral converges then the series converges.

> **Important note**: If the integral and series converge, the series **does not** (necessarily) converge to the same value as the integral. We are just testing to see <u>if</u> the series converges at all.

The Integral Test - proof

Much of this unit is about whether or not a series converges and if so to what sum. We've seen that sometimes we can recognize a given series as a particular form for which there are specific rules (like telescoping or geometric series) and we can sometimes use theorems (like the n-th term test) for at least determining convergence. In this section, we expand our list of recognizable series and theorems about convergence with **The Integral Test**:

If you have a function decreasing from x = 1 onward and represent the area between x = 1 and x = n, we could set the rectangle height using the RHS, but would mean starting at $i = 2$ and we would obtain the *inscribed area*:

$$\text{inscribed area} = \sum_{n=2}^{n} f(i)$$
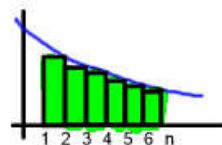
Using the LHS for rectangle height we would start at $i = 1$ and obtain the *circumscribed area*:

$$\text{circumscribed area} = \sum_{n=1}^{n-1} f(i)$$

The exact area is between these two values:

$$\sum_{n=2}^{n} f(i) \leq \int_1^n f(x)\, dx \leq \sum_{n=1}^{n-1} f(i)$$

Using the *n*th partial sum: $S_n = f(1) + f(2) + f(3) + \ldots + f(n)$

the previous area statement can be written as: $S_n - f(1) \leq \int_1^n f(x)\, dx \leq S_{n-1}$

Assuming that the integral converges to a sum which we will call L, then: $S_n - f(1) \leq L$

$$S_n \leq L + f(1)$$

Therefore $\{a_n\}$ is bounded and monotonic, and by previous theorem it therefore converges, so $\sum a_n$ converges.

Examples using the Integral Test

Use the Integral Test to determine if the series converges or diverges:

#1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

#2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$

$\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

The series doesn't have to start at n=1

Use the Integral Test to determine if the series converges or diverges:

#3. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2+1}$

p-Series and Harmonic Series

Here are some more recognizable forms of infinite series:

p-Series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$

"The Harmonic Series": $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

"The Harmonic Series" is called that because it relates to music theory. If a string (or column of air in a wind or brass instrument) is vibrating, it can vibrate in different modes:

pitch (frequency)

0

this mix of frequencies gives the instrument its timbre
(unique sound quality)
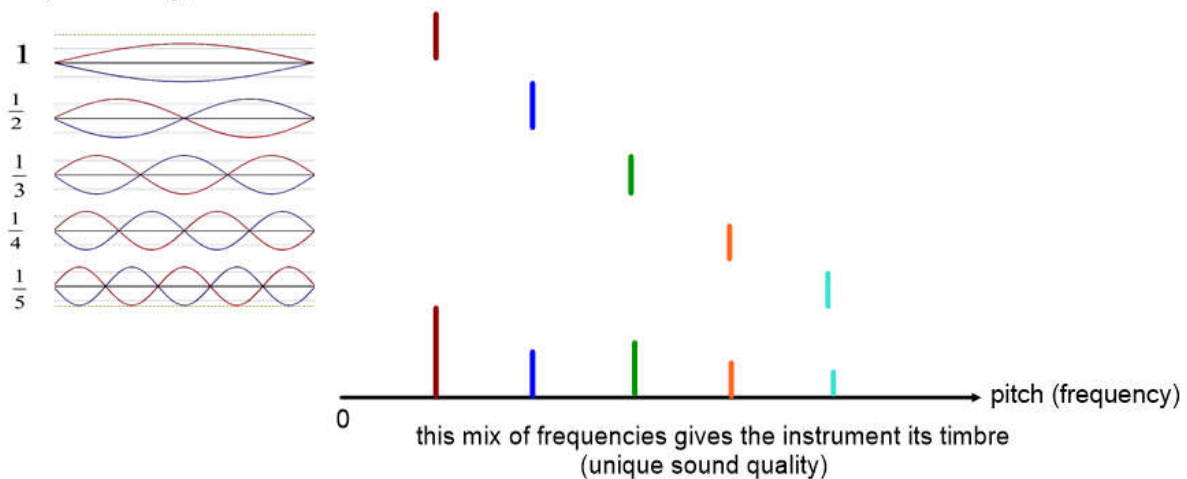
Convergence of *p*-Series

There is a theorem for determining convergence of *p*-series:

The *p*-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^p} = \dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \dfrac{1}{4^p} + \ldots$ converges for $p > 1$, and diverges for $0 < p \le 1$.

Proof: Using the Integral Test, $f(x) = \dfrac{1}{x^p}$ and $\displaystyle\int_1^{\infty} \dfrac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx$

$$= \left[\dfrac{x^{-p+1}}{-p+1} \right]_1^{\infty}$$

$$= \left[\lim_{b \to \infty} \dfrac{b^{-p+1}}{-p+1} \right] - \left[\dfrac{(1)^{-p+1}}{-p+1} \right]$$

$$= \left[\lim_{b \to \infty} \dfrac{b^{-p+1}}{-p+1} \right] + \dfrac{1}{p} - 1$$

If the exponent is positive, $-p+1 > 0$, then b^{-p+1} will increase without bound and the integral and series diverge.
The integral (and series) will only converge if the exponent is negative, because this moves the b^{-p+1} term to the bottom of the fraction. Therefore, the *p*-series will converge when $-p+1 < 0$

which is when $p > 1$

Examples using the convergence of *p*-series theorem

Determine if the series converges or diverges:

#4. $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n}$

#5. $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^2}$

More examples

Determine if the series converges or diverges:

#6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

#7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

#8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$

#9. Determine if the series converges or diverges using both the integral test and the p-series theorem: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

#10. Find the values of p for which the series converges: $\sum_{n=1}^{\infty} n\left(1+n^2\right)^p$

Unit 7-4: Comparison of Series

Comparison Tests

We've seen a number of ways to check whether a series converges or diverges which involve either recognizing the form of the series or performing some calculation using the series itself or a function representing the series. But some series are difficult to analyze in this way.

So instead, we sometimes determine if a series converges or diverges by comparing it in some way to another series for which it is easier to determine divergence/converge. There are called <u>comparison tests</u> and in this section we will example two of these:

    <u>**The Direct Comparison Test**</u>: Used when we know that one term is always higher than another, term by term.

    <u>**The Limit Comparison Test**</u>: Used when the ratio of two series is always positive as n approaches infinity (meaning that, eventually for large n, one series converges to a higher value than another, even though we can't say every term in one series is higher than the corresponding term in the other series).

The Direct Comparison Test

Here is the formal statement:

> Let $0 < a_n \le b_n$ for all n.
>
> If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
>
> If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Instead of a proof, let's understand what this is saying intuitively. If you have two series, and comparing each term, term-by-term, one series (the *b* series) terms are always higher than the *a* series terms, then one of two things can happen:
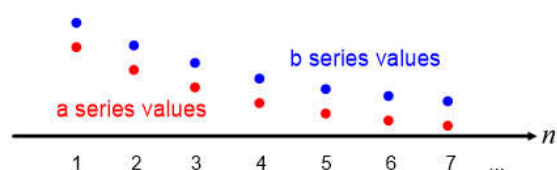
If the b series is converging to a value, it 'squeezes' the a between this value and zero, so a must also be converging (although not to any particular value)
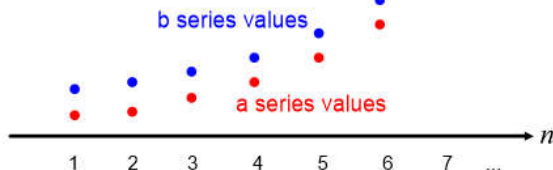
If the a series diverges, then it 'pushes' the b values up too, so the b series must also diverge

The way we use direct comparison is for our given series, we need to identify a different series which we can show is always above or below our series (term by term) but for which it is easier to determine convergence.

Example: Determine if the following series converges or diverges: $\sum_{n=1}^{\infty} \dfrac{1}{2+3^n}$

This series is similar in form to $\sum_{n=1}^{\infty} \dfrac{1}{3^n}$ and we can write out a few terms of each series...

n	1	2	3	4	
$\sum_{n=1}^{\infty}\dfrac{1}{2+3^n}$	$\dfrac{1}{5}$	$\dfrac{1}{11}$	$\dfrac{1}{29}$	$\dfrac{1}{83}$	$\sum_{n=1}^{\infty}\dfrac{1}{2+3^n}=\sum_{n=1}^{\infty}a_n$
$\sum_{n=1}^{\infty}\dfrac{1}{3^n}$	$\dfrac{1}{3}$	$\dfrac{1}{9}$	$\dfrac{1}{27}$	$\dfrac{1}{81}$	$\sum_{n=1}^{\infty}\dfrac{1}{3^n}=\sum_{n=1}^{\infty}b_n$

...to convince ourselves that $0 < a_n \le b_n$

Our new series is a geometric series $\sum_{n=1}^{\infty}\dfrac{1}{3^n}=\sum_{n=1}^{\infty}1\left(\dfrac{1}{3}\right)^n$ with $r=\dfrac{1}{3}$ and since $|r|<1$ $\sum_{n=1}^{\infty}\dfrac{1}{3^n}$ converges.

Since this series is always higher than our original series, $\sum_{n=1}^{\infty}\dfrac{1}{2+3^n}$ converges.

The Direct Comparison Test

Here is an example that shows the divergent case:

Example: Determine if the following series converges or diverges: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$

This series is similar in form to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ and because the denominator of our original series is always 2 higher, this new series is always higher then the original series: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}=\sum_{n=1}^{\infty}a_n \qquad \sum_{n=1}^{\infty}\frac{1}{n^{1/2}}=\sum_{n=1}^{\infty}b_n \qquad 0<a_n \le b_n$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ is a p-Series with $p=\frac{1}{2}$ which diverges (p must be > 1 for convergence).

Since this new series is higher series is above our original series, it diverging doesn't tell us anything about our original series - turns out this was a bad choice for the other series.

We will need to pick a different series to compare to, and maybe one which is instead lower than our original series.

To stay under our original series, we need the denominator to grow more rapidly with n, so let's try the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$

n	1	2	3	4	5	6	7	
$\displaystyle\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$	0.33	0.29	0.27	0.25	0.24	0.22	0.215	(using calculator table function to check)
$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$	1	0.5	0.33	0.25	0.20	0.17	0.143	

This new series starts with terms higher than the original, but for n=5 and higher, it remains below our original series. It doesn't really matter what happens with the first terms - this is all about convergence/divergence as $n \to \infty$

So we will now call the new series *a* and the original series *b* so we have:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\sum_{n=1}^{\infty}a_n \qquad\qquad \sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}=\sum_{n=1}^{\infty}b_n \qquad 0<a_n \le b_n \quad [for\ n \ge 4]$$

This new series is a p-Series with $p = 1$, so it diverges.

Since this series is always lower than our original series (for n > 4), $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$ *diverges*.

#1. Example: Determine if the following series converges or diverges: $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{2^n-1}$

The Limit Comparison Test

Sometimes a series closely resembles another series, but it is difficult to establish, term-by-term, that even after some value of n one series is always higher than the other. In this case, we can use a different test:

$$\text{If } a_n > 0, \ b_n > 0, \ \text{and } \lim_{n \to \infty} \frac{a_n}{b_n} = L \text{ and } L \text{ is } finite \text{ and } positive, \text{ then}$$

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \quad \text{either } both \ converge, \text{ or } both \ diverge.$$

(See the textbook if you want formal proof of this or the Direct Comparison Test.)

Let's consider a few examples to see how it works...

#2. Example: Determine if the following series converges or diverges: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$

For a similar form series, consider ignoring the constants in the denominator: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Both series have positive terms, so now we need to evaluate the limit $\displaystyle\lim_{n \to \infty} \frac{a_n}{b_n}$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\left(\dfrac{1}{3n^2 - 4n + 5}\right)}{\left(\dfrac{1}{n^2}\right)}$$

$$= \lim_{n \to \infty} \frac{n^2}{3n^2 - 4n + 5} = \frac{\infty}{\infty}$$

by L'Hopital:

$$= \lim_{n \to \infty} \frac{2n}{6n - 4} = \frac{\infty}{\infty}$$

by L'Hopital:

$$= \lim_{n \to \infty} \frac{2}{6} = \frac{1}{3} \quad converges$$

This limit ($L = 1/3$) is *finite* and *positive* so these two series will either both diverge or both converge. Evaluating the easier series:

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a *p*-Series with $p = 2$ so it converges, therefore

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$ also converges.

#3. Example: Determine if the following series converges or diverges: $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Let's try this series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\left(\sin\left(\dfrac{1}{n}\right)\right)}{\left(\dfrac{1}{n}\right)} = \frac{\sin(0)}{0} = \frac{0}{0}$$

by L'Hopital:

$$= \lim_{n \to \infty} \frac{\cos\left(\dfrac{1}{n}\right) \bullet \left(-n^{-2}\right)}{-n^{-2}} = \lim_{n \to \infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$$

So $L = 1$ is finite and positive, both series converge or diverge together.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is a *p*-Series with $p = 1$ so it diverges, therefore

$\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ also diverges.

The Limit Comparison Test will often work in place of the Direct Comparison Test

#4. Example: Determine if the following series converges or diverges: $\sum\limits_{n=1}^{\infty} \dfrac{1}{2+\sqrt{n}}$

...and determined that the series diverges. It was difficult to find a series that was term-by-term larger (and we had to select one which only became larger after a certain number of terms). We could have instead just used the Limit Comparison Test:

Select another series, perhaps by ignoring the constant in the denominator: $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1/2}}$

Evaluate the limit of the ratio of terms: $\sum\limits_{n=1}^{\infty} \dfrac{a_n}{b_n} = \sum\limits_{n=1}^{\infty} \dfrac{\left(\dfrac{1}{2+\sqrt{n}}\right)}{\left(\dfrac{1}{\sqrt{n}}\right)} = \sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n}}{2+\sqrt{n}} \quad \left(\dfrac{\infty}{\infty}\right)$ (use L'Hopital)

$$\sum\limits_{n=1}^{\infty} \dfrac{a_n}{b_n} = \dfrac{\dfrac{1}{2} n^{-\frac{1}{2}}}{\dfrac{1}{2} n^{-\frac{1}{2}}} = 1 \quad \text{which is finite and positive so the two series are 'locked together'}$$

$\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1/2}}$ is a p-series with $p \le 1$ so it diverges, therefore the original series also diverges.

More examples...   Determine if the following series converge or diverge

#5. $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3 - 1}$

#6. $\sum\limits_{n=1}^{\infty} \dfrac{5}{2 + 3^n}$

#7. $\sum\limits_{n=1}^{\infty} \dfrac{4 + 3^n}{2^n}$

#8. $\sum\limits_{n=1}^{\infty} \dfrac{n}{\sqrt{n^5 + 4}}$

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\displaystyle\sum_{n=1}^{\infty} a_n$		$\displaystyle\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series $(r \neq 0)$	$\displaystyle\sum_{n=0}^{\infty} ar^n$	$|r| < 1$	$|r| \geq 1$	Sum: $S = \dfrac{a}{1-r}$
Telescoping Series	$\displaystyle\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\displaystyle\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Alternating Series $(a_n > 0)$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$a_{n+1} \leq a_n$ and $\displaystyle\lim_{n\to\infty} a_n = 0$		Remainder: $|R_N| \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\displaystyle\sum_{n=1}^{\infty} a_n,$ $a_n = f(n) \geq 0$	$\displaystyle\int_{1}^{\infty} f(x)\, dx$ converges	$\displaystyle\int_{1}^{\infty} f(x)\, dx$ diverges	Remainder: $0 < R_N < \displaystyle\int_{N}^{\infty} f(x)\, dx$
Root	$\displaystyle\sum_{n=1}^{\infty} a_n$	$\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$	$\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $= \infty$	Test is inconclusive when $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.
Ratio	$\displaystyle\sum_{n=1}^{\infty} a_n$	$\displaystyle\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1$	$\displaystyle\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| > 1$ or $= \infty$	Test is inconclusive when $\displaystyle\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$.
Direct Comparison $(a_n, b_n > 0)$	$\displaystyle\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison $(a_n, b_n > 0)$	$\displaystyle\sum_{n=1}^{\infty} a_n$	$\displaystyle\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges	$\displaystyle\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges	

#9. $\displaystyle\sum_{n=1}^{\infty} \dfrac{4+3^n}{2^n}$ (use Limit comparison)

Unit 7-5: Alternating Series, Telescoping Series

The Alternating Series Test

An **Alternating Series** is a series in which the signs of the terms alternative from term to term:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}..$$

We can test whether or not an alternating series converges by using the **Alternating Series Test**:

> Let $a_n > 0$. The alternating series
>
> $$\sum_{n=1}^{\infty}(-1)^{n} a_n \quad and \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$
>
> converge when the following are both true:
>
> 1) $\lim_{n\to\infty} a_n = 0$
>
> 2) $a_{n+1} \leq a_n$ for all n

#1. Example: $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{2^{n-1}}$

Alternating Series Test can only show convergence

If the two conditions apply for the Alternating Series Test, then the series converges, but technically if these do not apply, then we can't use the Alternating Series Test and must choose another test to verify that the series diverges.

#2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^2+4}$

#3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln(n)}{n}$

#4. $-\dfrac{1}{3}+\dfrac{2}{4}-\dfrac{3}{5}+\dfrac{4}{6}-\dfrac{5}{7}+\ldots$

Telescoping series

Telescoping series are series which expand to the form: $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + ...$

so that cancellation occurs between terms...

leaving only the first and last term left: $b_1 - b_{n+1}$

For telescoping series, the nth partial sum is: $S_n = b_1 - b_{n+1}$

Since the first term is just a number, telescoping series will converge if the last term here converges: $\lim_{n \to \infty}(b_{n+1})$

...and will converge to the sum: $S = b_1 - \lim_{n \to \infty}(b_{n+1})$

We usually need to do some manipulation of the expression to see that a series is telescoping and also need to be very careful with how the cancellation actually occurs. This is best seen by looking at a more complicated example.

Determine if the series converges and if so to what sum: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

This a_n has a factored denominator, which suggests we can expand using Partial Fractions:

$\dfrac{1}{n(n+2)} = \dfrac{A}{n} + \dfrac{B}{n+2}$

$A(n+2) + B(n) = 1$

$(A+B)n + (2A) = (0)n + 1$

$\begin{cases} A + B = 0 \\ 2A = 1 \end{cases}$

$A = \dfrac{1}{2}, \quad B = -\dfrac{1}{2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\frac{1}{n} - \frac{1}{2}\frac{1}{n+2}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

two terms with subtraction = telescoping series

Now, write out some terms to see how the cancellation works:

$\dfrac{1}{2}\left[\left(\dfrac{1}{1} - \dfrac{1}{3}\right) + \left(\dfrac{1}{2} - \dfrac{1}{4}\right) + \left(\dfrac{1}{3} - \dfrac{1}{5}\right) + \left(\dfrac{1}{4} - \dfrac{1}{6}\right) + \left(\dfrac{1}{5} - \dfrac{1}{7}\right) + \left(\dfrac{1}{6} - \dfrac{1}{8}\right) + ...\right]$

cancellation happens, but one term is skipped

$\dfrac{1}{2}\left[\left(\dfrac{1}{1} - \dfrac{1}{3}\right) + \left(\dfrac{1}{2} - \dfrac{1}{4}\right) + \left(\dfrac{1}{3} - \dfrac{1}{5}\right) + \left(\dfrac{1}{4} - \dfrac{1}{6}\right) + \left(\dfrac{1}{5} - \dfrac{1}{7}\right) + \left(\dfrac{1}{6} - \dfrac{1}{8}\right) + ...\right]$

this leave two terms at beginning and end

so the expression for the nth partial sum is: $S_n = \dfrac{1}{2}\left[1 + \dfrac{1}{2} - \dfrac{1}{n+1} - \dfrac{1}{n+2}\right]$

and the limit is: $\displaystyle\lim_{n \to \infty}\frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right] = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{\infty} - \frac{1}{\infty}\right] = \frac{1}{2}\left[1 + \frac{1}{2} - 0 - 0\right] = \frac{3}{4}$ **converges**

#5. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$

Unit 7-6: Absolute vs. Conditional Convergence

Absolute Convergence

Sometimes, a series may have both positive and negative terms (but may or may not be an alternating series) for example:

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$$

$$= .84147 + .22732 + .01588 - .0473 - .0384 - .0078 + .01341...$$

One way to learn things about the convergence of such a series is to investigate the series with absolute value of a_n :

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^2} \right|$$ Using the Direct Comparison Test, this series can be compared to $\sum_{n=1}^{\infty} \frac{1}{n^2}$
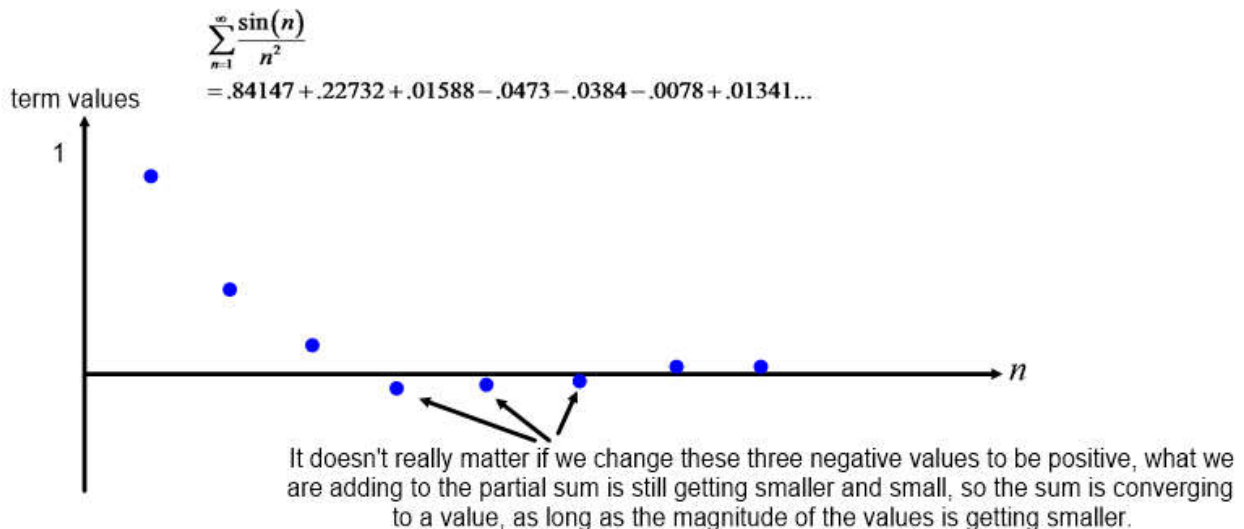
$\left| \frac{\sin(n)}{n^2} \right| \le \frac{1}{n^2}$ ...and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-Series with p = 2 which converges, so this 'squeezes' the left series to also converge.

$0 < a_n \le b_n$ Therefore the absolute value of our original series, $\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^2} \right|$ converges.

How does this help us evaluate the convergence of $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$?   We need another theorem...

> If a series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.   Therefore $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ also converges.

The formal proof of this theorem is in our textbook, but consider this intuitively with our example:

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$$

term values

$$= .84147 + .22732 + .01588 - .0473 - .0384 - .0078 + .01341...$$

It doesn't really matter if we change these three negative values to be positive, what we are adding to the partial sum is still getting smaller and small, so the sum is converging to a value, as long as the magnitude of the values is getting smaller.

But the converse is not true...

Okay, so if the absolute value of a series converges, then the original series also converges. But the converse would be, if a series converges, then the absolute value of the series also converges, and this is <u>not necessarily true</u>. Consider this example of an alternating harmonic series:

By the alternating series test...

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}...$$

$$\lim_{n \to \infty} \frac{1}{n} = 0$$ $a_{n+1} \le a_n$ this series converges.

$$\frac{1}{n+1} \le \frac{1}{n}$$

But if we took the absolute value: $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n} \right|$

This would be equivalent to this series: $\sum_{n=1}^{\infty} \frac{1}{n}$ which is a p-Series with p = 1, which diverges.
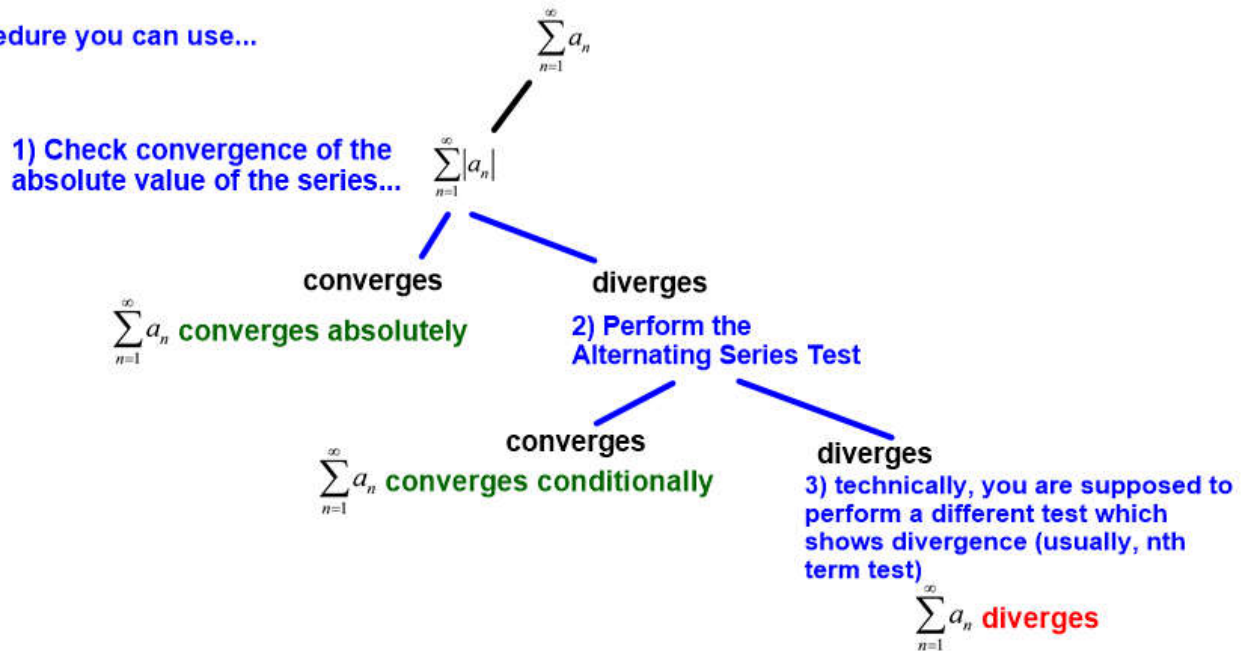
Absolute vs. Conditional Convergence

There are specific terms for cases related to this, defined as follows:

> A series is called **<u>Absolutely Convergent</u>** if $\sum_{n=1}^{\infty}|a_n|$ converges (and therefore $\sum_{n=1}^{\infty}a_n$ also converges.)
>
> A series is called **<u>Conditionally Convergent</u>** if $\sum_{n=1}^{\infty}a_n$ converges, but $\sum_{n=1}^{\infty}|a_n|$ diverges.

A procedure you can use...

$$\sum_{n=1}^{\infty}a_n$$

1) Check convergence of the absolute value of the series... $\sum_{n=1}^{\infty}|a_n|$

converges

$\sum_{n=1}^{\infty}a_n$ **converges absolutely**

diverges

2) Perform the Alternating Series Test

converges

$\sum_{n=1}^{\infty}a_n$ **converges conditionally**

diverges

3) technically, you are supposed to perform a different test which shows divergence (usually, nth term test)

$\sum_{n=1}^{\infty}a_n$ **diverges**

Examples: Determine whether the series coverges absolutely or conditionally, or diverges.

#1. $\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^2}$

#2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+3}$

#3. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n+3)}{n+10}$

(minor, but interesting fact): Rearrangement of Series

Later in this unit we may encounter series where it is advantageous to rearrange the order of the terms, and determine the sum of the series. Counter-intuitively, when you rearrange the terms of an infinite series it can actually change the sum to which a series (if convergent) converges to. We can use the fact that the series is absolutely or conditional convergent to predict things about the effect of term rearrangement.

This is not an issue with a finite series:

$$1+3-2+5-4 \quad S=3$$
$$1+3+5-2-4 \quad S=3$$

But consider this series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\frac{1}{13}-\frac{1}{14}\ldots$$

here are the partial sums adding 1, then 2, then 3, etc. terms:

$$1, .5, .83, .58, .78, .62, .76, .63, .75, .6, .74, .6532, .7301, .6587 \to \ln 2$$

However, if you rearrange the terms like this:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=\left(\frac{1}{1}-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)\ldots$$

now you get these partial sums:

$$.5, .25, 417, .292, .392, .3083, .4869 \to \frac{1}{2}\ln 2$$

...the partial sums are converging on a different sum for the series, just be regrouping the terms!

This is counter-intuitive but is a real effect, one of the many strange things that happen as we consider things involving infinity.

The reason this is important to us in this course, is that later we may want to rearrange terms while finding a sum, but we don't want to do that if rearrangement would affect the series sum. Fortunately, this effect doesn't happen for all series, and we can predict when it will happen:

If a series is **absolute convergent**, then its terms can be rearranged in any order without changing the sum of the series.

If a series is **conditionally convergent**, then rearranging its terms will change the sum of the series.

Unit 7-7: The Ratio and Root Tests

The Ratio Test

Let $\sum a_n$ be a series with nonzero terms:

1) The series $\sum a_n$ converges absolutely when $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| < 1$

2) The series $\sum a_n$ diverges when $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| > 1$ *or* $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \infty$

3) The Ratio Test is inconclusive when $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$

** The Ratio Test is especially good for factorials and for series expressed as a function of the previous term.

#1. $\displaystyle\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$

#2. $a_1 = 2, \quad a_{n+1} = \dfrac{5n+1}{4n+3} a_n$

2, 2, 2.13333333, 2.357894....

If these tests are inconclusive, you must use another method to evaluate convergence...

#3. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+1}$

The Root Test

For a series $\sum a_n$:

1) The series $\sum a_n$ converges absolutely when $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$

2) The series $\sum a_n$ diverges when $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$

3) The Root Test is inconclusive when $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$

** The Root Test is especially good for series involving nth powers.

#4. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

#5. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\displaystyle\sum_{n=1}^{\infty} a_n$		$\displaystyle\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series $(r \neq 0)$	$\displaystyle\sum_{n=0}^{\infty} ar^n$	$|r| < 1$	$|r| \geq 1$	Sum: $S = \dfrac{a}{1-r}$
Telescoping Series	$\displaystyle\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\displaystyle\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Alternating Series $(a_n > 0)$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$a_{n+1} \leq a_n$ and $\displaystyle\lim_{n\to\infty} a_n = 0$		Remainder: $|R_N| \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\displaystyle\sum_{n=1}^{\infty} a_n,$ $a_n = f(n) \geq 0$	$\displaystyle\int_1^{\infty} f(x)\,dx$ converges	$\displaystyle\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \displaystyle\int_N^{\infty} f(x)\,dx$
Root	$\displaystyle\sum_{n=1}^{\infty} a_n$	$\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$	$\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $= \infty$	Test is inconclusive when $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.
Ratio	$\displaystyle\sum_{n=1}^{\infty} a_n$	$\displaystyle\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1$	$\displaystyle\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| > 1$ or $= \infty$	Test is inconclusive when $\displaystyle\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$.
Direct Comparison $(a_n, b_n > 0)$	$\displaystyle\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison $(a_n, b_n > 0)$	$\displaystyle\sum_{n=1}^{\infty} a_n$	$\displaystyle\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges	$\displaystyle\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges	

Unit 7-8: Error for Alternating Series

Alternating Series Remainder

For a convergent alternating series, the partial sum S_n can be a useful approximation to the series sum S.
You can use a calculator Series function to either write out the terms of a series, or to find partial sums:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$$

To write out <u>the terms</u> of the series, enter a_n as if you are going to graph and use a table:

a_1	a_2	a_3	a_4	...
1	$-.25$	$.11111$	$-.0625$	...

To write out <u>the partial sums</u> of the series, use the Summation function (under MATH):

S_1	S_2	S_3	S_4	...
1	.75	.861111	.798611	...

...or you can just write out the terms, them sum them up manually: $S_3 = 1-.25+.111111=.861111$

Series Terms...

a_1	a_2	a_3	a_4	...
1	$-.25$	$.11111$	$-.0625$	...

Term values

Series Partial Sums...

S_1	S_2	S_3	S_4	...
1	.75	.861111	.798611	...
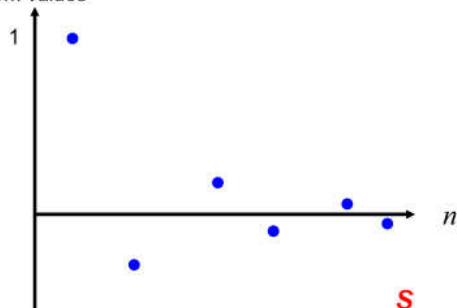
Partial Sums (the sum if we truncate the series after this many terms)

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$$

S The true sum value

'Error' or 'Remainder'
= difference between the actual series sum and the partial sum of the truncated series
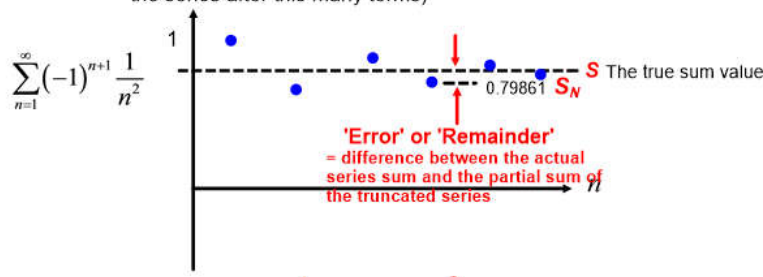
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} \approx 1 - .25 + .11111 - .0625 + ... = 0.79861$$

truncating the series after 4 terms

Alternating Series Remainder

If you go far enough out in the terms (and the alternating series is converging) you can get as close as you wish to the actual sum...you get closer to the actual sum the more terms you include in the partial sum. The difference between the actual sum and your approximation is called the **Remainder** and represents how much error there is in your approximation:

$$remainder, R_N = S - S_N \quad \textit{for an N term partial sum}$$

The following theorem is helpful for approximating an alternating series sum:

> If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than or equal to the first neglected term:
>
> $$\left|S-S_N\right|=\left|R_N\right|\le a_{N+1}$$

#1. Example: Approximate the sum of the series by its first six terms: $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n!}$

Finding N for a given allowable error

We can also determine the minimum number of terms a partial sum must have to approximate the series sum of an alternating series to a given maximum error:

#2. Example: Determine the number of terms required to approximate the sum of the series with an error of less than 0.001: $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n^4}$

#3. How many terms of the series do we need to approximate the sum to an error of < 0.002? $\sum_{n=1}^{\infty}\dfrac{(-1)^n n}{4^n}$

Unit 7 Part 1: Strategies for testing Series

Different textbooks suggest different 'orders' of tests to try. My approach is...

Try these, in order:
1) Does the nth term approach 0? If not, the series diverges by the nth term test.
2) Is the series form Geometric or p-Series or an Alternating Series? Use matching test.
3) Factorials or rules where next term is a function of previous term? Use Ratio Test
4) Does dropping a constant make it Geometric or p-Series? Limits (or maybe Direct) Comparison
5) Is the whole a_n to a power? Root test
6) Denominator factorable? Partial Fraction expansion to Telescoping Series
7) Integral test as a last resort

Only need to check for conditional vs. absolute convergence if the question directly asks for this and the series has a mix of positive and negative terms.

Unit 7-9: Power Series

Power Series

In this section, we will study **Power Series** which are defined as follows:

If x is a variable, then an infinite series of the form:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

is called a **Power Series**. More generally, an infinite series of the form:

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots + a_n(x-c)^n + \ldots$$

is called a **Power Series centered at c**, where c is a constant.

Examples of Power Series

A Power Series centered at 0: $\quad \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

A Power Series centered at -1: $\quad \sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 - (x+1)^3 + \ldots$

A Power Series centered at 5: $\quad \sum_{n=0}^{\infty} \dfrac{1}{n}(x-5)^n = (x-5) + \dfrac{1}{2}(x-5)^2 + \dfrac{1}{3}(x-5)^3 + \ldots$

Radius and Interval of Convergence

A Power Series can be thought of as a function of x: $\quad f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

where the domain of f is the set of all x for which the power series converges.

The constant c always lies in the domain of f, and there are 3 possibilities for the domain:
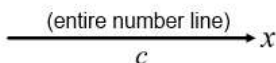
1) A single point: series converges only at $x = c$

2) an interval with radius R: series converges absolutely for $|x-c| < R$
   diverges for $|x-c| > R$
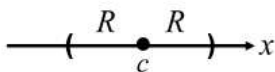
3) all x: (entire number line) series converges absolutely for all x

$R =$ **Radius of convergence** of the power series: 1) A single point: $R = 0$
2) an interval: $R = R$
3) all x: $R = \infty$

The set of all x for which the series converges is called the **interval of convergence**.

Determining the Radius and Interval of Convergence

Although we may use any method to determine convergence of a power series, our main tool is the Ratio Test.

Examples: Find the radius of convergence for the given series:

#1. $\sum_{n=0}^{\infty} 3(x-2)^n$

#2. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

#3. $\sum_{n=0}^{\infty} n! x^n$

ratio test: $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$\lim_{n \to \infty} \left| \frac{3(x-2)^{n+1}}{3(x-2)^n} \right|$

$\lim_{n \to \infty} \left| \frac{(x-2)(x-2)^n}{(x-2)^n} \right|$

$\lim_{n \to \infty} |x-2|$

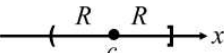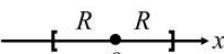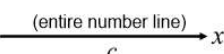$|x-2| < 1$

$-1 < x-2 < 1$
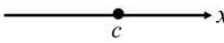
$1 < x < 3$

center = 2

R = 1

Endpoint Convergence

The Ratio Test is inconclusive if the limit = 1, so that means that we must test for convergence at $x - R$, and $x + R$ by hand using others tests of convergence. Each endpoint may converge or diverge independently of the other, which means there are really 6 possible cases for the interval of convergence:

1) A single point: series converges only at $x = c$

2a)   series converges for $(x-R, \ x+R)$

2b)   series converges for $[x-R, \ x+R)$

2c)   series converges for $(x-R, \ x+R]$

2d)   series converges for $[x-R, \ x+R]$

3) all x: (entire number line)   series converges for $(-\infty, \ \infty)$

Endpoint Convergence

Examples: Find the interval of convergence for the given series:

#4. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

#5. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$

#6. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

ratio test : $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ converges

$\displaystyle\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \frac{n}{x^n} \right|$

$\displaystyle\lim_{n\to\infty} \frac{n}{n+1} |x|$

$(1)|x|$ converges when :

$|x| < 1 \quad so \quad -1 < x < 1$

Now test each endpoint...

<u>when x = -1:</u>

$\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

is an alternating series,

so use alternating series test :

$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$a_{n+1} \overset{?}{<} a_n \quad \frac{1}{n+1} < \frac{1}{n}$

converges for $x = -1$

<u>when x = 1:</u>

$\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$

is a p – Series with $p = 1$,

so it diverges, and the

power series diverges at $x = 1$

$\boxed{interval\ of\ convergence = [-1, 1)}$

Derivatives and Integrals of Power Series

Since a power series can be thought of as equivalent to a function, you can also take derivatives and integrals of power series.

If the function $f(x)=\sum_{n=0}^{\infty}a_n(x-c)^n=a_0+a_1(x-c)+a_2(x-c)^2+...a_3(x-c)^3+...$

has a radius of convergence of $R>0$, then, on the interval $(c-R,\ c+R)$

f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are:

$$f'(x)=\sum_{n=0}^{\infty}na_n(x-c)^{n-1}=a_1+2a_2(x-c)+3a_3(x-c)^2+...$$

$$\int f(x)\,dx=C+\sum_{n=0}^{\infty}a_n\frac{(x-c)^{n+1}}{n+1}=C+a_0(x-c)+a_1\frac{(x-c)^2}{2}+a_2\frac{(x-c)^3}{4}...$$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The interval of convergence, however, may differ as a result of behavior at the endpoints.

Example: Find the intervals of convergence of $f(x)$, $f'(x)$, and $\int f(x)\,dx$ for $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

First, let's just find the series for the derivative and integral:

#7. $f'(x)$ #8. $\int f(x)\,dx$

Derivatives and Integrals of Power Series

Example: Find the intervals of convergence of $f(x)$, $f'(x)$, and $\int f(x)\,dx$ for $f(x)=\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

$$\boxed{f(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}}$$

$\text{ratio test}:\lim\limits_{n\to\infty}\left|\dfrac{(x-1)^{n+2}}{(n+2)}\dfrac{(n+1)}{(x-1)^{n+1}}\right|$

$\lim\limits_{n\to\infty}\left|\dfrac{(n+1)}{(n+2)}\dfrac{(x-1)(x-1)^{n+1}}{(x-1)^{n+1}}\right|$

$\lim\limits_{n\to\infty}\dfrac{n+1}{n+2}|(x-1)|$

$(1)|(x-1)|<1$

$0<x<2$

$test\ x=0...$

$\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}(0-1)^{n+1}}{n+1}$

$\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}(-1)^{n+1}}{n+1}$

$\sum\limits_{n=0}^{\infty}\dfrac{(1)^{n+1}}{n+1}=\sum\limits_{n=0}^{\infty}\dfrac{1}{n+1}$

$positive,\ decreasing$

$integral\ test:$

$\int\limits_{1}^{\infty}\dfrac{1}{x+1}dx=\Big[\ln|x+1|\Big]_{1}^{\infty}$

$\lim\limits_{b\to\infty}\ln|b+1|-\ln|1|$

$\infty-0\ \ diverges\ at\ x=0$

$test\ x=2...$

$\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}(2-1)^{n+1}}{n+1}$

$\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}}{n+1}$

$alternating\ series\ test$

$\lim\limits_{n\to\infty}\dfrac{1}{n+1}=0$

$a_{n+1}\overset{?}{<}a_n\quad \dfrac{1}{n+2}<\dfrac{1}{n+1}$

$converges\ at\ x=2$

$$\boxed{interval\ of\ convergence=(0,2]}$$

$$\boxed{\begin{aligned}f'(x)&=\sum_{n=0}^{\infty}\frac{(n+1)(-1)^{n+1}(x-1)^{n}}{n+1}\\&=\sum_{n=0}^{\infty}(-1)^{n+1}(x-1)^{n}\end{aligned}}$$

$same\ R,\ so$

$0<x<2,\ recheck\ endpoints$

$test\ x=0...$

$\sum\limits_{n=0}^{\infty}(-1)^{n+1}(0-1)^{n}$

$\sum\limits_{n=0}^{\infty}(-1)(-1)^{n}(-1)^{n}$

$\sum\limits_{n=0}^{\infty}(-1)(1)^{n}$

$\sum\limits_{n=0}^{\infty}(-1)\ diverges\ at\ x=0$

$test\ x=2...$

$\sum\limits_{n=0}^{\infty}(-1)^{n+1}(2-1)^{n}$

$\sum\limits_{n=0}^{\infty}(-1)(-1)^{n}(1)^{n}$

$\sum\limits_{n=0}^{\infty}(-1)(-1)^{n}\ diverges\ at\ x=2$

$$\boxed{interval\ of\ convergence=(0,2)}$$

$$\boxed{\int f(x)\,dx=C+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}}$$

$same\ R,\ so$

$0<x<2,\ recheck\ endpoints$

$test\ x=0...$

$\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}(0-1)^{n+2}}{(n+1)(n+2)}$

$\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}(-1)(-1)^{n+1}}{(n+1)(n+2)}$

$\sum\limits_{n=0}^{\infty}\dfrac{(-1)(1)^{n+1}}{(n+1)(n+2)}$

$\sum\limits_{n=0}^{\infty}\dfrac{(-1)}{(n+1)(n+2)}$

$(-1)\sum\limits_{n=0}^{\infty}\dfrac{1}{n^2+3n+2}$

$limit\ comparison\ with\ \sum\limits_{n=0}^{\infty}\dfrac{1}{n^2}$

$(p-series\ w/\ p=2,\ converges)$

$\lim\limits_{n\to\infty}\left|\dfrac{\left(\dfrac{1}{n^2+3n+1}\right)}{\left(\dfrac{1}{n^2}\right)}\right|$

$\lim\limits_{n\to\infty}\left|\dfrac{n^2}{n^2+3n+1}\right|=1$

$finite,\ positive\ so\ series\ are\ "linked"$

$converges\ at\ x=2$

$test\ x=2...$

$\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}(2-1)^{n+2}}{(n+1)(n+2)}$

$\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}(1)^{n+2}}{(n+1)(n+2)}$

$\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}}{(n+1)(n+2)}$

$alternating\ series\ test:$

$\lim\limits_{n\to\infty}\dfrac{1}{(n+1)(n+2)}=0$

$a_{n+1}\overset{?}{>}a_n\quad \dfrac{1}{(n+2)(n+3)}<\dfrac{1}{(n+1)(n+2)}$

$converges\ at\ x=2$

$$\boxed{interval\ of\ convergence=[0,2]}$$

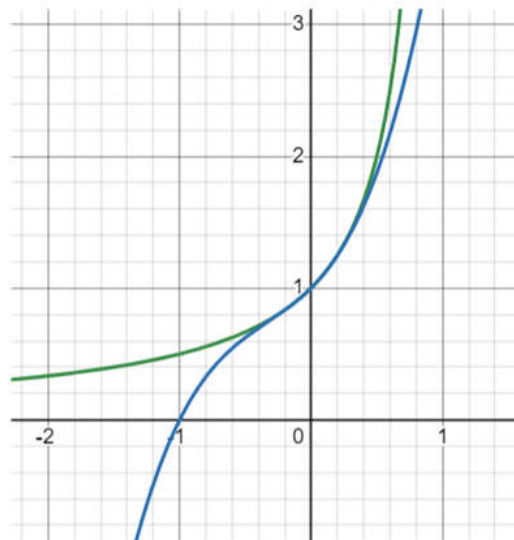# Unit 7-10: Power Series Representation of Functions

Power Series Representation of a Function

https://www.desmos.com/calculator/98tegqkpy2

$$y = \frac{1}{1-x} \approx a + bx + cx^2 + dx^3 + \ldots$$

Playing around with the parameters, we end up with this for a 3rd degree power series approximation to the function:

$$y = \frac{1}{1-x} \approx 1 + x + x^2 + x^3 + \ldots$$

This is only a perfect match at the centering value, x = 0, and the difference between the function and the power series approximation (the 'error') gets larger the farther we are from the centering value in x.

Geometric Power Series

It would be good if we had a procedure to come up with the term coefficients without having to try different values experimentally. For some functions, we can do this by matching the given function to the sum of a geometric series:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1$$

If we have a function of the form $f(x) = \frac{1}{1-x}$ this closely resembles the sum of a geometric series

So if we match a = 1 and r = x, then

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 1(x)^n = 1 + x + x^2 + x^3 + \ldots$$

over the interval of convergence for this series.

Therefore, a function may be represented by a power series (over its interval of convergence), although this representation only works at the x where the series is centered. Here, that would be x = 0.

If we wanted a power series to represent this function around x = -1, that would require a different series centered at -1.

That means in place of x we need (x+1). Here are the steps needed to convert from centering at 0 to -1:

#1. Find a power series centered at x = -1 to represent $\dfrac{1}{1-x}$

$$\frac{1}{1-x} = \frac{1}{1-x+1-1} = \frac{1}{1+1-x-1} = \frac{1}{2-(x+1)} = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{x+1}{2}\right)}$$

Now we match to the form $\dfrac{a}{1-r}$ to find a and r: $\quad a = \dfrac{1}{2}, \quad r = \dfrac{x+1}{2}$

Then we build the corresponding geometric series: $\displaystyle\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x+1}{2}\right)^n$

$$= \frac{1}{2} + \frac{1}{4}(x+1) + \frac{1}{8}(x+1)^2 + \frac{1}{16}(x+1)^3 + \ldots$$

Geometric Power Series

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x+1}{2}\right)^n$$

We need to also find the interval of convergence. Since this is a geometric series...

$$|r| < 1$$

$$\left|\frac{x+1}{2}\right| < 1$$

$$-1 < \frac{x+1}{2} < 1$$

$$-2 < x+1 < 2$$

$$-3 < x < 1$$

$$(-3, 1)$$

(We don't need to test endpoints because when a geometric series always diverges for r = 1)

Examples of converting functions to geometric form

We can sometimes also convert other function forms to geometric form to find corresponding geometric power series.

#2. $f(x) = \dfrac{4}{x+2}$ centered at 0

$$\frac{4}{2-(-x)}$$

$$\frac{2}{1-\left(-\dfrac{x}{2}\right)}$$

$$a = 2, \quad r = -\frac{x}{2}$$

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 2\left(-\frac{x}{2}\right)^n$$

$$= 2\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^n$$

$$\left|\frac{x}{2}\right| < 1$$

$$(-2, 2)$$

#3. $f(x) = \dfrac{1}{x}$ centered at 1

$$\frac{1}{1-x} \quad \frac{1}{1-1+x}$$

$$\frac{1}{1-1-(-x)}$$

$$\frac{1}{1-(-x+1)}$$

$$a = 1, \quad r = -x+1$$

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 1(-x+1)^n$$

$$= \sum_{n=0}^{\infty} (-1(x-1))^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$|x-1| < 1$$

$$-1 < x-1 < 1$$

$$0 < x < 2$$

$$(0, 2)$$

Using polynomial division to find power series

For some functions, we can use polynomial division to find the power series representation.

$$f(x) = \frac{4}{x+2}$$

$$\begin{array}{r} 2 - x + \frac{1}{2}x - \frac{1}{4}x^3 \cdots \\ 2+x \overline{)\, 4 } \\ -(4+2x) \\ \hline -2x \\ -(2x-x^2) \\ \hline x^2 \\ -\left(x^2 + \frac{1}{2}x^3\right) \\ \hline -\frac{1}{2}x^3 \cdots \end{array}$$

Previous example, this is represented by:

$$2\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^n = 2 - x + \frac{1}{2}x^2 - \frac{1}{4}x^3 + \cdots$$

Operations with Power Series

Some properties related to operations on power series turn out to be helpful...

$$\text{Let } f(x)=\sum_{n=0}^{\infty}a_n x^n \text{ and } g(x)=\sum_{n=0}^{\infty}b_n x^n$$

1. $f(kx)=\sum_{n=0}^{\infty}a_n k^n x^n$

2. $f(x^N)=\sum_{n=0}^{\infty}a_n x^{nN}$

3. $f(x)\pm g(x)=\sum_{n=0}^{\infty}(a_n \pm b_n)x^n$

But operations of any kind may affect the interval of convergence. With these properties, if using more than one function, the interval of convergence is the intersection of the individual intervals:

$$\sum_{n=0}^{\infty}x^n + \sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\left(1\pm\frac{1}{2^n}\right)x^n$$

$$(-1,1) \quad (-2,2) \quad so \quad (-1,1)$$

An example where properties are helpful

#4. $f(x)=\dfrac{3x-1}{x^2-1}$

centered at 0

First, this can be separated using Partial Fraction Expansion:

$$\frac{3x-1}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1} \qquad A(x+1)+B(x-1)=3x-1$$

$$(A+B)x+(A-B)=(3)x-1$$

$$\begin{cases} A+B=3 \\ A-B=-1 \end{cases} \quad A=1,\ B=2$$

$$\frac{3x-1}{x^2-1}=\frac{1}{x-1}\ +\ \frac{2}{x+1}$$

$$=\frac{-1}{1-x}\ +\ \frac{2}{1-(-x)}$$

$$\sum_{n=0}^{\infty}(-1)(x)^n\ +\ \sum_{n=0}^{\infty}(2)(-x)^n$$

$$\sum_{n=0}^{\infty}(-1)(x)^n\ +\ \sum_{n=0}^{\infty}(2)(-1)^n(x)^n$$

$$\sum_{n=0}^{\infty}\left(2(-1)^n-1\right)x^n$$

Using derivatives and integrals to find power series representations of other form function

So far, we've been restricted to rational function forms, but what if we wanted a power series representation for $f(x)=\ln(x)$?

$$f(x)=\ln(x)?$$

Note that the derivative of this is: $f'(x)=\dfrac{1}{x}$ centered at 1

Which we previous saw was represented by $\sum_{n=0}^{\infty}(-1)^n(x-1)^n$ over $(0,2)$

Since $f(x)=\ln(x)=\int\dfrac{1}{x}dx$

$$f(x)=\int\sum_{n=0}^{\infty}(-1)^n(x-1)^n\,dx$$

$$=C+\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{n+1}}{n+1}$$

So $\boxed{\ln(x)=\sum_{n=0}^{\infty}(-1)^n\dfrac{(x-1)^{n+1}}{n+1}}$

to establish C we use the x value where the series was centered here that was 1:

$$\ln(x)=C+\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{n+1}}{n+1}$$

$$\ln(1)=C+\sum_{n=0}^{\infty}(-1)^n\frac{((1)-1)^{n+1}}{n+1}$$

$$0=C+\sum_{n=0}^{\infty}(-1)^n\frac{(0)^{n+1}}{n+1}$$

$$0=C+0,\quad C=0$$

Using derivatives and integrals to find power series representations of other form function

When we use this procedure, we should recheck the endpoints. Sometimes, even if they were divergent before, now they may converge:

$$\ln(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \qquad 0,2$$

$$(0,2]$$

$\underline{x=0}$

$$\sum_{n=0}^{\infty} (-1)^n \frac{((0)-1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)(-1)^n}{n+1}$$

$$\sum_{n=0}^{\infty} (-1)(1)^n \frac{1}{n+1}$$

$$-\sum_{n=0}^{\infty} \frac{1}{n+1}$$

$\sum_{n=0}^{\infty} \frac{1}{n+1}$ *is positive, decreasing*

use integral test :

$$\int_1^{\infty} \frac{1}{x+1} dx = \lim_{b\to\infty} \left[\ln|x+1| \right]_1^b$$

$\infty - 0 \ \ diverges$

$\underline{x=2}$

$$\sum_{n=0}^{\infty} (-1)^n \frac{((2)-1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

use alternating series test :

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

$$a_{n+1} \overset{?}{<} a_n$$

$$\frac{1}{n+2} < \frac{1}{n+1}$$

converges

Another example (we'll use the result on the next slide)

$f(x) = \ln(x+1)$ centered at 0

$$f'(x) = \frac{1}{x+1} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (1)(-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$so \ f(x) = \ln(x+1) = C + \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$$

$$= \ln(x+1) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

at x = 0: $\ln((0)+1) = C + \sum_{n=0}^{\infty} (-1)^n \frac{(0)^{n+1}}{n+1}$

$$0 = C$$

$$so \ f(x) = \ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

interval of convergence

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+2}}{(n+2)} \frac{(n+1)}{x^{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{n+2} \left| \frac{xx^{n+1}}{x^{n+1}} \right| = (1)|x| < 1$$

$$-1 < x < 1$$

check endpoints:

$\underline{x=-1}$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(-1)(-1)^n}{n+1} = \sum_{n=0}^{\infty} (-1) \frac{(1)^n}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1}$$

$\sum_{n=0}^{\infty} \frac{1}{n+1}$ *is positive, decreasing - integral test*

$$\int_1^{\infty} \frac{1}{x+1} dx = \lim_{b\to\infty} \left[\ln|x+1| \right]_1^b = \infty - 0$$

diverges

$\underline{x=1}$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

alternating series test :

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

$$a_{n+1} \overset{?}{<} a_n$$

$$\frac{1}{n+2} < \frac{1}{n+1} \quad converges$$

so interval of convergence:

$$(-1,1]$$

You can sometimes find the sum of a convergent series by evaluating the function it represents

<u>Example</u>: Find the sum of the convergent series $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{3^n(n+1)}$

In the last example we found: $\displaystyle\ln(x+1)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{3^n(n+1)}$$

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\left(\dfrac{1}{3^n}\right)}{(n+1)}$$

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\left(\dfrac{1}{3}\right)^n}{(n+1)}$$

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(3)\left(\dfrac{1}{3}\right)\left(\dfrac{1}{3}\right)^n}{(n+1)}$$

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(3)\left(\dfrac{1}{3}\right)^{n+1}}{(n+1)}$$

$$(-3)\sum_{n=0}^{\infty}(-1)^n\frac{\left(\dfrac{1}{3}\right)^{n+1}}{(n+1)}\quad matches\ form\quad \sum_{n=0}^{\infty}(-1)^n\frac{(x)^{n+1}}{(n+1)}=\ln(x+1)$$

Therefore, $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{3^n(n+1)}=-3\ln\left(\frac{1}{3}+1\right)$

$$=-3\ln\left(\frac{4}{3}\right)$$

Unit 7-11: Taylor and Maclaurin Polynomials and Series

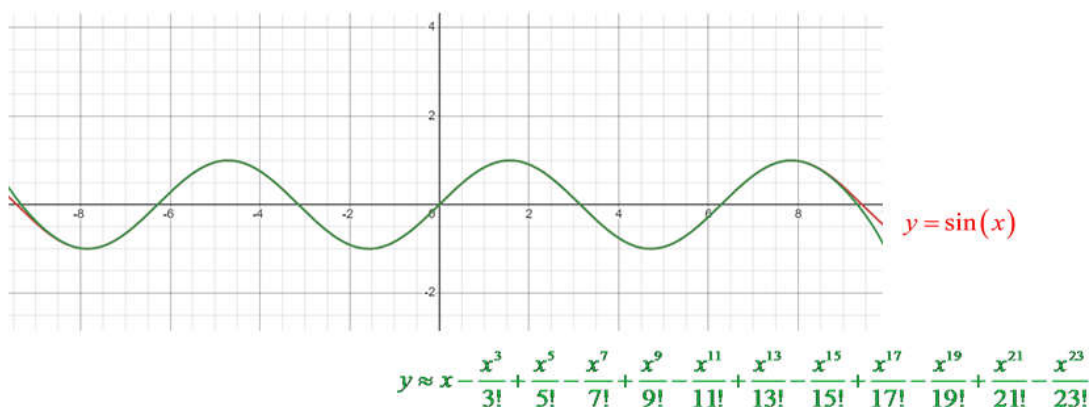## Power Series can approximate a function over a region

https://www.desmos.com/calculator/rhvv6txz9c

We've seen that we can build a power function to correspond to a given function that is perfect at the centering x value and whose error increases the further we are from that x:

$$y \approx x - 0.167x^3 + 0.00833x^5 - 0.0001984x^7$$
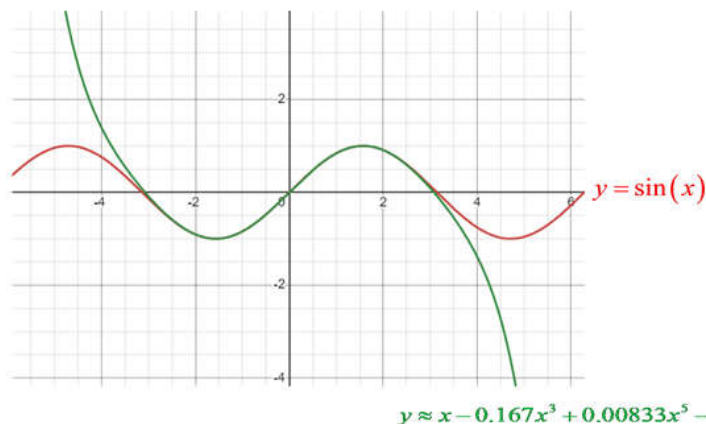
If you include an infinite number of terms, the power series can actually perfectly match the given function if the coefficients are set correctly. Here is sin(x) being approximated by a power series with only

$$y \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \frac{x^{17}}{17!} - \frac{x^{19}}{19!} + \frac{x^{21}}{21!} - \frac{x^{23}}{23!}$$
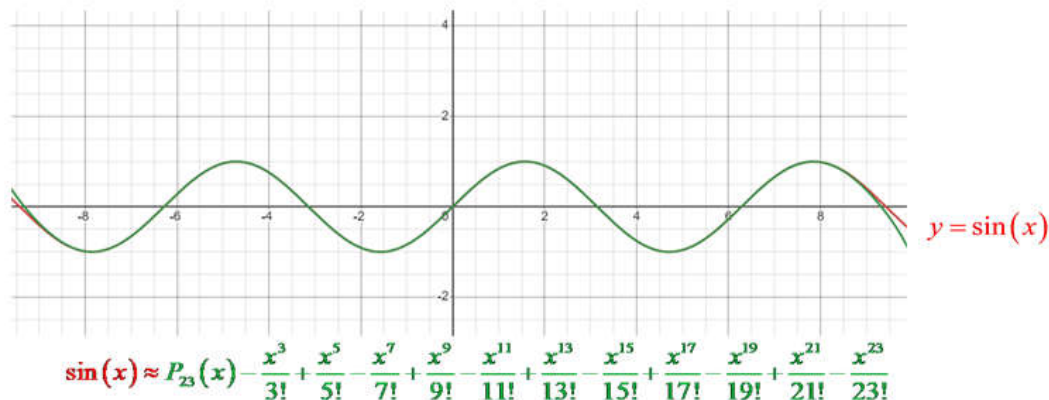
Taylor/Maclaurin Polynomials and Series

We can try matching the function to a Geometric series to find the coefficients, but this doesn't work very often. There is a different procedure we can use to find the coefficients that works with all functions.

> The result is called a **Taylor Series** if we are using an infinite series and a **Taylor Polynomial** if we truncate and only use some of the terms.
>
> If we are centering at x = 0, then the results is also called a **Maclaurin Series (or Polynomial)**.

$$\sin(x) \approx P_{23}(x) - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \frac{x^{17}}{17!} - \frac{x^{19}}{19!} + \frac{x^{21}}{21!} - \frac{x^{23}}{23!}$$

This is called the 23rd-degree Taylor Polynomial for sin(x)
(It could also be called the 23rd-degree Maclaurin Polynomial, because it is centered at x = 0)

Derivation of the Taylor Polynomial procedure

The way we form such a polynomial is to first require that the y-values of the original function, $f(x)$, and the polynomial approximation function, $P(x)$, match at the desired x-value, which we label c:

$$P(c) = f(c)$$

To add terms in order to get a higher-degree polynomial approximation, we then also require one or more <u>derivatives</u> to be equal between the original function and the polynomial function. Wherever we decide to stop, that polynomial is called the 'nth order polynomial approximation', labeled $P_n(c)$ where c is the x-value we are approximating around.

For example, to create the fourth-degree approximating polynomial $P_4(x)$ we need:

$$P(c) = f(c)$$
$$P'(c) = f'(c)$$
$$P''(c) = f''(c)$$
$$P'''(c) = f'''(c)$$
$$P^{(4)}(c) = f^{(4)}(c)$$

This is easiest to understand by looking at a specific example...

Let's find a 3rd-order polynomial to approximate $f(x) = e^x$ around x=0.

We can first postulate a 3rd-degree polynomial with unknown coefficients:

$$P_3(x) = a_0 + a_1(x-0) + a_2(x-0)^2 + a_3(x-0)^3$$
$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

We can then take 3 derivatives of both the original function and the polynomial approximation:

original function	polynomial approximation function
$f(x) = e^x$	$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$
$f'(x) = e^x$	$P_3'(x) = a_1 + 2a_2 x + 3a_3 x^2$
$f''(x) = e^x$	$P_3''(x) = 2a_2 + 6a_3 x$
$f'''(x) = e^x$	$P_3'''(x) = 6a_3$

Now we can replace all x values with the particular c value (in this case 0) and set the original function and polynomial function equal for each derivative (and y expressions). This will form a system of equations where the values to solve for are the unknown coefficients:

original function	polynomial approximation function	
$f(0) = e^0 = 1$	$P_3(0) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 = a_0$	$a_0 = 1$
$f'(0) = e^0 = 1$	$P_3'(0) = a_1 + 2a_2(0) + 3a_3(0)^2 = a_1$	$a_1 = 1$
$f''(0) = e^0 = 1$	$P_3''(0) = 2a_2 + 6a_3(0) = 2a_2$	$2a_2 = 1$
$f'''(0) = e^0 = 1$	$P_3'''(0) = 6a_3$	$6a_3 = 1$

system:

$$\begin{cases} a_0 = 1 \\ a_1 = 1 \\ 2a_2 = 1 \\ 6a_3 = 1 \end{cases}$$

solution to the system: $a_0 = 1, \; a_1 = 1, \; a_2 = \dfrac{1}{2}, \; a_3 = \dfrac{1}{6}$

There is a pattern to these coefficients: $\left(a_2 = \dfrac{1}{2!}, \; a_3 = \dfrac{1}{3!} \right)$

That means $\boxed{f(x) = e^x \approx P_3(x) = 1 + 1x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 \;\; (in\,the\,vicinity\,of\;x=0)}$

Taylor and Maclaurin Polynomials

Defining this process more generally provides the definition of **nth-degree Taylor and Maclaurin Polynomial Approximations**:

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the **nth-degree Taylor polynomial for f at c.**

If $c = 0$, then the polynomial

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

is called the **nth-degree Maclaurin polynomial for f at c.**

(the full derivation of this more general definition is shown in our textbook if you are interested in seeing it)

Taylor and Maclaurin Series

Generalizing this procedure for an infinite number of terms gives the definition of a **Taylor Series**:

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

is called the **Taylor Series for f at c.** If c = 0, then the series is called the **Maclaurin Series for f**.

#1. Find the n=3 Taylor polynomial for $f(x) = \sqrt{x}$ at c = 4

#2. Find the nth Maclaurin polynomial for $f(x) = e^x$

#3. Find the n=4 Taylor
polynomial for $f(x)=\ln x$
at c = 1

Find the Maclaurin Series for the function $f(x)=\sin x$

First, write out the function, and a few derivatives, and solving for the values at x = c (c = 0 for Maclaurin)...

$$f(x)=\sin x \qquad f(0)=0$$
$$f'(x)=\cos x \qquad f'(0)=1$$
$$f''(x)=-\sin x \qquad f''(0)=0$$
$$f'''(x)=-\cos x \qquad f'''(0)=-1$$
$$f^{(4)}(x)=\sin x \qquad f^{(4)}(0)=0$$
$$f^{(5)}(x)=\cos x \qquad f^{(5)}(0)=1$$

Next, use the definition of Taylor Series to build some terms of the series...

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+...\frac{f^{(n)}(c)}{n!}(x-c)^n+...$$

$$=0+1(x-0)+\frac{0}{2!}(x-0)^2+\frac{-1}{3!}(x-0)^3+\frac{0}{4!}(x-0)^4+\frac{1}{5!}(x-0)^5+...$$

Then, eliminate any zero terms, and write the term number under each term...

$$=1(x-0)+\frac{-1}{3!}(x-0)^3+\frac{1}{5!}(x-0)^5+...$$
$$n=0 \qquad n=1 \qquad n=2$$

Now we need to figure out an expression for the nth term...

$$\frac{(-1)^n}{(2n+1)!}(x-0)^{2n+1}$$

Finally write out the series:

$$\boxed{\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+...}$$

Taylor and Maclaurin Series Examples

Find the Taylor Series for the function $f(x)=\dfrac{1}{1-x}$ centered at c = 2.

First, write out the function. and a few derivatives. and solving for the values at x = c (c = 0 for Maclaurin)...

$$f(x)=\frac{1}{1-x}=(1-x)^{-1} \qquad\qquad f(2)=\frac{1}{1-2}=-1$$

$$f'(x)=-(1-x)^{-2}(-1)=(1-x)^{-2}=\frac{1}{(1-x)^2} \qquad f'(2)=1$$

$$f''(x)=-2(1-x)^{-3}(-1)=2(1-x)^{-3}=\frac{2}{(1-x)^3} \qquad f''(2)=-2$$

$$f'''(x)=2(-3)(1-x)^{-4}(-1)=6(1-x)^{-4}=\frac{6}{(1-x)^4} \qquad f'''(2)=6$$

$$f^{(4)}(x)=6(-4)(1-x)^{-5}(-1)=24(1-x)^{-5}=\frac{24}{(1-x)^5} \qquad f^{(4)}(2)=-24$$

Next, use the definition of Taylor Series to build some terms of the series...

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+...\frac{f^{(n)}(c)}{n!}(x-c)^n+...$$

$$=-1+1(x-2)+\frac{-2}{2!}(x-2)^2+\frac{6}{3!}(x-2)^3+\frac{-24}{4!}(x-2)^4+...$$

Then, eliminate any zero terms, and write the term number under each term and find an expression for the nth term...

$$=\quad -1 \quad +(x-2)+\frac{-2}{2!}(x-2)^2+\frac{6}{3!}(x-2)^3+\frac{-24}{4!}(x-2)^4+...$$

$$n=0 \qquad n=1 \qquad n=2 \qquad n=3 \qquad n=4$$

$$\frac{(-1)^{n+1}n!}{n!}(x-2)^n=(-1)^{n+1}(x-2)^n$$

Finally write out the series:
$$\boxed{\frac{1}{1-x}=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n=-1+(x-2)-(x-2)^2+(x-2)^3+...}$$

We could also have used our Geometric Series procedure instead...

$$\frac{1}{1-x}\rightarrow\frac{a}{1-r} \qquad centered\ at\ c=2,\ x\rightarrow(x-2)$$

$$\frac{1}{1-x+2-2}$$

$$\frac{1}{1-(x-2)-2}$$

$$\frac{1}{-1-(x-2)}$$

$$\frac{1}{-(1-(-x+2))}$$

$$\frac{-1}{1-(-x+2)}=\frac{a}{1-r} \qquad a=-1,\ r=-x+2$$

$$\sum_{n=0}^{\infty}ar^n=\sum_{n=0}^{\infty}(-1)(-x+2)^n$$

$$=\sum_{n=0}^{\infty}(-1)((-1)(x-2))^n$$

$$=\sum_{n=0}^{\infty}(-1)(-1)^n(x-2)^n$$

$$=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$$

...which matches what we got with the Taylor procedure:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$$

Binomial Series

If the function we are representing is of the form: $f(x) = (1+x)^k$

then the series is called a **Binomial Series**.

Example: Find the Maclaurin Series to represent $f(x) = (1+x)^k$

$f(x) = (1+x)^k$ $f(0) = 1$

$f'(x) = k(1+x)^{k-1}$ $f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$ $f''(0) = k(k-1)$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$ $f'''(0) = k(k-1)(k-2)$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

$$= 1 + k(x-0) + \frac{k(k-1)}{2!}(x-0)^2 + \frac{k(k-1)(k-2)}{3!}(x-0)^3 \dots$$

$$\boxed{= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 \dots}$$

which can be shown to have an interval of convergence of (-1, 1)

With more complicated functions, using the Taylor procedure can be difficult...

Example: Find the power series for $f(x) = \sin(x^2)$

Taking derivatives... $f(x) = \sin(x^2)$

$$f'(x) = \cos(x^2)(2x)$$

$$f''(x) = \cos(x^2)(2) + (2x)\left(-\sin(x^2)(2x)\right)$$

...we can see that the derivatives quickly get very complicated, and are not of the same form, so it will not be possible in a case like this to use the definition of the Taylor Series to find the series.

In cases like this, we can employ a list of pre-calculated power series for elementary functions.

Power Series for Elementary Functions

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \le x \le 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3 x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)! x^{2n+1}}{(2^n n!)^2 (2n+1)} + \cdots$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1^*$

memorize these!

*The convergence at $x = \pm 1$ depends on the value of k.

Using the basic list to find power series

#1. Find the power series for $f(x) = \cos\left(\sqrt{x}\right)$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$$

Just substitute the expression for the argument in place of the x...

$$\cos\left(\sqrt{x}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\sqrt{x}\right)^{2n}}{(2n)!}$$

$$\cos\left(\sqrt{x}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{\frac{1}{2}}\right)^{2n}}{(2n)!}$$

$$\boxed{\cos\left(\sqrt{x}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 1 - \frac{1}{2!}x + \frac{1}{4!}x^2 - \frac{1}{6!}x^3 \ldots}$$

Using the basic list to find power series

#2. Find the power series for $f(x) = e^x \arctan x$

You can also combine basic list power series using arithmetic operation such as multiplication...

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Note: you cannot just multiply the expressions for the nth term because of the term combination interactions.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$e^x \arctan x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)$$

$$= 1\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) + x\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) + \frac{x^2}{2!}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) + \frac{x^3}{3!}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \frac{x^3}{2} - \frac{x^5}{6} + \frac{x^7}{10} - \frac{x^9}{14} + \frac{x^4}{6} - \frac{x^4}{18} + \frac{x^8}{30} - \frac{x^{10}}{42} + \dots$$

$$\boxed{= x + x^2 + \frac{1}{6}x^3 - \frac{2}{9}x^4 + \frac{1}{30}x^5 + \frac{1}{5}x^6 - \frac{3}{70}x^7 - \frac{23}{210}x^8 - \frac{1}{14}x^9 - \frac{1}{42}x^{10} + \dots}$$

<u>Example</u>: Find the power series for $f(x) = \tan x$

You can use division to find a series for tanx...

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots}$$

using polynomial division...

$$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots \overline{\smash{\big)}\, x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots} \quad\Big[\, x + \frac{1}{3}x^3 + \frac{2}{15}x^5 \dots \Big]$$

$$-\left(x - \frac{1}{2}x^3 + \frac{1}{24}x^5\right)$$

$$\frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots$$

$$-\left(\frac{1}{3}x^3 - \frac{1}{6}x^5 + \frac{1}{72}x^7\right)$$

$$\frac{2}{15}x^5 + \dots$$

$$\boxed{\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 \dots}$$

Using the basic list to find power series

Find the power series for $f(x) = \sin^2 x$

You can use identities to convert things into forms in the basic list: $\sin^2 x = \dfrac{1 - \cos 2x}{2} = \dfrac{1}{2} - \dfrac{1}{2}\cos(2x)$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}$$

We can just work with the nth term here because we are just multiplying every term by the same thing.

#3. Use a power series to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$

First expand out terms using the basic list form:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots$$

$$e^{(-x^2)} = 1 + (-x^2) + \frac{1}{2}(-x^2)^2 + \frac{1}{6}(-x^2)^3 + \frac{1}{24}(-x^2)^4 + \frac{1}{120}(-x^2)^5 + \dots$$

$$e^{(-x^2)} = 1 + -x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10} + \dots$$

Then integrate term by term:

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(1 + -x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10} + \dots\right)dx$$

$$= \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 - \frac{1}{1320}x^{11} + \dots\right]_0^1$$

$$= \left[(1) - \frac{1}{3}(1)^3 + \frac{1}{10}(1)^5 - \frac{1}{42}(1)^7 + \frac{1}{216}(1)^9 - \frac{1}{1320}(1)^{11} + \dots\right] - \left[(0) - \frac{1}{3}(0)^3 + \frac{1}{10}(0)^5 - \frac{1}{42}(0)^7 + \frac{1}{216}(0)^9 - \frac{1}{1320}(0)^{11} + \dots\right]$$

$$= (1) - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \dots \approx 0.746729$$

Unit 7-13: Lagrange Error for Taylor Polynomials

Remember Alternating Series Error?

If you go far enough out in the terms (and the alternating series is converging) you can get as close as you wish to the actual sum...you get closer to the actual sum the more terms you include in the partial sum. The difference between the actual sum and your approximation is called the **Remainder** and represents how much error there is in your approximation:

$$remainder, R_N = S - S_N \quad \text{for an } N \text{ term partial sum}$$

The following theorem is helpful for approximating an alternating series sum:

> If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than or equal to the first neglected term:
>
> $$|S - S_N| = |R_N| \le a_{N+1} \quad \text{(the value of the first neglected term)}$$

Remainder/Error for a Taylor Polynomial

The error (or remainder) between a Taylor Polynomial and the function it represents is similar in concept:
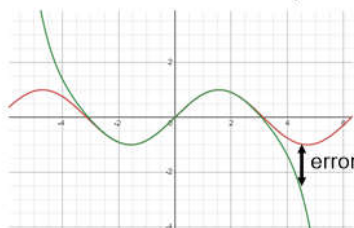
Alternating Series
$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 ... = S$$
$$a_1 + a_2 + ... + a_N = S_n$$
$\lvert Alternating\ Series\ Remainder \rvert \le \lvert a_{N+1} \rvert$
$\lvert Alternating\ Series\ Remainder \rvert \le \lvert 1st\ neglected\ term \rvert$

Taylor Polynomial

Remainder/Error depends upon 2 things:

- How many terms are included (degree)
- The x-value you are evaluating the polynomial and function at, and how far away this x-value is from the centering value.

Like the Alternating Series Remainder, the Taylor Polynomial Remainder is still determined by the first neglected term, but now the expression for calculating this depends upon that term's derivative and the particular x-value you plug in, and it is called the **Lagrange Remainder**:

$$\lvert Lagrange\ Remainder \rvert \le \left\lvert \frac{f^{(N+1)}(z)}{(N+1)!}(x-c)^{N+1} \right\rvert$$

Where $f^{(N+1)}(z)$ is the maximum value of the derivative in the interval of interest, between the centering x-value and the x-value where you are evaluating the function.

Examples to help us see how this works in different cases...

#1. For the series $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n!}$

(a) Approximate the sum of the series by using the first 6 terms.

(b) Find the upper bound for the remainder for the approximation in part a.

(c) Find upper and lower bounds for the actual sum of the series.

#2. For the series $\sum_{n=0}^{\infty}(-1)^{n}\dfrac{1}{(2n)!}$

(a) Approximate the sum of the series with an error of less than 0.001.

(b) Which memorized Power Series matches this series form?
Use the function for the matching series to find the actual value of the given series.

#3. (a) Estimate e^2 using a Maclaurin polynomial of degree 10 for e^x

(b) Use the Lagrange form of the remainder (error) to find the upper bound of the error using this partial sum.

(c) What is the <u>actual</u> error $|f(x) - P(x)|$?

#4. If $f^{(5)}(x) = 700\sin(x)$ and if $x = 0.7$ is in the convergence interval for the power series of f centered at $x = 0$, find an upper limit for the error when the fourth-degree Taylor polynomial is used to approximate $f(0.7)$

#5. If $f^{(6)}(x)$ is a positive, decreasing function, find the error bound when a 5th degree Taylor polynomial centered at $x = 4$ is used to approximate $f(4.1)$

(Assume the series converges for $x = 4.1$)

#6. The Taylor series for $\ln(x)$, centered at $x = 1$ is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$

Let f be the function given by the sum of the first three nonzero terms of this series.

The maximum value of $|\ln(x) - f(x)|$ for $0.3 \leq x \leq 1.7$ is:

(A) 0.030

(B) 0.039

(C) 0.145

(D) 0.153

(E) 0.529

- First, try Lagrange error.
- Next, try Alternating Series Error
- Finally, can we just compute the actual error over this x interval?