AP Calc BC – Lesson Notes – Unit 7: Integration Techniques, Improper Integrals

Unit 7-1: Review of Integration Techniques
Larsen: 7.1

Integration Shortcuts are the inverse of the Derivative Shortcuts

$\frac{d}{dx}[C]=0$	$\int 0\,dx=C$
$\frac{d}{dx}[kx]=k$	$\int k\,dx=kx+C$
$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$	$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C\quad (n\neq -1)$
$\frac{d}{dx}\left[e^x\right]=e^x$	$\int e^x\,dx=e^x+C$
$\frac{d}{dx}\left[a^x\right]=(\ln a)\,a^x$	$\int a^x\,dx=\left(\frac{1}{\ln a}\right)a^x+C$
$\frac{d}{dx}[\ln x]=\frac{1}{x}\quad (x>0)$	$\int \frac{1}{x}\,dx=\ln\lvert x\rvert+C$
$\frac{d}{dx}[\sin x]=\cos x$	$\int \cos x\,dx=\sin x+C$
$\frac{d}{dx}[\cos x]=-\sin x$	$\int \sin x\,dx=-\cos x+C$
$\frac{d}{dx}[\tan x]=\sec^2 x$	$\int \sec^2 x\,dx=\tan x+C$
$\frac{d}{dx}[\sec x]=\sec x\tan x$	$\int \sec x\tan x\,dx=\sec x+C$
$\frac{d}{dx}[\csc x]=-\csc x\tan x$	$\int \csc x\cot x\,dx=-\csc x+C$
$\frac{d}{dx}[\cot x]=-\csc^2 x$	$\int \csc^2 x\,dx=-\cot x+C$

$$\frac{d}{dx}[\arcsin x]=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arccos x]=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan x]=\frac{1}{1+x^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} x]=\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{arccsc} x]=\frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{arccot} x]=\frac{-1}{1+x^2}$$

$$\int \frac{du}{\sqrt{a^2-u^2}}=\arcsin\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{a^2+u^2}=\frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C$$

Integration by Substitution

Some integrals cannot be evaluated by using the basic integration formulas, so we need other integration techniques. One of these is **integration by substitution** which is based on the Chain Rule.

Derivative using Chain Rule

$$y = \left(x^2+5\right)^4$$

useful when one function is 'inside' another

$$u = x^2+5 \qquad y = u^4$$

$$\frac{du}{dx} = 2x \qquad \frac{dy}{du} = 4u^3$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = 4u^3(2x)$$

$$\frac{dy}{dx} = 4\left(x^2+5\right)^3(2x)$$

Integration by Substitution

$$\int 4\left(x^2+5\right)^3 2x\,dx$$

1) define the 'inside' function to be u.

2) Find du/dx and solve for du to get a 'toolkit' with du and u.

3) Substitution all expressions with x and dx in the original integral to obtain a new integral using u as the variable.

4) Integrate, then resubstitute u to use the original x variable.

$$u = x^2+5 \qquad 4\int u^3 du$$

$$\frac{du}{dx} = 2x \qquad 4\left[\frac{1}{4}u^4\right]+C$$

$$du = 2x\,dx \qquad u^4 + C$$

$$\left(x^2+5\right)^4 + C$$

Splitting into multiple integrals

Sometimes, u-sub won't work as initially stated, but we can split the integral into multiple integrals:

$$\int \frac{x+2}{\sqrt{4-x^2}}dx$$

$$\int \frac{x}{\sqrt{4-x^2}}dx \quad + \quad \int \frac{2}{\sqrt{4-x^2}}dx$$

u-sub for this one...

$$u = 4-x^2$$

$$du = -2x\,du$$

$$xdx = -\frac{1}{2}du$$

$$-\frac{1}{2}\int u^{-\frac{1}{2}}du$$

$$-\frac{1}{2}\left[2u^{\frac{1}{2}}\right]$$

$$-\sqrt{4-x^2}$$

this one matches an inverse sine...

$$2\int \frac{1}{\sqrt{4-x^2}}dx$$

$$\int \frac{1}{\sqrt{a^2-u^2}}du = \arcsin\left(\frac{u}{a}\right) \quad with\, u = x,\ a = 2$$

$$2\arcsin\left(\frac{x}{2}\right)$$

$$\boxed{-\sqrt{4-x^2} + 2\arcsin\left(\frac{x}{2}\right) + C}$$

Completing the Square

Sometimes completing the square with a quadratic will result in matching a memorized/table form...

$$\int \frac{1}{x^2 - 4x + 7} dx$$

$$x^2 - 4x + 7$$
$$(x^2 - 4x + 4) + 7 - 4$$
$$(x-2)^2 + 3$$

$$\int \frac{1}{3 + (x-2)^2} dx$$

now, u-sub... $u = x - 2$

$du = dx$

$$\int \frac{1}{3 + u^2} du$$

and this matches arctan form... $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)$

$$\boxed{\frac{1}{\sqrt{3}}\arctan\left(\frac{x-2}{\sqrt{3}}\right) + C}$$

Here is another technique (new)...

Sometimes adding and subtracting something in the integral will allow a useful split...

$$\int \frac{1}{1+e^x} dx$$

Quotient form suggests maybe Log Rule, so try to make a port of the numerator match the denominator by adding and subtracting the same term...

$$\int \frac{1 + e^x - e^x}{1+e^x} dx$$

Now split into 2 integrals (remember you can split a numerator but not a denominator - must retain the 'common denominator')...

$$\int \frac{1+e^x}{1+e^x} dx - \int \frac{e^x}{1+e^x} dx$$

Simplify, then u-sub for 2nd integral...

$$\int 1\, dx - \int \frac{e^x}{1+e^x} dx$$

u=1+e^x

$$du = e^x dx$$

$$\int 1\, dx - \int \frac{1}{u} du$$

$$x \quad - \ln|u|$$

$$\boxed{x \quad - \ln\left|1+e^x\right| \; + \; C}$$

Unit 7-2: Integration by Parts
Larsen: 7.2

Integration by Parts

Integration by Substitution is one of our main techniques for evaluating integrals. Another very widely used technique is **Integration by Parts**. Where Integration by Substitution is based on the Chain Rule, Integration by Parts is based on the Product Rule**:**

Derivative using Product Rule

$$\frac{d}{dx}[uv] = u\frac{d}{dx}[v] + v\frac{d}{dx}[u]$$

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

← integrating this on both sides:

Integration by Parts

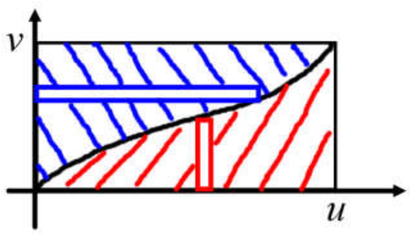
$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$uv = \int u\,dv + \int v\,du$$

...produces the 'integration by parts formula':

$$\boxed{\int u\,dv = uv - \int v\,du}$$

We can also derive the integration by parts formula by using our ideas of integration representing area:



blue area red area total area

$$\int u\,dv + \int v\,du = uv$$

Now solve for the first term: $\boxed{\int u\,dv = uv - \int v\,du}$

In Integration by Parts, the entire integrand must be divided into two factors which are multiplied together, u and dv

$$\int u\,dv = uv - \int v\,du$$

given integral two factors

...then you can instead use the equivalent right side formula. This effectively removes some of the integrand out of the integral into the *uv* and the remaining integral should be simpler to evaluate than the original integral.

Integration by Parts

1) separate the integrand into two factors, *u* and *dv*.
The *dv* factor must contain the differential and be something you can integrate.

2) Take the derivative of *u* and solve for *du* to obtain du.

3) Take the antiderivative of *dv* to obtain *v*.

4) Substitute *u*, *v*, and *dv* into the integration by parts formula.

5) Integrate the remaining (simpler) integral.

Comparing Integration by Part with Integration by Substitution

Integration by Parts

1) separate the integrand into two factors, *u* and *dv*.
The *dv* factor must contain the differential and be something you can integrate.

2) Take the derivative of *u* and solve for *du* to obtain du.

3) Take the antiderivative of *dv* to obtain *v*.

4) Substitute *u*, *v*, and *dv* into the integration by parts formula.

5) Integrate the remaining (simpler) integral.

Integration by Parts is used when the integrand is easily separated into two things which are multiplied. There is a *u* and a *v*, and we substitute into a formula (not the original integral). The resulting integral still uses the original integration variable.

Integration by Substitution

1) define the 'inside' function to be u.

2) Find du/dx and solve for du to get a 'toolkit' with du and u.

3) Substitution all expressions with x and dx into the original integral to obtain a new integral using u as the variable.

4) Integrate, then resubstitute u to use the original x variable.

Integration by Substitution is used when one function is 'inside' the other (a composition of functions) and we make the 'inside' function *u*. There is no *v*, and we substitute back into the original integral so that the variable changes to *u*.

$$\int 3xe^x\,dx$$

$$\int e^{(3x)}\,dx$$

More examples

$$\int \ln x\,dx$$

Integration by Parts - guidelines for choose *u* and *dv*

- **Make sure that *dv* is something you can integrate.** For example, if the integrand contains natural log, we don't have a shortcut to integrate that, so that portion should be part of *u*.
- **If you can, select something for *u* that will get simpler when you take the derivative.** For example, if the integrand contains x^3 that would be best to put into *u*, because when you take the derivative it becomes $2x^2$ (a lower degree and therefore simpler).
- **Try letting *dv* be the more complicated portion of the integrand.**

If you happen to choose wrong, it just means that the resulting remaining integral won't be simpler than the original to integrate. In that case, just choose *u* and *dv* differently and try again (or maybe try a different procedure, like Integration by Substitution).

Try it...

$$\int x\sin(4x)\,dx$$

Integration by Parts

1) separate the integrand into two factors, *u* and *dv*.
The *dv* factor must contain the differential and be something you can integrate.

2) Take the derivative of *u* and solve for *du* to obtain du.

3) Take the antiderivative of *dv* to obtain *v*.

4) Substitute *u*, *v*, and *dv* into the integration by parts formula.

5) Integrate the remaining (simpler) integral.

You might even do integration by parts after doing integration by substitution

$$\int 2x^3\cos\left(x^2\right)dx$$

Sometimes, you need to use integration by parts more than once

Try it... $\int x^2 \sin x \, dx$ u = x^2 (why is this the right choice?)

You may find that after repeated integration by parts, you obtain the original integral...

$$\int e^{4x} \cos(2x)\, dx$$

The "tabular method"

If you must use Integration by Parts more than once (and the result doesn't cycle back around to the original integral) this results in a noticable pattern in the terms of the answer. Some textbooks (including ours) point this out and suggest a quicker method call the "tabular method". It isn't anything official, and won't be asked about on the AP Exam, but you could employ this when it is appropriate if you like:

$$\int x^2 e^{2x}\, dx$$

The usual method...

The "tabular" method...

$$\int x^6 e^x\, dx$$

Using Integration by Parts requires a lot of practice, but gets easier with experience. It is probably the most powerful integration technique we have in our toolbox, and it comes up often, so definitely worth spending the time to get good at it :)

Unit 7-3: Trigonometric Integrals
Larsen: 7.3

Trigonometric Integrals

Integrals which involve powers of one or more trigonometric functions are referred to as **trigonometric integrals**. For example, the following integrals are trigonometric integrals:

$$\int \sin^3 x \cos^4 x \, dx \qquad \int \tan^5 x \, dx$$

Integrating such integrals involves changing the form of the integral as given using trigonometric identities in order to produce a form which splits off a portion to form the 'du' for a u-substitution. Often (but not always) the result is that the integral is split into multiple, simpler integrals.

Each case is a little different, and becoming adept at this requires practice, but there are some guiding 'rules of thumb' that suggest techniques which work in different situations.

Things to review that you will need

You should review and memorize some trig identities:

$$\sin^2 u + \cos^2 u = 1 \qquad \sin^2 u = \frac{1-\cos 2u}{2}$$
$$\sec^2 u = \tan^2 x + 1$$
$$\csc^2 u = \cot^2 x + 1 \qquad \cos^2 u = \frac{1+\cos 2u}{2}$$

You should also review memorize the trig derivatives and integrals:

$$\frac{d}{dx}[\sin x] = \cos x \qquad \int \cos x \, dx = \sin x + C$$
$$\frac{d}{dx}[\cos x] = -\sin x \qquad \int \sin x \, dx = -\cos x + C$$
$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \int \sec^2 x \, dx = \tan x + C$$
$$\frac{d}{dx}[\cot x] = -\csc^2 x \qquad \int \csc^2 x \, dx = -\cot x + C$$
$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \int \sec x \tan x \, dx = \sec x + C$$
$$\frac{d}{dx}[\csc x] = -\csc x \cot x \qquad \int \csc x \cot x \, dx = -\csc x + C$$

An example

The general idea behind integrating trigonometric integrals is to try to split the integrand into factors so that one part can become 'u' for a u-substitution and the du needed is present in the integral.

$$\int \sin^3 x \cos^4 x \, dx$$

$$\int \sin^2 x \cos^4 x \sin x \, dx$$ ← **split off one sinx to use for 'du'**

$$\int (1-\cos^2 x)\cos^4 x \sin x \, dx$$ ← **trig indentity to convert other sines to cosines**

$$\int (\cos^4 x - \cos^6 x)\sin x \, dx$$ ← **...so that you can split into two 'cosine' integrals**

$$\int \cos^4 x \sin x \, dx - \int \cos^6 x \sin x \, dx$$

$$u = \cos x \qquad du = -\sin x \, dx$$

$$\int u^4 (-du) \qquad - \int u^6 (-du)$$ **Now, each integral has the 'du' required to do a u-substitution**

$$-\int u^4 \, du \qquad + \int u^6 \, du$$

$$-\frac{1}{5}u^5 \qquad + \frac{1}{7}u^7 + C$$

$$-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

Guidelines for Integrating Trigonometric Integrals

Here are some guidelines that can help in rearranging the integrand for integration:

Sines/Cosines

- Power of sine odd and positive? Reserve one sine for 'du' and convert the rest to cosines.
- Power of cosine odd and positive? Reserve one cosine for 'du' and convert the rest to sines.
- Powers of both sine and cosine even? Repeatedly use $\sin^2 u = \frac{1-\cos 2u}{2}$ and $\cos^2 u = \frac{1+\cos 2u}{2}$

Secants/Tangents

- Power of secant even and positive? Reserve one $\sec^2 x$ for 'du' and convert the rest to tangents.
- Power of tangent odd and positive? Reserve one secx tanx for 'du' and convert the rest to secants.
- Tangent only with even power? Split out one $\tan^2 x$ and convert it to $\sec^2 x - 1$, then expand.
- Secant only with odd power? Use integration by parts by splitting off $\sec^2 x$ for dv.
- Try converting to sines and cosines.

In general

- Radicals in the integrand? Convert everything to exponents.

Examples

$$\int \sin^3 x \cos^3 x \, dx$$

If you forget $\int \tan x \, dx$ ...

If you forget $\int \sec x \, dx$ ...

Examples

$\int \tan^2 x \sec^2 x \, dx$

$\int \cos^2 x \, dx$

$\int \sec^3 x \, dx$

$\int x \sin^3 \left(x^2\right) dx$

Examples

$$\int \cot^5 \theta \sin^4 \theta \, d\theta$$

$$\int \tan^4 x \, dx$$

$$\int \frac{\cos x + \sin x}{\sin(2x)} dx$$

$$\int_0^{\pi/3} \tan^5 x \sec x \, dx$$

$$\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx$$

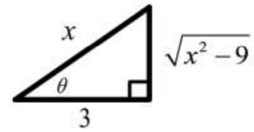
Unit 7-4: Trigonometric Substitution
Larsen: 7.4

Trigonometric Substitution

Some integrals have a form in which a part of the integrand can be represented as one side of a right triangle, and then trig functions of angle θ for this triangle can be used to substitute everything in the original integral. This procedure is called **trigonometric substitution**. Let's look at an example to see the concept...

$$\int \frac{\sqrt{x^2-9}}{x}\,dx$$

The part under the radical could be thought of as being a leg of a right triangle:



Then we can use the other 2 sides to form a substitution: $\cos\theta = \frac{3}{x}$ $\quad x = \frac{3}{\cos\theta}$ $\quad \boxed{x = 3\sec\theta}$

Using the 2 sides with the radical and the constant we get another substitution: $\tan\theta = \frac{\sqrt{x^2-9}}{3}$ $\quad \boxed{\sqrt{x^2-9} = 3\tan\theta}$

The last thing we would need to substitute is *dx* so we take the derivative of the x substitution:

$$x = 3\sec\theta$$
$$\frac{dx}{d\theta} = 3\sec\theta\tan\theta$$
$$\boxed{dx = 3\sec\theta\tan\theta\,d\theta}$$

Now we substitute everything back into the original integral and evaluate:

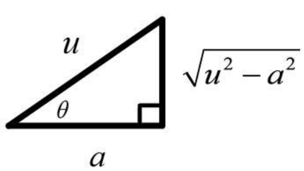
$$\int \frac{\sqrt{x^2-9}}{x}\,dx$$
$$\int \frac{3\tan\theta}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta$$
$$3\int \tan^2\theta\,d\theta$$
$$3\int \left(\sec^2\theta - 1\right)d\theta$$
$$3\int \sec^2\theta\,d\theta + 3\int 1\,d\theta$$
$$3\tan\theta + 3\theta + C$$

Finally, we can substitute out the θ we added:

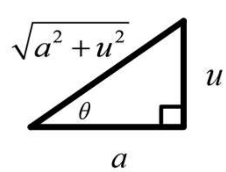
$\sqrt{x^2-9} = 3\tan\theta$ $\quad \theta = \arctan\left(\frac{\sqrt{x^2-9}}{3}\right)$

$$\boxed{\sqrt{x^2-9} + 3\arctan\left(\frac{\sqrt{x^2-9}}{3}\right) + C}$$

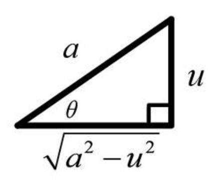
There are 3 triangle structures which work in this way to produce an integral which can be evaluated:



$$u = a\sec\theta$$



$$u = a\tan\theta$$



$$u = a\sin\theta$$

Examples

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx$$

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt$$

Examples

$$\int_0^2 x^3 \sqrt{x^2+4}\, dx$$

When evaluating definite integrals, you can resubsitute all way back to x before you plug in the limits of integration, or you can convert the limits of integration into equivalent θ values to substitute...

$$\int \frac{dt}{\sqrt{t^2-6t+13}}$$

Unit 7-5: Partial Fractions
Larsen: 7.5

Partial Fraction Expansion

Another technique we can use to evaluate integrals is **Partial Fraction Expansion**. This technique works when we have a rational function with degree higher in the denominator, and we can factor the denominator into multiple linear or quadratic factors. We then expand the integral into multiple smaller integrals, one for each factor to evaluate.

Partial Fractions Procedure

1) First, if degree in numerator is not lower than denominator, use polynomial division to divide, producing a polynomial plus a remainder with lower degree in numerator.

2) Factor the denominator into linear and quadratic factors of the form: $(px+q)^m$ and $(ax^2+bx+c)^n$

3) For each (possibly multiple) linear factor, create terms of the form:

$$(px+q)^m \rightarrow \frac{A}{(px+q)}+\frac{B}{(px+q)^2}+....+\frac{F}{(px+q)^m}$$ ...where A,B,...F are (currently unknown) constants.

4) For each (possibly multiple) quadratic factor, create terms of the form:

$$(ax^2+bx+c)^n \rightarrow \frac{Gx+H}{(ax^2+bx+c)}+\frac{Ix+J}{(ax^2+bx+c)^2}+....+\frac{Mx+N}{(ax^2+bx+c)^n}$$ ...where G,H,...N are (currently unknown) constants.

5) Add all these new terms and set equal to the original integrand. Then multiply each new numerator by whatever is needed so that all terms have the same common denominator.

6) Equate the sum of all the numerators with the original integrand's numerator to form the 'basic equation'. The gather all the x^2 terms together, x terms together, and constants together on each side, and form the equations of a system (which have the unknown constants as 'variables') by matching coefficients.

7) Solve the resulting system for the constants A, B.... then fill these in to form the partial fraction expansion.

Partial Fraction Expansion - Simple Example

$$\int \frac{1}{x^2-5x+6}\,dx$$

What if numerator degree is equal to or higher than denominator?

...then you use polynomial division to divide out a polynomial, leaving a remainder to do the partial fractions procedure on...

$$\int \frac{x^3 - x + 3}{x^2 + x - 2}\,dx$$

What if there are multiple copies of factors?

...then you include a term for each 'multiplicity' with its own constants...

$$\frac{1}{(x-5)^3\left(x^2+x+1\right)^2}$$

A very complex example

$$\int \frac{1}{x^3 - 1}\, dx$$

More examples

try these...

$$\int \frac{x+3}{\left(x^2+2x+7\right)\left(x-2\right)}dx$$

$$\int \frac{5x+3}{x^2+x-2}dx$$

$$\int \frac{x^2+1}{x^2-1}dx$$

Unit 7-6: Strategies for Integration
(not in the Larsen textbook)

Summary of Integration Strategies

We've now learned a number of strategies for integration, so let's consider these and when they are applicable.

1) Simplify algebraically / Use Antiderivative Shortcuts

We should always first consider if we can split the integrand into multiple terms, each of which can be integrated using a shortcut:

$$\int \frac{x^3-2x}{\sqrt{x}}\,dx = \int \frac{x^3-2x}{x^{1/2}}\,dx = \int\left(\frac{x^3}{x^{1/2}} - \frac{2x}{x^{1/2}}\right)dx = \int x^{5/2}\,dx - 2\int x^{1/2}\,dx$$

2) Use trig identities

When trig functions are involved, sometimes trig identities will simplify things:

$$\int\left(2\cos^2 x + 2\sin^2 x\right)dx = 2\int 1\,dx$$

$$\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\tan\theta}{1}\frac{1}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\frac{\cos^2\theta}{1}\,d\theta = \int \sin\theta\cos\theta\,d\theta = \frac{1}{2}\int \sin 2\theta\,d\theta$$

3) u-substitution

When one function is 'inside' another or considering the entire denominator as 'inside' 1/x, if we substitute u for the 'inside' function, and the derivative of u appears in the integrand, try u-substitution:

$$\int 3e^{-x^2}x\,dx$$

$$u=-x^2,\ du=-2x\,dx,\ \ x\,dx=-\frac{1}{2}du$$

take derivative to find du, substitute back into original integral:

$$\int 3e^{-x^2}x\,dx = -\frac{3}{2}\int e^u\,du$$

4) Classify integrand by form: trig function powers

If there are trig functions to exponential powers, use the rules for trigonometric integrals. The main idea is to reserve some of the trig function(s) to build the 'du' and write everything else as powers of one trig function.

$$\int \tan^5 x\sec^7 x\,dx = \int \tan^4 x\sec^6 x\sec x\tan x\,dx$$

$$= \int\left(\tan^2 x\right)^2 \sec^6 x\sec x\tan x\,dx = \int\left(\sec^2 x-1\right)^2\sec^6 x\sec x\tan x\,dx$$

$$u=\sec x,\ du=\sec x\tan x\,dx$$

$$= \int\left(u^2-1\right)^2 u^6\,du = \int\left(u^{10}-2u^8+u^6\right)du$$

Sines/Cosines

- Power of sine odd and positive? Reserve one sine for 'du' and convert the rest to cosines.
- Power of cosine odd and positive? Reserve one cosine for 'du' and convert the rest to sines.
- Powers of both sine and cosine even? Repeatedly use $\sin^2 u = \frac{1-\cos 2u}{2}$ and $\cos^2 u = \frac{1+\cos 2u}{2}$

Secants/Tangents

- Power of secant even and positive? Reserve one $\sec^2 x$ for 'du' and convert the rest to tangents.
- Power of tangent odd and positive? Reserve one secx tanx for 'du' and convert the rest to secants.
- Tangent only with even power? Split out one $\tan^2 x$ and convert it to $\sec^2 x$-1, then expand.
- Secant only with odd power? Use integration by parts by splitting off $\sec^2 x$ for dv.
- Try converting to sines and cosines.

In general

- Radicals in the integrand? Convert everything to exponents.

5) Classify integrand by form: rational functions - use partial fractions

For rational functions, try partial fraction expansion:

$$\int \frac{x^3+x}{x-1}\,dx = \int\left(x^2+x+2+\frac{2}{x-1}\right)dx$$

Partial Fractions Procedure

1) First, if degree in numerator is not lower than denominator, use polynomial division to divide, producing a polynomial plus a remainder with lower degree in numerator.

2) Factor the denominator into linear and quadratic factors of the form: $(px+q)^m$ and $(ax^2+bx+c)^n$

3) For each (possibly multiple) linear factor, create terms of the form:

$$(px+q)^m \rightarrow \frac{A}{(px+q)}+\frac{B}{(px+q)^2}+\dots+\frac{F}{(px+q)^m}$$ ...where A,B,...F are (currently unknown) constants.

4) For each (possibly multiple) quadratic factor, create terms of the form:

$$(ax^2+bx+c)^n \rightarrow \frac{Gx+H}{(ax^2+bx+c)}+\frac{Ix+J}{(ax^2+bx+c)^2}+\dots+\frac{Mx+N}{(ax^2+bx+c)^n}$$ ...where G,H,...N are (currently unknown) constants.

5) Add all these new terms and set equal to the original integrand. Then multiply each new numerator by whatever is needed so that all terms have the same common denominator.

6) Equate the sum of all the numerators with the original integrand's numerator to form the 'basic equation'. The gather all the x^2 terms together, x terms together, and constants together on each side, and form the equations of a system (which have the unknown constants as 'variables') by matching coefficients.

7) Solve the resulting system for the constants A, B.... then fill these in to form the partial fraction expansion.

6) Classify integrand by form: products - use integration by parts

If the integrand can be split into two factors multiplied together, try integration by parts:

$$\int \underbrace{x}\,\underbrace{e^x\,dx}$$

deriv. to find du

$$u = x, \qquad dv = e^x dx$$

antiderivative to find dv

$$\frac{du}{dx} = 1 \qquad \int dv = \int e^x dx$$

$$du = 1dx \qquad v = e^x$$

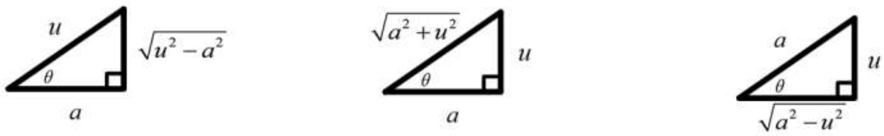
substitute into

$$uv - \int v\,du$$

$$xe^x - \int e^x dx$$

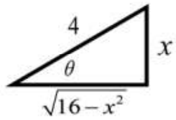
7) Classify integrand by form: radicals

When integrand contains radicals, sometimes you can use trigonometric substitution:



$u = a\sec\theta$ $\qquad$ $u = a\tan\theta$ $\qquad$ $u = a\sin\theta$

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx$$



$$\sin\theta = \frac{x}{4},\quad x = 4\sin\theta,\ dx = 4\cos\theta d\theta$$

$$\cos\theta = \frac{\sqrt{16-x^2}}{4},\quad \sqrt{16-x^2} = 4\cos\theta$$

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = \int \frac{(4\sin\theta)^3}{4\cos\theta} 4\cos\theta d\theta = 64\int \sin^3\theta d\theta$$

then...

$$64\int \sin^3\theta d\theta = 64\int \sin^2\theta \sin\theta\, d\theta = 64\int \left(1-\cos^2\theta\right)\sin\theta\, d\theta$$

$$= 64\int \sin\theta d\theta - 64\int \cos^2\theta \sin\theta d\theta$$

(u-sub)

8) Completing the square in the denominator

When you have a quadratic polynomial in the denominator of a rational function, you can try completing the square to obtain a form which matches the arctan integration shortcut (this sometimes happens for one part of a partial fraction expansion):

$$\int \frac{1}{x^2+x+1}\,dx$$

complete the square...

$$x^2 + x + \tfrac{1}{4} + 1 - \tfrac{1}{4}$$

$$\left(x+\tfrac{1}{2}\right)^2 + \tfrac{3}{4}$$

$$\int \frac{1}{\left(x+\tfrac{1}{2}\right)^2 + \tfrac{3}{4}}\,dx$$

...now matches the arctan shortcut:

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)$$

with $u = x + \tfrac{1}{2}$, $a = \sqrt{3}/2$

$$\frac{2}{\sqrt{3}}\arctan\left(\frac{x+\tfrac{1}{2}}{\left(\sqrt{3}/2\right)}\right) + C$$

If your first strategy doesn't work, try other strategies until you find one that works!

Sometimes, more than one method is required...the first produces multiple integrals which each require further, different, strategies.

Examples

$$\int \frac{1+\cos x}{\sin x}\,dx$$

$$\int \frac{\sqrt{9-x^2}}{x}\,dx$$

$$\int e^{x+e^x}\,dx$$

$$\int_0^1 \left(1+\sqrt{x}\right)^8 dx$$

$$\int \frac{3x^2 - 2}{x^3 - 2x - 8} dx$$

$$\int \frac{1}{x^3 - 8} dx$$

$$\int \sqrt{\frac{1 + x}{1 - x}} dx$$

$$\int \frac{x^4}{x^{10}+16}$$

$$\int \frac{1}{x\sqrt{4x^2+1}}\,dx$$

Unit 7-7: Integration using Tables
(Larsen 7.6)

Integration using Tables

Even though we now have many integration strategies at our disposal, applying these to evaluate more advanced integrals can be time consuming and very difficult. Once we are done with calculus as a class, we may need the results on an integration and we want to take advantage of any way possible to accomplish this.

In order to speed integration of more advanced integrals, mathematicians have pre-solved a number of standard integral forms which appear in **integration tables**. We have an integral table in Appendix B of our textbook, and in today's section we will practicing using integration tables to evaluate integrals.

Must find a matching general structure in the table

First, we try to find an integral in the table whose main form matches the integral under consideration. This often involves identifying one or more constants, and frequently involves first needing to make a u-substitution. This is best seen by looking at examples...

$$\int x\sqrt{x^4-9}\,dx$$

In our textbook's integral table, entry number 26 is: $\int \sqrt{u^2 \pm a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2 \pm a^2} \pm a^2 \ln\left|u+\sqrt{u^2 \pm a^2}\right|\right)+C$

...and we can make the given integral conform to this by making: $u = x^2$

Then: $du = 2xdx \quad so \quad xdx = \frac{1}{2}du$ ...and if we define: $a = 3$

$$\int x\sqrt{x^4-9}\,dx = \int \sqrt{u^2-a^2}\,\frac{1}{2}du = \frac{1}{2}\int \sqrt{u^2-a^2}\,du$$

Now, by table... $= \frac{1}{2}\left[\frac{1}{2}\left(u\sqrt{u^2 \pm a^2} \pm a^2 \ln\left|u+\sqrt{u^2 \pm a^2}\right|\right)\right]+C$

$$= \frac{1}{2}\left[\frac{1}{2}\left(x^2\sqrt{x^4-9} - 9\ln\left|x^2+\sqrt{x^4-9}\right|\right)\right]+C$$

$$\boxed{= \frac{1}{4}\left(x^2\sqrt{x^4-9} - 9\ln\left|x^2+\sqrt{x^4-9}\right|\right)+C}$$

Reduction Formulas

Sometimes, the table doesn't provide the final answer, but removes a portion from the integral and includes a new, simpler integral. These integral forms are known as **reduction formulas**:

$$\int x^3 \sin x\,dx$$

Integral table entry number 54 is: $\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du$

...so to match the table, $u = x,\ n = 3$

and: $\int x^3 \sin x\,dx = -x^3\cos x + 3\underline{\int x^2 \cos x\,dx}$

For the remaining integral, it matches table form 54: $\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du$

...so to match the table, $u = x,\ n = 2$

and: $\int x^2 \cos x\,dx = x^2 \sin x - 2\underline{\int x \sin x\,dx}$

For the last integral, it matches table form 52: $\int u \sin u\,du = \sin u - u\cos u$

and: $\int x \sin x\,dx = \sin x - x\cos x$

Putting this all back together...

$$\int x^3 \sin x\,dx = -x^3\cos x + 3\left(x^2\sin x - 2\left(\sin x - x\cos x\right)\right)+C$$

$$\boxed{= -x^3\cos x + 3x^2\sin x - 6\sin x + 6x\cos x + C}$$

Unit 7-8: Improper Integrals

There are some conditions required in order to use the Fund. Theorem for definite integrals

When we evaluate definite integrals using the Fundamental Theorem of Calculus $\int_a^b f(x)\,dx = F(b) - F(a)$

there are a few conditions that must be met:

- The interval [a,b] must be finite.
- *f(x)* must be continuous over the entire interval [a,b].

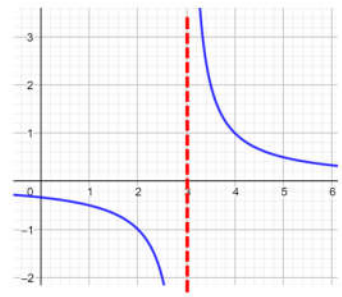
Integrals which do not meet these conditions are called **Improper Integrals**.

Examples of improper integrals:

$$\int_2^{\infty} \frac{1}{2-x^2}\,dx$$

improper because the interval is not finite

$$\int_2^{4} \frac{1}{x-3}\,dx$$



improper because the function is not continuous over [2,4]

Convergence and Divergence of Improper Integrals

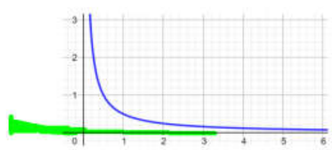
In this section, we will learn techniques for attempting to evaluate improper integrals. Some improper integrals will evaluate to a finite, numerical value and these integrals are said to **converge** to this value. Other improper integrals will evaluate to an infinite value and these integrals are said to **diverge**.

Evaluate by replacing infinite limits or x value at a discontinuity with a constant

There are a couple of variations on the theme, but the main idea for evaluating improper integrals is to replace any limit of integration x-value with a constant, evaluate the integral using the constant, then take the limit as that constant approaches the problem value.

A first example: $\int_1^{\infty} \frac{1}{x^2}\,dx$

First, change to proper integral form: $= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$



This is equivalent to finding the area under this curve from 1 out to infinity. Seemingly, this area might be infinite, but on the other hand, the curve is rapidly approaching zero.

Then, we'll evaluate the indefinite integral... $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \left[-x^{-1}\right] = \left[-\frac{1}{x}\right]$

Now, when we plug in the limits of integration we will use a constant for the infinity and use a limit to evaluate:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]^b - \left[-\frac{1}{x}\right]_1$$

$$= \lim_{b\to\infty} \left[-\frac{1}{b}\right] - \left[-\frac{1}{(1)}\right]$$

$$= 0 - [-1] = \boxed{1}$$

Interestingly, even though we are integrating forever in x, the area **converges** to a finite area of 1.

Another example

$$\int_1^{\infty} \frac{1}{x}\,dx \quad = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx$$

First, we'll evaluate the indefinite integral... $\int \frac{1}{x}\,dx = \ln|x|$

Now, when we plug in the limits of integration we will use a constant for the infinity and use a limit to evaluate:

$$\lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx = \lim_{b\to\infty} \Big[\ln|x|\Big]_1^b$$

$$= \lim_{b\to\infty} \Big[\ln|x|\Big]^b - \Big[\ln|x|\Big]_1$$

$$= \lim_{b\to\infty} \Big[\ln|b|\Big] - \Big[\ln|1|\Big]$$

$$= \infty - [0] = \infty$$

This improper integral **diverges**. The 1/x curve doesn't approach zero as fast as the $1/x^2$ curve, so as x increases to infinity, the area does not stay finite, but grows infinitely.

Forms with infinite integration limits that produce improper integrals

$$\int_a^{\infty} f(x)\,dx$$

to evaluate...

$$\lim_{b\to\infty}\big[F(b)\big] - \big[F(a)\big]$$

$$\int_{-\infty}^{a} f(x)\,dx$$

to evaluate...

$$\big[F(a)\big] - \lim_{b\to\infty}\big[F(b)\big]$$

$$\int_{-\infty}^{\infty} f(x)\,dx$$

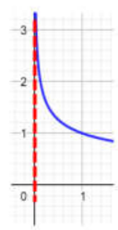
to evaluate, split the integral at any convenient x – value, c...

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

If there is any x-value where f(x) is discontinuous, replace that x value with a constant, use limit

$$\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$$

$$\int \frac{1}{\sqrt[3]{x}}\,dx = \int x^{-1/3}\,dx = \left[\frac{3}{2}x^{2/3}\right]_0^1$$



When we plug in the zero, we must use a limit as x approaches zero from the right side...

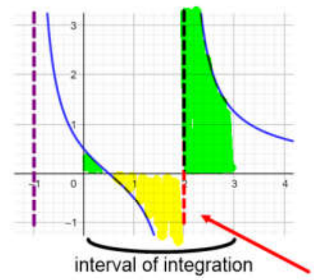
$$\left[\frac{3}{2}(1)^{2/3}\right] - \lim_{b\to 0^+}\left(\frac{3}{2}(b)^{2/3}\right)$$

$$\frac{3}{2}(1) - \frac{3}{2}(0) = \boxed{\frac{3}{2}}$$

This area ***converges*** to 3/2.

Always check to make sure the function is continuous over the integral of integration

$$\int_0^3 \frac{2x-1}{x^2-x-2}\,dx$$



at x=2 is a problem, so we must split the integration here and use limits approaching 2

We must split the integral at x=2...

$$\int_0^2 \frac{2x-1}{x^2-x-2}\,dx + \int_2^3 \frac{2x-1}{x^2-x-2}\,dx$$

Evaluate the indefinite integral (using u-substitution):

$$\int \frac{2x-1}{x^2-x-2}\,dx \quad u = x^2-x-2 \quad du = (2x-1)\,dx$$

$$\int \frac{1}{u}\,du = \ln|u| = \ln|x^2-x-2|$$

Now plug in limits of integration and use limits for the x=2 values:

$$\int_0^2 \frac{2x-1}{x^2-x-2}\,dx + \int_2^3 \frac{2x-1}{x^2-x-2}\,dx$$

$$\left[\ln|x^2-x-2|\right]_0^2 + \left[\ln|x^2-x-2|\right]_2^3$$

$$\lim_{b\to 2^-}\left(\ln|b^2-b-2|\right) - \ln|0| + \left[\ln|3^2-3-2|\right]^3 - \lim_{b\to 2^+}\left(\ln|b^2-b-2|\right)$$

$$-\infty - \mathit{undef} + \ln|3| - (-\infty)$$

$$-\infty - \infty + \ln|3| + \infty$$

The ln0 is undefined but we could use a limit to evaluate this as well because we approaching x from the right:

$$\ln|0| = \lim_{x\to 0^+} \ln|x| = -\infty$$

Positive and negative infinities are fighting for control, but we just say that this improper integral **diverges**.

We can use all of our previous techiques when evaluating the indefinite integral...

We may still need any of our previous methods in order to evaluate the indefinite integral. With an improper integral, what changes is that at any problematic x values, we evaluate using a limit as x is approaching those values.

$$\int_{-\infty}^{1} xe^{2x}\, dx$$