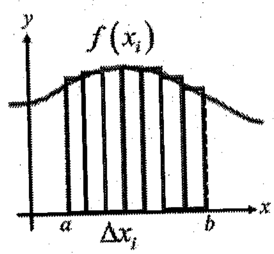
AP Calc BC – Lesson Notes – Unit 6: Integration to find Area, Distance, Volume

Unit 6-1: Area between curves
Larsen: 6.1 (Stewart: 5.1)

Integrals are 'summing machines'

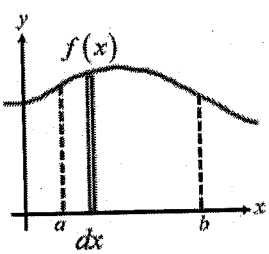
One of our definitions using the integral symbol nicely illustrates the idea that, in general, what an integral does is 'sum up' or 'accumulate' things.

When we first looked at area under a function curve, we did so by adding up subinterval rectangle areas with a Riemann Sum:



$$area = \sum_{i=1}^{n} (rectangle\ area)_i$$
$$= \sum_{i=1}^{n} (height)_i \cdot (height)_i$$
$$= \sum_{i=1}^{n} f(x_i) \cdot \Delta x_i$$

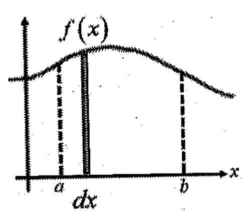
Then we got the definite integral by using an infinite number of infinitely narrow rectangles:



$$area = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x_i$$

$$area = \int_a^b f(x)\,dx$$

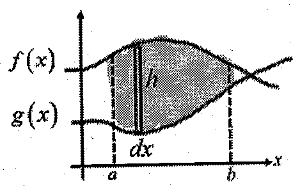
It turns out that integrals can be used to sum up an infinite number of a variety of different things, not just rectangles to produce an area. The only requirement is that one element of the 'thing' being summed must be an infinitely small quantity which can be represented by a differential.



$$area = \int_a^b f(x)\,dx$$

Area between two curves

For example, we could find the area between two function curves:



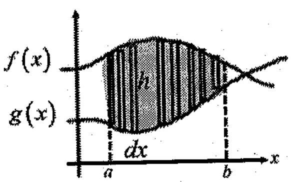
$$area = \int height \bullet width$$

$$= \int_a^b \left(f(x) - g(x)\right) dx$$

most positive curve   most negative curve

Some things to note:

- We want to find a positive area, so the height must be positive, which means we need to be careful to take the 'top' (most positive) curve and subtract the 'bottom' (most negative) curve to form the height.



- We need to imagine that we are 'sliding' the summing rectangle across different x-values to fill the area. This means that each rectangle is at its own unique x-value position, so:
  - Limits of integration are x-values: the lowest and highest x-value we need to 'cover' the region.
  - The function curves must be expressed as functions of the x-variable.

We call this '<u>integrating with respect to x</u>'.

Examples

Find the area enclosed by $y = -x + 1$ and $y = -x^2 + 3x + 1$

intersections: $-x+1 = -x^2+3x+1$

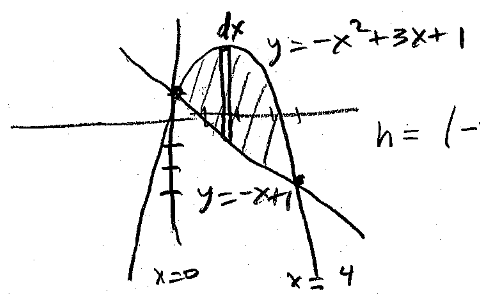
$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0 \quad x = 4$

$y = 1 \quad y = -3$

$(0,1) \quad (4,-3)$



$h = (-x^2+3x+1) - (-x+1)$

$$\int_0^4 \left[(-x^2+3x+1) - (-x+1)\right] dx$$

$$\int_0^4 (-x^2+4x)\, dx$$

$$\left[-\frac{1}{3}x^3 + 2x^2\right]_0^4$$

$$\left(-\frac{64}{3} + 32\right) - (0+0)$$

$$\boxed{\frac{32}{3}}$$

intersections: $-y=-y^2+2y$

$y^2-3y=0$

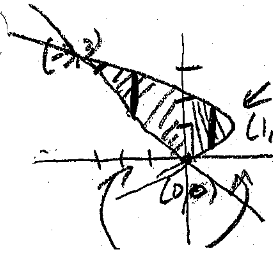
$y(y-3)=0$ $\quad (0,0)\ (-3,3)$

$y=0 \quad y=3$

$x=0 \quad x=-3$

Examples

Find the area enclosed by $x=-y$ and $x=-y^2+2y$



2 regions

where is this? (max y)

$\frac{dx}{dy}=-2y+2=0$

when $y=1$

$x=-(1)^2+2(1)=1$

$x=-y \rightarrow y=-x$

$x=-y^2+2y$

$x-1=-(y^2-2y+1)$

$x-1=-(y-1)^2$

$(y-1)^2=1-x$

$y-1=\pm\sqrt{1-x}$

$y=1\pm\sqrt{1-x}$

$\int_{-3}^{0}[(1+\sqrt{1-x})-(-x)]dx+\int_{0}^{1}[(1+\sqrt{1-x})-(1-\sqrt{1-x})]dx$

$\int_{-3}^{0}1\,dx+\int_{-3}^{0}\sqrt{1-x}\,dx+\int_{-3}^{0}x\,dx+2\int_{0}^{1}\sqrt{1-x}\,dx$

$u=1-x$ $\qquad u=1-x$

$du=-dx$ $\qquad du=-dx$

$-\int_{4}^{1}u^{1/2}du$ $\qquad -2\int_{1}^{0}u^{1/2}du$

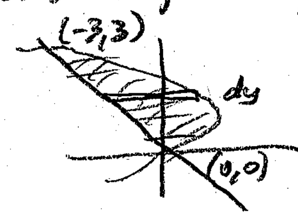
$\int_{1}^{4}u^{1/2}du$ $\qquad +2\int_{0}^{1}u^{1/2}du$

$[x]_{-3}^{0}+\left[\frac{2}{3}u^{3/2}\right]_{1}^{4}+\left[\frac{1}{2}x^2\right]_{-3}^{0}+2\left[\frac{2}{3}u^{3/2}\right]_{0}^{1}$

$0+3+\frac{16}{3}-\frac{2}{3}+0-\frac{9}{2}+\frac{4}{3}-0$

$\boxed{\frac{9}{2}}$

... but what if we made the rectangles horizontally?



new height $=$

$(-y^2+2y)-(-y)$

$\int_{0}^{3}[(-y^2+2y)-(-y)]dy$

$\int_{0}^{3}(-y^2+3y)dy$

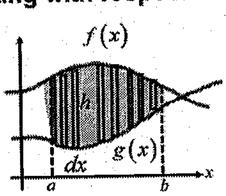
$\left[-\frac{1}{3}y^3+\frac{3}{2}y^2\right]_{0}^{3}$

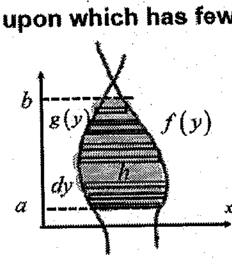
$\left(-9+\frac{27}{2}\right)-(0-0)$

$\boxed{\frac{9}{2}}$

much easier!

Select integrating with respect to x or y depending upon which has fewer regions





$area=\int_{a}^{b}(f(x)-g(x))\,dx$ $\qquad area=\int_{a}^{b}(f(y)-g(y))\,dy$

__Remember__:

Everything in the integral is either all x or all y (the variables and the limits of integration)

Use a graph to determine which function is more positive (may be different in different regions)

Ex) Find the area enclosed by the functions: $f(x)=\sqrt[3]{x-1}$

$g(x)=x-1$

intersections: $\sqrt[3]{x-1}=x-1$

$x-1=(x-1)^3=(1)(x)^3(-1)^0+(3)(x)^2(-1)^1+(3)(x)^1(-1)^2+(1)(x)^0(-1)^3$

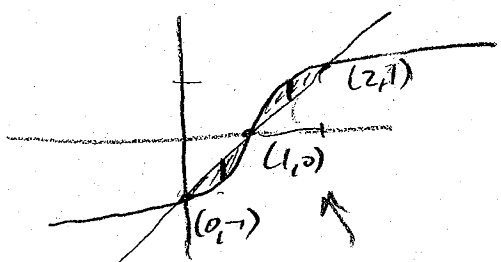
$x-1 \quad x^3-3x^2+3x-1$ $\qquad x^3-3x^2+2x=0$

$x(x^2-3x+2)=0$

$x(x-1)(x-2)=0$

$x=0 \quad x=1 \quad x=2 \qquad (0,-1)\ (1,0)\ (2,1)$

$y=-1 \quad y=0 \quad y=1$



this region:

top: $y=\sqrt[3]{x-1}$

bottom: $y=x-1$

this region:

top: $y=x-1$

bottom: $y=\sqrt[3]{x-1}$

$$\int_0^1 [(x-1)-\sqrt[3]{x-1}]dx+\int_1^2(\sqrt[3]{x-1}-(x-1))dx$$

$$\int_0^1(x-1)dx-\int_0^1(x-1)^{1/3}dx+\int_1^2(x-1)^{1/3}dx-\int_1^2(x-1)dx$$

$u=x-1, \quad dx=du$

$$\int_{-1}^0 u\,du-\int_{-1}^0 u^{1/3}du+\int_0^1 u^{1/3}du-\int_0^1 u\,du$$

$$\left[\tfrac{1}{2}u^2\right]_{-1}^0-\left[\tfrac{3}{4}u^{4/3}\right]_{-1}^0+\left[\tfrac{3}{4}u^{4/3}\right]_0^1-\left[\tfrac{1}{2}u^2\right]_0^1$$

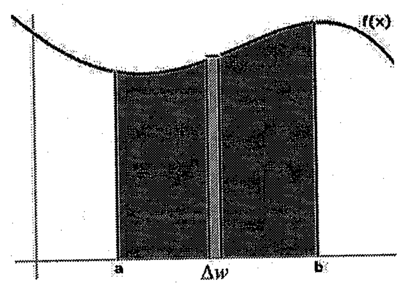
$$\left(0-\tfrac{1}{2}\right)-\left(0-\tfrac{3}{4}\right)+\left(\tfrac{3}{4}-0\right)-\left(\tfrac{1}{2}-0\right)$$

$$=\boxed{\tfrac{1}{2}}$$

Unit 6-2: Volumes of Solids of Revolution (Disc/Washer), Cross-Sections
Larsen: 6.2 (Stewart: 6.2)

Integrals are 'summing machines'

We can use an integral to find the area under a function curve (between the curve and the x-axis):

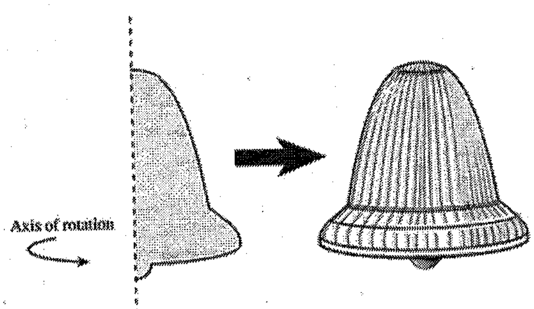


The area is a summation of an infinite number of small rectangles:

$$A = \sum (\textit{area of rectangle})$$
$$A = \sum \textit{height} \bullet \textit{width}$$
$$A = \int_a^b h \bullet \Delta w$$
$$A = \int_a^b f(x)\,dx$$

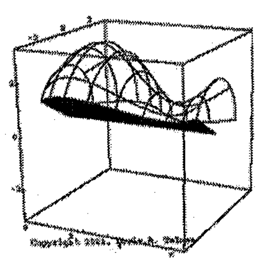
Instead of area elements, we can sum volume elements

Have you ever seen one of those tissue paper accordion-style decorations? They start out as a two-dimensional cardboard shape, but as you open them, you get a three-dimensional shape:

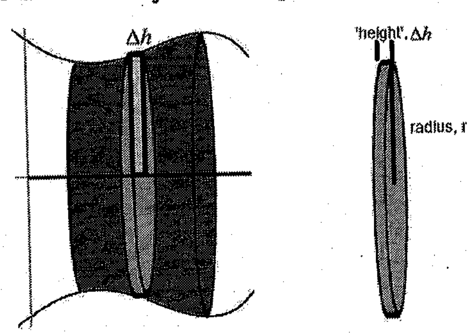


Volumes of Solids of Revolution - Disc Method

When we rotate a 2-D area around an axis, it forms a 3-D **solid of revolution**:

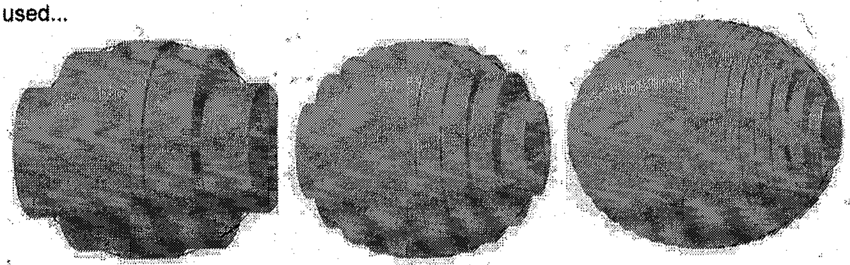


We can then calculate the area by summing a series of infinitely thin 'discs' (cylinders):



Volumes of Solids of Revolution - Disc Method

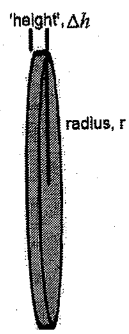
Of course, the volume becomes closer to the actual volume when more cylinders are used...



(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

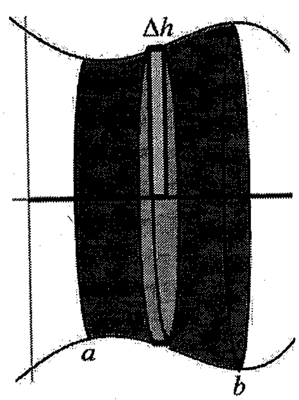
...so we will sum an infinite number of infinitely thin cylinders using an integral.

From geometry, the volume of a right circular cylinder is $V_{cylinder} = \pi r^2 h$
so the volume in our small (infinitely thin) cylinder is:



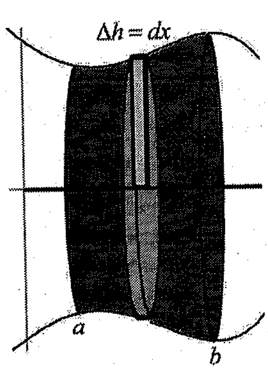
$$V_{thin\ cylinder} = \pi r^2 \Delta h$$

We can therefore use an integral to find the summation of a series of these cylinder volumes to find the volume of the solid of revolution:



$$V = \sum (volume\ of\ cylinder)$$
$$V = \sum \pi r^2 \cdot height$$
$$V = \int_a^b \pi r^2 \Delta h$$

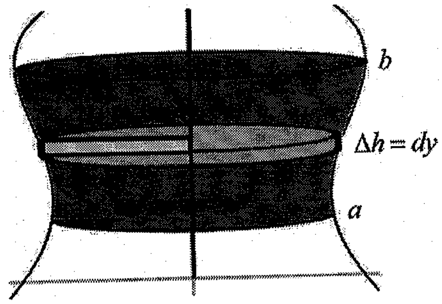
For this solid of revolution, the radius, r, is also 'y' which is f(x), and the small height, Δh, is a small change in 'x', which we would write as 'dx':



$$V = \sum (volume\ of\ cylinder)$$
$$V = \sum \pi r^2 \cdot height$$
$$V = \int_a^b \pi r^2 \Delta h$$
$$V = \int_a^b \pi y^2 dx$$
$$\boxed{V = \int_a^b \pi [f(x)]^2 dx}$$

Volumes of Solids of Revolution - Disc Method

Instead of the function y=f(x), we could have a function x=f(y) and the solid could be revolving around the y-axis. In that case, the radius would be an 'x' value, and the Δh would be a change in y, dy:



$$V = \sum (\textit{volume of cylinder})$$

$$V = \sum \pi r^2 \cdot \textit{height}$$

$$V = \int_a^b \pi r^2 \Delta h$$

$$V = \int_a^b \pi x^2 dy$$

$$\boxed{V = \int_a^b \pi \left[f(y) \right]^2 dy}$$

This suggests a procedure we could use to find the volume of a solid of revolution:

1) Draw a sketch (and show 3-D rotation into a solid).
2) Select method ('disc').
3) Draw the rectangle (disc: rectangle is Δh $\perp$ to axis of rotation).
4) Determine if this Δh is *dx* or *dy*.
5) Rotate the rectangle to make the cylinder shape and use geometry to write *r* in terms of *x* or *y*.
6) Build an integral for the volume by summing cylinder areas:

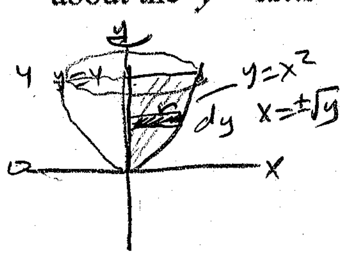
   disc method: $V = \int_a^b \pi r^2 dh$

7) Substitute to get everything in terms of the integration variable (set by *dx* or *dy*).
8) Evaluate the integral.

Volume Examples (Disc Method)

Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

$y=x^2$, y-axis, $y=4$, in the first quadrant; about the $y-axis$



$$V = \int \pi r^2 dh$$

$$= \int \pi x^2 dy$$

$$= \int \pi (\sqrt{y})^2 dy$$

$$= \int_0^4 \pi\, y\, dy$$

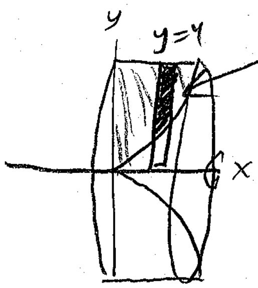
$$= \pi \left[\tfrac{1}{2} y^2 \right]_0^4$$

$$= \pi (8 - 0)$$

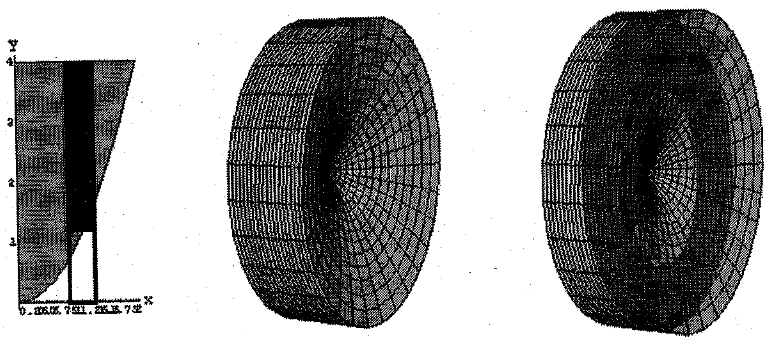
$$= \boxed{8\pi}$$

Volumes of Solids of Revolution - Washer Method

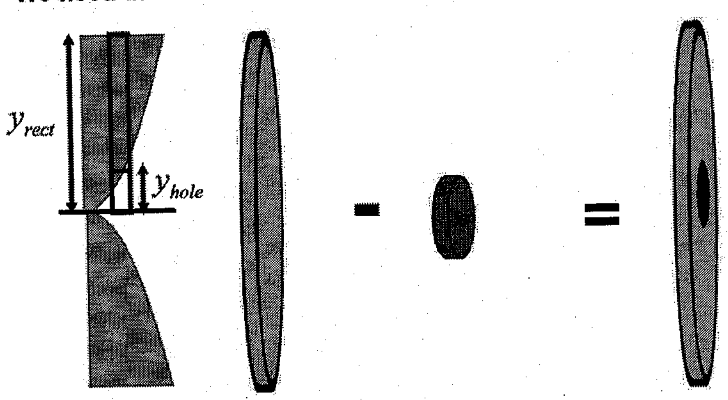
$y = x^2$, y-axis, $y = 4$, in the first quadrant;
about the x-axis



only part of the rectangle is filled....



We need the Washer Method when only part of the rectangle is filled with material.



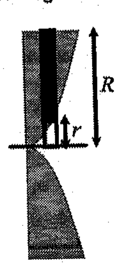
We find the volume of the solid 'washer' by finding the volume of the entire disc, and then subtracting the volume in the hole.

These volumes have the same 'width' but different 'heights'.

$$V = V_{entire\ cylinder} - V_{hole}$$

$$V = \int_a^b \pi {y_{rect}}^2 dx - \int_a^b \pi {y_{hole}}^2 dx$$

A better and more flexible way to think of this is to build a single integral for the volume but for the height use the height of just the portion of the rectangle which is filled with material, defining two radii: *R* for the larger 'outer' radius, and *r* for the smaller 'inner' radius:



$$V = \int_a^b \left[\pi R^2 dh - \pi r^2 dh\right]$$

Volumes of Solids of Revolution - Disc/Washer Methods

Here is an updated procedure covering both cases:

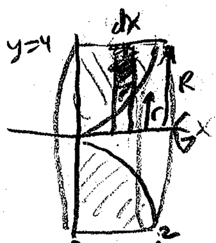
1) Draw a sketch (and show 3-D rotation into a solid).
2) Draw the rectangle (disc: rectangle is $\Delta h \perp$ to axis of rotation).
3) Select method: disc if rectangle is completely filled, washer if partially filled.
4) Determine if this Δh is *dx* or *dy*.
5) Rotate the rectangle to make the cylinder shape and use geometry to write *r* in terms of *x* or *y*.
6) Build an integral for the volume by summing cylinder areas:

   disc method: $V=\int_a^b \pi r^2 dh$ washer method: $V=\int_a^b \pi\left[R^2-r^2\right] dh$

7) Substitute to get everything in terms of the integration variable (set by *dx* or *dy*).
8) Evaluate the integral.

Volumes Example (Washer Method)

$y=x^2$, y-axis, $y=4$, in the first quadrant; about the x-axis



$R=4$
$r=x^2$

$$V=\int \pi[R^2-r^2]dh$$

$$=\int_0^2 \pi[4^2-(x^2)^2]dx$$

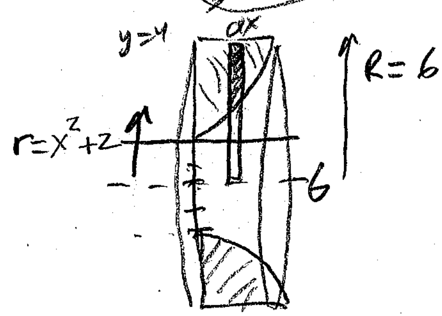
$$=\pi\int_0^2(16-x^4)dx$$

$$=\pi\left[16x-\tfrac{1}{5}x^5\right]_0^2$$

$$=\pi\left[\left(32-\tfrac{32}{5}\right)-0\right]$$

$$=\boxed{\frac{128\pi}{5}}\approx 80.425$$

$y=x^2$, y-axis, $y=4$, in the first quadrant; about $y=-2$



$$V=\int \pi[R^2-r^2]dh$$

$$=\int_0^2 \pi[6^2-(x^2+2)^2]dx$$

$$=\pi\int_0^2(36-x^4-4x^2-4)dx$$

$$=\pi\left[32x-\tfrac{1}{5}x^5-\tfrac{4}{3}x^3\right]_0^2$$

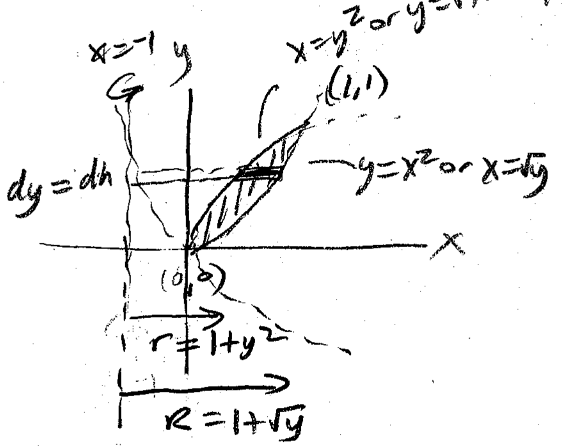
$$=\pi\left[\left(64-\tfrac{32}{5}-\tfrac{32}{3}\right)-0\right]$$

$$=\boxed{\frac{704\pi}{15}}\approx 147.485$$

(larger because the same cross sectional area is sweeping around a larger circle)

Volumes Examples

$y = x^2$, $x = y^2$; about $x = -1$



intersections: (treat as a system)

$$\begin{cases} y=x^2 \\ x=y^2 \end{cases}$$

$y=x^2$

$y=(y^2)^2$

$y=y^4$

$y-y^4=0$

$y(1-y^3)=0$

$y=0$ or $1-y^3=0$

$x=0^2=0$ $\quad y^3=1$

$(0,0)$ $\quad y=1$

$x=(1)^2=1$

$(1,1)$

V 'hole', 2 integrals:

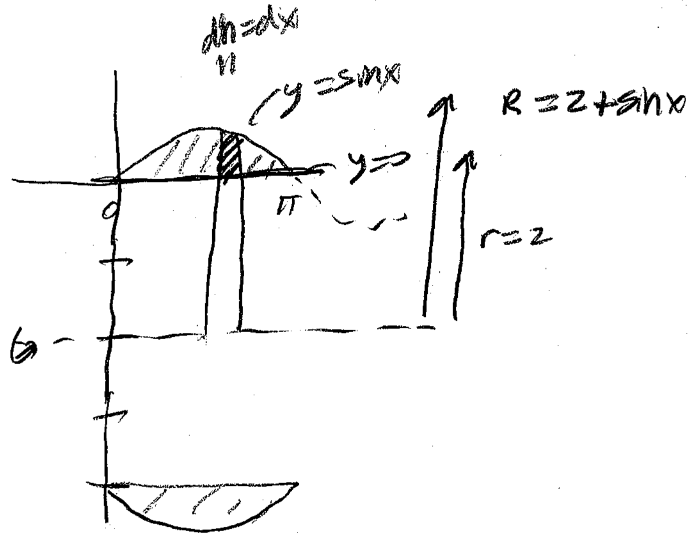
$$\int \pi R^2 dh - \int \pi r^2 dh$$

$$\boxed{\int_0^1 \pi(1+\sqrt{y})^2 dy - \int_0^1 \pi(1+y^2)^2 dy}$$

math 9 $\approx \boxed{3.0369}$

Set up, but **do not evaluate**, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

$y=0, \quad y=\sin x, \quad 0 \le x \le \pi; \quad$ about $y=-2$



$$V = \int \pi R^2 dh - \int \pi r^2 dh$$

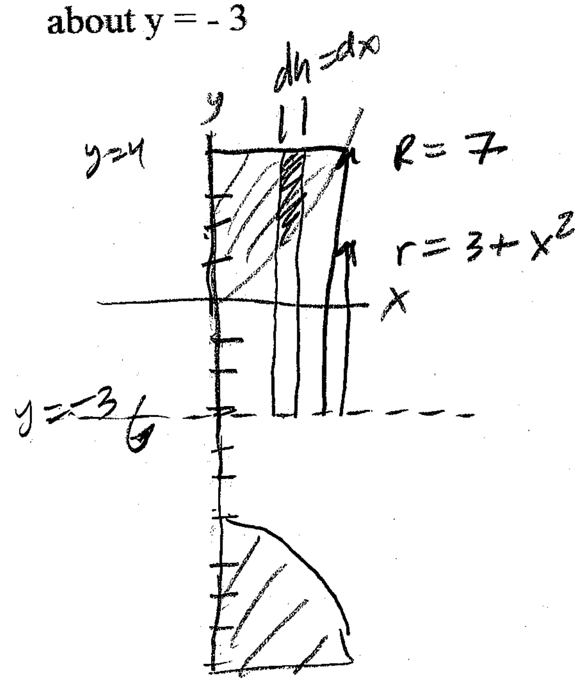
$$= \boxed{\int_0^{\pi} \pi(2+\sin x)^2 dx - \int_0^{\pi} \pi(2)^2 dx}$$

$$= \int_0^{\pi} \pi\left[(2+\sin x)^2 - 4\right] dx$$

math 9 $\approx \boxed{30.0675}$

$y=x^2$, y-axis, $y=4$, in the first quadrant;

about y = - 3



$$V = \int \pi R^2 dh - \int \pi r^2 dh$$

$$= \boxed{\int_0^{2} \pi(7)^2 dx - \int_0^{2} \pi(3+x^2)^2 dx}$$

$$= \int_0^{2} \pi\left[49 - (3+x^2)^2\right] dx$$ math 9

$\approx \boxed{180.956}$

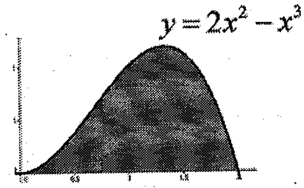
divide by π (math → fraction)

$= \boxed{\frac{288\pi}{5}}$

Unit 6-3: Volumes of Solids of Revolution (Shell)
Larsen: 6.3 (Stewart: 6.3)

Some problems are difficult to solve in the Disc/Washer method:

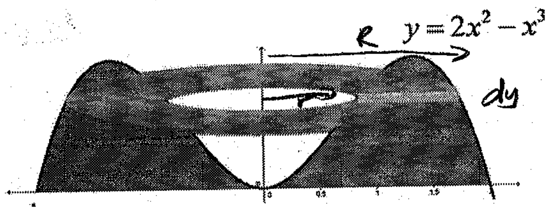
Find volume of solid obtained by rotating about the y-axis



R and r both come from $y = 2x^2 - x^3$ and are x-values, so we would have solve...

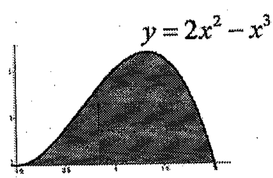
$y = 2x^2 - x^3$

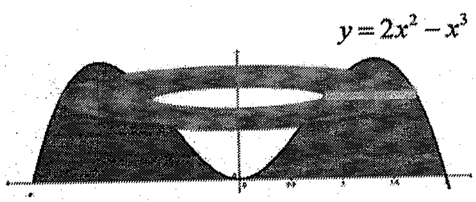
... for x (which is not possible)



Volumes - Shell Method

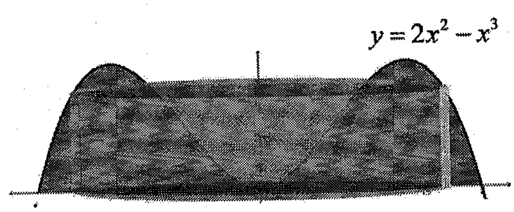
Find volume of solid obtained by rotating about the y-axis





'Disc/Washer Method'

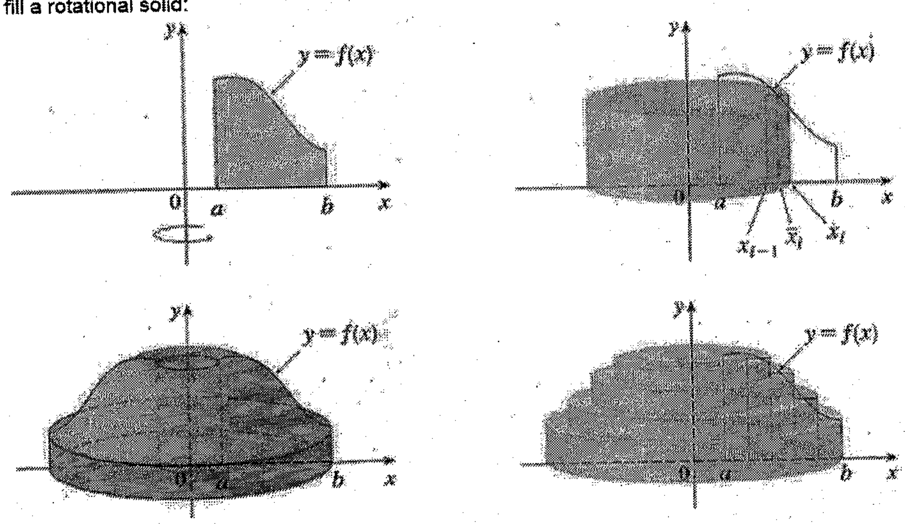
rectangle $\perp$ *axis of rotation*



'Shell Method'

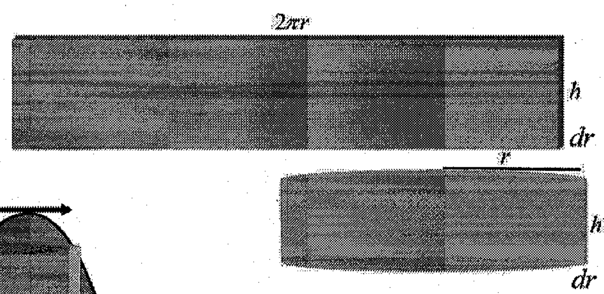
rectangle $\parallel$ *axis of rotation*

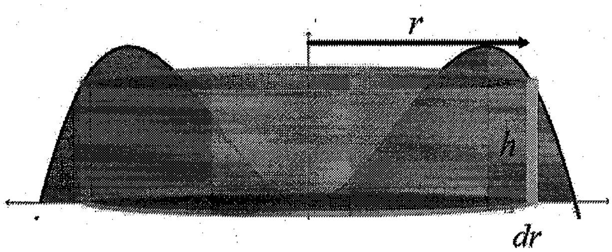
Rotating a rectangle parallel to axis of rotation around creates a 'shell' and these shells can be nested to fill a rotational solid:



Volumes - Shell Method

What is the volume of the shell element? A thin-walled cylindrical shell 'folds out' to become a rectangular box:





$$V_{shell} = 2\pi r h dr$$

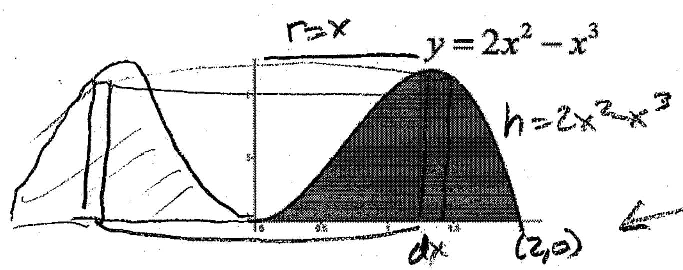
Volumes - Shell Method Example

Find volume of solid obtained by rotating about the y-axis

$2x^2 - x^3 = 0$

$x^2(2-x) = 0$

$x = 0 \quad x = 2$



$$V = \int 2\pi r h\, dr$$

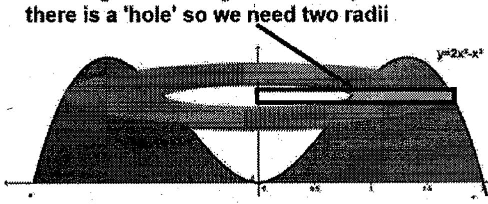
$$= \int_0^2 2\pi x(2x^2-x^3)\,dx = 2\pi \int_0^2 (2x^3-x^4)\,dx$$

$$= 2\pi\left[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5\right]_0^2$$

$$= 2\pi\left(8 - \tfrac{32}{5}\right) = \boxed{\tfrac{16\pi}{5}}$$

Shell is better when there is 'no hole' or when you can't solve for an expression

Rectangle doesn't go all the way to axis there is a 'hole' so we need two radii

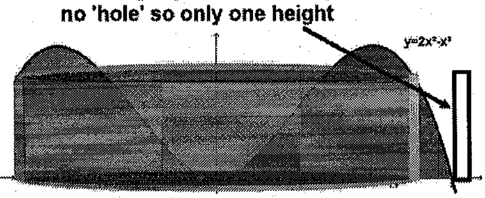


'Disc/Washer Method'

rectangle $\perp$ *axis of rotation*

$$V = \int \pi\left[R^2 - r^2\right] dh$$

Rectangle goes all the way to axis no 'hole' so only one height

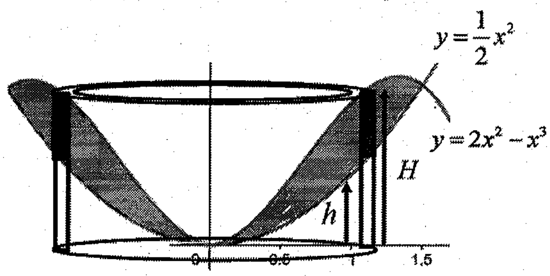


'Shell Method'

rectangle $\parallel$ *axis of rotation*

$$V = \int 2\pi r h dr$$

...although you can need two heights for shell in some circumstances...

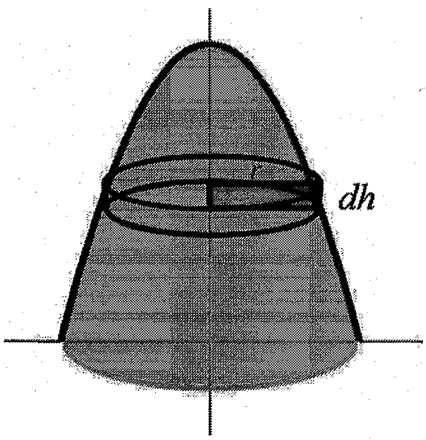


'Shell Method with two curves'

$$V = \int 2\pi r\left(H-h\right)dr$$

Summary of Methods for Volume of Solids of Rotation

'Disc/Washer Method'

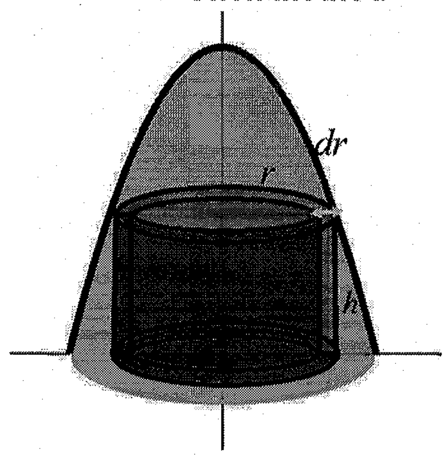


rectangle $\perp$ *axis of rotation*

rectangle filled: $V = \int \pi r^2 dh$

rectangle not filled: $V = \int \pi\left[R^2 - r^2\right]dh$

'Shell Method'



rectangle $\parallel$ *axis of rotation*

rectangle filled: $V = \int 2\pi rh\,dr$

rectangle not filled: $V = \int 2\pi r\left(H-h\right)dr$

Volumes of Solids of Revolution - Disc/Washer/Shell Methods

1) Draw a sketch (and show 3-D rotation into a solid).
2) Select method: disc/washer or shell based upon writing expressions or which has only 1 radii or height.
3) Draw the rectangle (disc: rectangle $\perp$, shell: rectangle $\parallel$ to axis of rotation).
4) Determine if the rectangle width is *dx* or *dy* (this establishes variable of integration).
5) Rotate the rectangle to make the cylinder shape and use geometry to write radii and heights in terms of *x* or *y*.
6) Build an integral for the volume by summing cylinder areas...

disc/washer method:

rectangle filled: $V = \int \pi r^2 dh$

rectangle not filled: $V = \int \pi\left[R^2 - r^2\right]dh$

shell method:

rectangle filled: $V = \int 2\pi rh\,dr$

rectangle not filled: $V = \int 2\pi r\left(H-h\right)dr$

7) Substitute to get everything in terms of the integration variable (set by *dx* or *dy*).
8) Evaluate the integral.

Volumes Examples

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis. Sketch the region and a typical shell.

$$y = x^2 - 6x + 10, \quad y = -x^2 + 6x - 6$$

intersections:

$x^2 - 6x + 10 = -x^2 + 6x - 6$

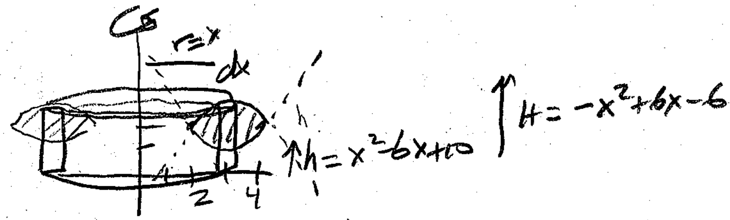
$2x^2 - 12x + 16 = 0$

$2(x^2 - 6x + 8) = 0$

$2(x-2)(x-4) = 0$

$x = 2 \quad x = 4$

$y = 2 \quad y = 2$



$V = \int 2\pi r (H - h)\, dr$

$= \int_2^4 2\pi x [(-x^2+6x-6) - (x^2 - 6x + 10)]\, dx = 2\pi \int_2^4 (-2x^3 + 12x^2 - 16x)\, dx$

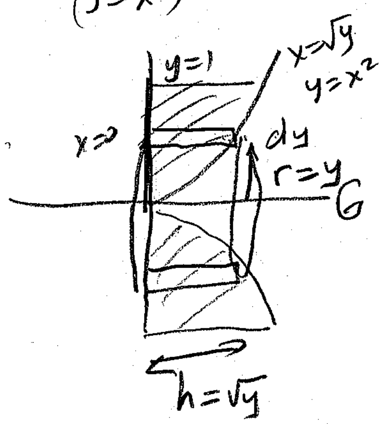
$= 2\pi \left[-\frac{1}{2}x^4 + 4x^3 - 8x^2 \right]_2^4$

$= 2\pi [(0 - (-8)] = \boxed{16\pi}$

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the x-axis. Sketch the region and a typical shell.

$$x = \sqrt{y}, \quad x = 0, \quad y = 1$$

$(y = x^2)$



$V = \int 2\pi r h\, dr$

$= \int_0^1 2\pi y \sqrt{y}\, dy$

$= 2\pi \int_0^1 y^{3/2}\, dy$

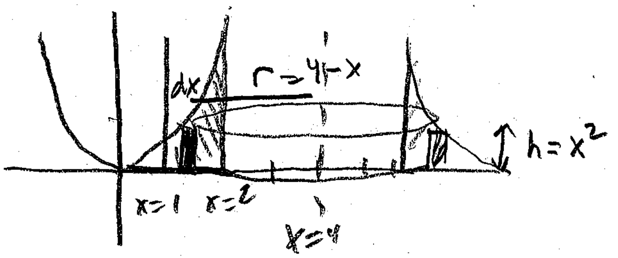
$= 2\pi \left[\frac{2}{5} y^{5/2} \right]_0^1$

$= \boxed{\frac{4\pi}{5}}$

Volumes Examples

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis. Sketch the region and a typical shell.

$y = x^2, \quad y = 0, \quad x = 1, \quad x = 2;$ **about** $x = 4$



$$V = \int 2\pi r h \, dr$$

$$= \int_1^2 2\pi (4-x) x^2 \, dx$$

$$= 2\pi \int_1^2 (4x^2 - x^3) \, dx$$

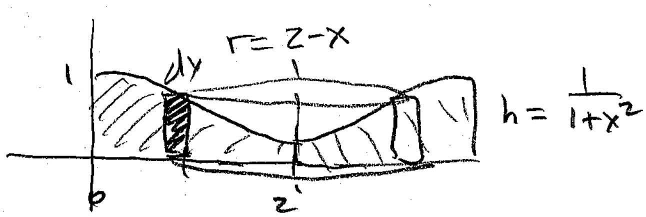
$$= 2\pi \left[\tfrac{4}{3}x^3 - \tfrac{1}{4}x^4 \right]_1^2$$

$$= 2\pi \left(\left[\tfrac{32}{3} - 4 \right] - \left[\tfrac{4}{3} - \tfrac{1}{4} \right] \right)$$

$$= \boxed{\frac{67\pi}{6}}$$

Set up, but **do not evaluate**, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

$y = \dfrac{1}{(1 + x^2)}, \quad y = 0, \quad x = 0, \quad x = 2;$ **about** $x = 2$



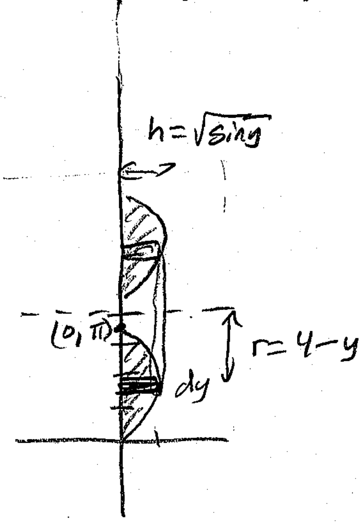
$$V = \int 2\pi r h \, dr$$

$$= \boxed{\int_0^2 2\pi (2-x)\left(\frac{1}{1+x^2}\right) dx}$$

Volumes Examples

Set up, but **do not evaluate**, an integral....

$x = \sqrt{\sin y}, \quad 0 \le y \le \pi, \quad x = 0; \quad$ **about** $y = 4$



$$V = \int 2\pi r h \, dr$$

$$= \boxed{\int_0^{\pi} 2\pi (4-y)\sqrt{\sin y}\, dy}$$

Use a graph to estimate the x-coordinates of the points of intersection of the given curves. Then use this information to estimate the volume of the solid obtained by rotating about the y-axis the region enclosed by these curves.

$y = x^4, \quad y = 3x - x^3$

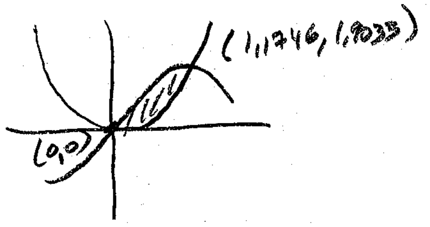
intersections:

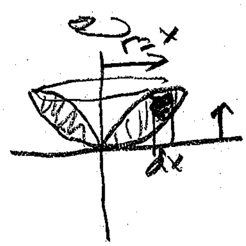
$x^4 = 3x - x^3$

$x^4 + x^3 - 3x = 0$

$x(x^3 + x^2 - 3) = 0$

$x = 0 \quad x = ?$ (graph w/calculator)





$h = x^4$ $H = 3x - x^3$

$$V = \int 2\pi r (H - h)\, dr$$

(math 9)

$$= \int_0^{1.1745594} 2\pi\, x(3x - x^3 - x^4)\, dx \approx \boxed{4.622}$$

Volumes Examples

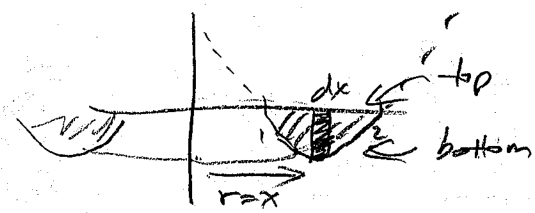
The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid by any method.

$y = x^2 - 3x + 2, \quad y = 0;$ about the y-axis

X ints: $x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

$x = 1 \quad x = 2$



$$V = \int 2\pi r (H - h)\, dr$$

$$= \int_1^2 2\pi \cdot x \left(0 - (x^2 - 3x + 2)\right) dx$$

$$= 2\pi \int_1^2 (-x^3 + 3x^2 - 2x)\, dx$$

$$= 2\pi \left[-\tfrac{1}{4}x^4 + x^3 - x^2 \right]_1^2$$

$$= 2\pi \left((-4 + 8 - 4) - (-\tfrac{1}{4} + 1 - 1) \right)$$

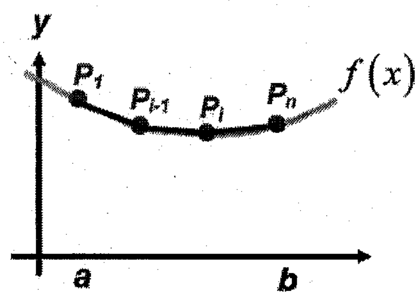
$$= \boxed{\frac{\pi}{2}}$$

Unit 6-4: Arc Length, Surface Areas of Revolution
Larsen: 6.4 (Stewart: 8.1/8.2)

Arc Length

We've been using integrals to add up elements of area and volume, but we can also use an integral to sum small linear segments to find the **arc length** along a function curve path:

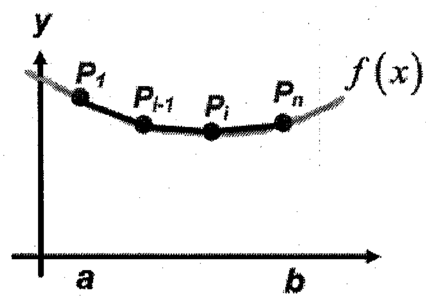
We can first imagine approximating the arc length as a series of small line segments, between points along the function curve from some starting x-value=a at point P_1 and ending a x-value=b at point P_n.



The arc length, L, would then be the summation of the individual line segment lengths. To find each segment length, we use the distance formula. Then, to make this approximation very close to the actual curve, we take the limit to use an infinite number of infinitely small line segments:

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}\left|P_{i-1}P_i\right|$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(\Delta x)^2 + (\Delta y)^2}$$



By the mean value theorem: $f'(x) = \dfrac{\Delta y}{\Delta x}$, $\Delta y = f'(x)\,\Delta x$ so...

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(\Delta x)^2 + \left[f'(x)\,\Delta x\right]^2}$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+\left[f'(x)\right]^2}\sqrt{(\Delta x)^2}$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+\left[f'(x)\right]^2}\;\Delta x$$

Finally, using an integral for the limit of the summation...

$$\boxed{L = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx}$$ **Arc Length Formula**

Arc Length Example

Graph the curve and visually estimate its length. Then find its exact length.

$$y=\frac{x^3}{6}+\frac{1}{2x},\quad \frac{1}{2}\le x\le 1$$

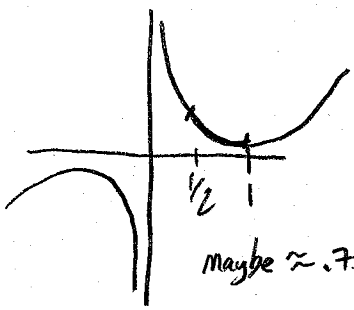
$y = \frac{1}{6}x^3+\frac{1}{2}x^{-1}$

$L = \int \sqrt{1+(f'(x))^2}\,dx$

$y' = \frac{1}{2}x^2 - \frac{1}{2}x^{-2} = \frac{1}{2}$

$= \int_{1/2}^{1} \sqrt{1+\left(\frac{1}{2}x^2-\frac{1}{2}\frac{1}{x^2}\right)^2}\,dx$

(math 9) $\approx \boxed{0.6458}$



maybe $\approx .75$?

if you need exact, or no calculator:

$1+\left(\frac{1}{2}x^2-\frac{1}{2}\frac{1}{x^2}\right)\left(\frac{1}{2}x^2-\frac{1}{2}\frac{1}{x^2}\right)$

$1+\frac{1}{4}x^4-\frac{1}{2}+\frac{1}{4}\frac{1}{x^4}$

$\frac{4}{4}+\frac{1}{4}x^4-\frac{2}{4}+\frac{1}{4}\frac{1}{x^4}$

$\frac{1}{4}\left(x^4-2+\frac{1}{x^4}\right)$

$\frac{1}{4}\left(x^2+\frac{1}{x^2}\right)^2$

$L = \int_{1/2}^{1} \sqrt{1+\left(\frac{1}{2}x^2-\frac{1}{2}\frac{1}{x^2}\right)^2}\,dx$

$= \int_{1/2}^{1} \sqrt{\frac{1}{4}\left(x^2+\frac{1}{x^2}\right)^2}\,dx$

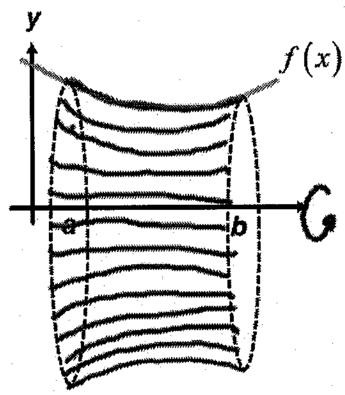
$= \frac{1}{2}\int_{1/2}^{1} (x^2+x^{-2})\,dx$

$= \frac{1}{2}\left[\frac{1}{3}x^3 - x^{-1}\right]_{1/2}^{1}$

$= \boxed{\frac{1}{2}\left[\left(\frac{1}{3}-1\right)-\left(\frac{1}{24}-2\right)\right]} = 0.6458$

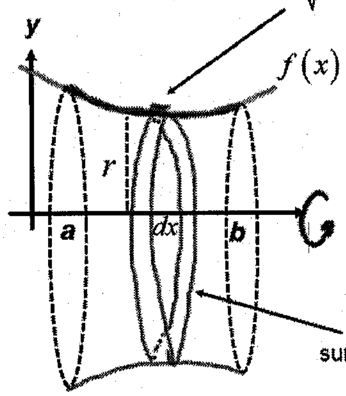
Surface Area of a Surface of Revolution

If you take an arc length and rotate it about an axis it forms a **surface of revolution**



We can imagine taking a small linear element on the arc length, and rotating this around the axis to form a strip in the form of a ring:

arc length element: $\sqrt{1+[f'(x)]^2}\,dx$



(circumference of circle is a length) x (width)

surface area element (ring): $2\pi r\sqrt{1+[f'(x)]^2}\,dx$

Summing these elements with an integral along the path:

$$\boxed{A_{surface}=\int_a^b 2\pi r\sqrt{1+[f'(x)]^2}\,dx}$$

Surface Area of Surface of Revolution Formula

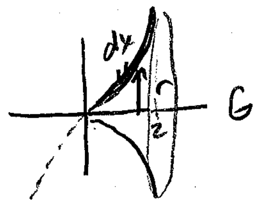
Surface Area of a Surface of Revolution Example

Find the area of the surface obtained by rotating the curve about the x-axis.

$y = x^3,\ 0 \le x \le 2$

$y = x^3$

$y' = 3x^2$



$$SA = \int 2\pi r \sqrt{1+(f'(x))^2}\,dx$$

$$= \int_0^2 2\pi\, x^3 \sqrt{1+(3x^2)^2}\,dx$$

$$= 2\pi \int_0^2 x^3 \sqrt{1+9x^4}\,dx$$

$u = 1+9x^4 \quad du = 36x^3\,dx$

$x^3\,dx = \frac{1}{36}\,du$

$$= \frac{2\pi}{36} \int_1^{145} u^{1/2}\,du$$

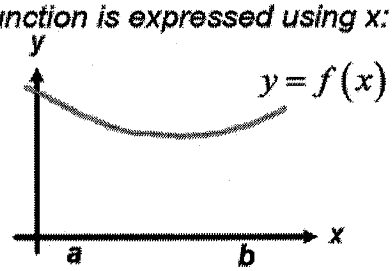
$$= \frac{2\pi}{36} \left[\frac{2}{3} u^{3/2}\right]_1^{145}$$

$$= \frac{\pi}{27}\left[145\sqrt{145} - 1\right] \approx \boxed{203.044}$$

Can use either dx or dy depending upon direction

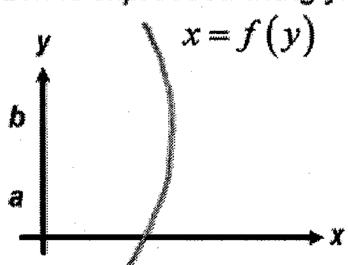
Arc Length

If function is expressed using x:



$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

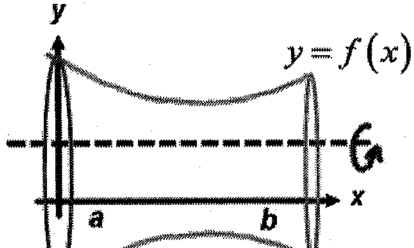
If function is expressed using y:



$$L = \int_a^b \sqrt{1+[f'(y)]^2}\,dy$$

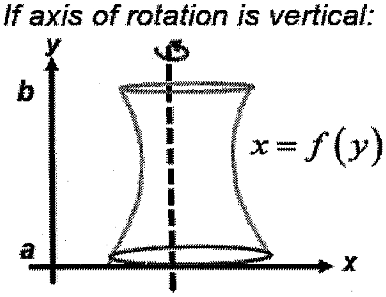
Surface Area of Surface of Revolution

If axis of rotation is horizontal:



$$A_{surface} = \int_a^b 2\pi r \sqrt{1+[f'(x)]^2}\,dx$$

If axis of rotation is vertical:



$$A_{surface} = \int_a^b 2\pi r \sqrt{1+[f'(y)]^2}\,dy$$

Examples

Find the length of the curve.

$$y=\frac{x^4}{4}+\frac{1}{8x^2},\quad 1\le x\le 3$$

$y = \frac{1}{4}x^4 + \frac{1}{8}x^{-2}$

$y' = x^3 - \frac{1}{4}x^{-3} = x^3 - \frac{1}{4}\frac{1}{x^3}$

$L = \int \sqrt{1+(f'(x))^2}\,dx$

$= \int_1^3 \sqrt{1+\left(x^3-\frac{1}{4}\frac{1}{x^3}\right)^2}\,dx$ math9 $\approx 20.111 = \boxed{\frac{181}{9}}$

$1+\left(x^3-\frac{1}{4}\frac{1}{x^3}\right)\left(x^3-\frac{1}{4}\frac{1}{x^3}\right)$

$1 + x^9 - \frac{1}{2} + \frac{1}{16}\frac{1}{x^9}$

$\cancel{1}x^9 + \frac{1}{2} + \frac{1}{16}\frac{1}{x^9}$

$\left(x^3 + \frac{1}{4}\frac{1}{x^3}\right)^2$

$= \int_1^3 \sqrt{\left(x^3+\frac{1}{4}\frac{1}{x^3}\right)^2}\,dx$

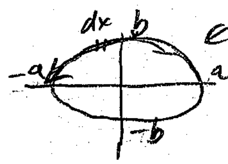
$= \int_1^3 \left(x^3 + \frac{1}{4}x^{-3}\right)dx$

$= \frac{1}{4}x^4 - \frac{1}{8}x^{-2} = \left[\frac{1}{4}x^4 - \frac{1}{8}\frac{1}{x^2}\right]_1^3$

$= \boxed{\left(\frac{81}{4}-\frac{1}{72}\right)-\left(\frac{1}{4}-\frac{1}{8}\right)} = \boxed{\frac{181}{9}}$

Set up, but do not evaluate, an integral for the length of the curve.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



let's do top half only...

$L = \int \sqrt{1+(f'(x))^2}\,dx$

$\frac{y^2}{b} = 1 - \frac{x^2}{a^2}$

$y^2 = b - \frac{b}{a^2}x^2$

$y = \pm\sqrt{b-\frac{b}{a^2}x^2} = \sqrt{b-\frac{b}{a^2}x^2} = \left(b-\frac{b}{a^2}x^2\right)^{1/2}$

$y' = \frac{1}{2}\left(b-\frac{b}{a^2}x^2\right)^{-1/2}\left(-2\frac{b}{a^2}x\right)$

$= \frac{-bx}{a^2\sqrt{b-\frac{b}{a^2}x^2}}$

$$\boxed{L = \int_{-a}^{a}\sqrt{1+\left[\frac{-bx}{a^2\sqrt{b-\frac{b}{a^2}x^2}}\right]^2}\,dx}$$

(ever wonder why there isn't a simple geometry formula for the distance around an ellipse?)

$C = 2\pi r$ for a circle but for an ellipse.

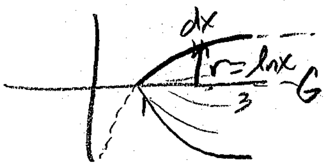
Examples

Set up, but do not evaluate, an integral for the area of the surface obtained by rotating the curve about the given axis.

$y = \ln x, \quad 1 \le x \le 3; \quad x - axis$

$y = \ln x$

$y' = \frac{1}{x}$

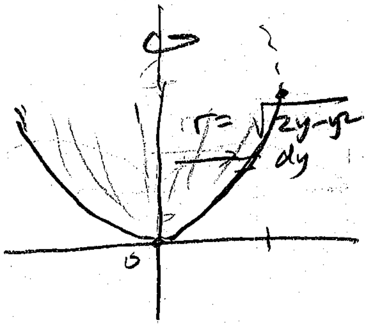


$$SA = \int 2\pi r \sqrt{1+[f'(x)]^2}\,dx$$

$$= \boxed{\int_1^3 2\pi \ln x \sqrt{1+\left(\frac{1}{x}\right)^2}\,dx}$$

The given curve is rotated about the y-axis. find the area of the resulting surface.

$x = \sqrt{2y - y^2}, \quad 0 \le y \le 1$



$x = (2y-y^2)^{1/2}$

$x' = \frac{1}{2}(2y-y^2)^{-1/2}(2-2y)$

$= \frac{2(1-y)}{2\sqrt{2y-y^2}} = \frac{1-y}{\sqrt{2y-y^2}}$

$$SA = \int 2\pi r \sqrt{1+[f'(y)]^2}\,dy$$

$$= 2\pi \int_0^1 \sqrt{2y-y^2}\sqrt{1+\frac{(1-y)^2}{2y-y^2}}\,dy$$

$\rightarrow \sqrt{2y-y^2}\sqrt{\frac{2y-y^2}{2y-y^2}+\frac{(1-y)^2}{2y-y^2}}$

$\frac{\sqrt{2y-y^2}}{\sqrt{2y-y^2}}\sqrt{2y-y^2+(1-y)^2}$

$(1)\sqrt{2y-y^2+1-2y+y^2}$

$\sqrt{1}$

$$= 2\pi \int_0^1 1\,dy$$

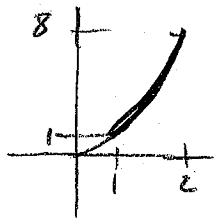
$$= 2\pi[1-0]$$

$$= \boxed{2\pi}$$

Examples

Find the arc length of the curve.

$y = x^3 \qquad 1 \le x \le 2$



$L = \int \sqrt{1 + [f'(x)]^2}\, dx$

$y = x^3$

$y' = 3x^2$

$= \int_1^2 \sqrt{1 + [3x^2]^2}\, dx = \int_1^2 \sqrt{1 + 9x^4}\, dx \approx \boxed{7.08246}$

or could use dy...

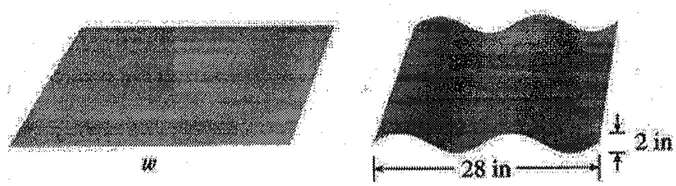
$x = \sqrt[3]{y} = y^{1/3}$

$x' = \frac{1}{3} y^{-2/3}$

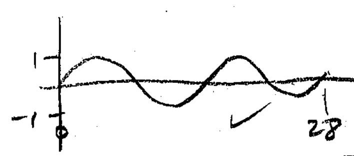
$L = \int_1^8 \sqrt{1 + [\frac{1}{3} y^{-2/3}]^2}\, dy$

$= \int_1^8 \sqrt{1 + \frac{1}{9} y^{-4/3}}\, dy \approx \boxed{7.08246}$

A manufacturer of corrugated metal roofing wants to produce panels that are 28 in. wide and 2 in. thick by processing flat sheets of metal as shown in the figure. The profile of the roofing takes the shape of a sine wave. Verify that the sine curve has the equation $y = \sin(\pi x / 7)$ and find the width w of a flat metal sheet that is needed to make a 28-inch panel.



$y = \sin(\frac{\pi}{7} x)$



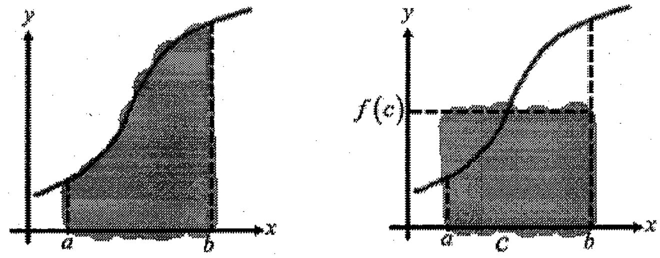
$y' = \cos(\frac{\pi}{7} x) \cdot \frac{\pi}{7}$

$w = L = \int_0^{28} \sqrt{1 + [\frac{\pi}{7} \cos(\frac{\pi}{7} x)]^2}\, dx$

$\approx \boxed{29.3607 \text{ in}}$

Unit 6-4: Average Value, Volumes of Solids - using Cross Sections

Average Value of a Function



From the previous theorem, at this value c, the height of the rectangle with the same area as the area under the function curve is *f(c)*. We define *f(c)* as the **average value of the function over this interval**, and can compute it as follows:

> If f is integrable on the closed interval $[a,b]$, then the **average value** of f on the interval is
>
> $$\textit{average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Examples

Find the average value of $f(x) = x^3 - 3x^2$ on the interval [1,4]

$$Av = \frac{1}{4-1}\int_1^4 (x^3-3x^2)\,dx$$

$$\frac{1}{3}\left[\frac{1}{4}x^4 - x^3\right]_1^4$$

$$\frac{1}{3}\left[\frac{1}{4}(4)^4-(4)^3\right] - \frac{1}{3}\left[\frac{1}{4}(1)^4-(1)^3\right] = \boxed{\frac{1}{4}}$$

Volumes with fixed Cross-Sectional Shapes

We can also use integrals to find volumes of solids whose shape is not rotationally symmetrical about an axis, but where the 'cross-sections' are constant shapes.

Here, we will still use infinitely thin cross-section 'slices' and the volume will be the infinitely thin differential width times a cross-sectional area:

$$V=\int_a^b A_{cross-section}\,dw$$

1) Sketch the 3-D solid.

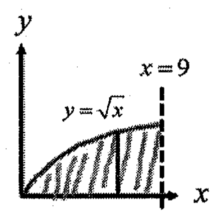
2) Summing along what variable? (perpendicular to cross-sections)

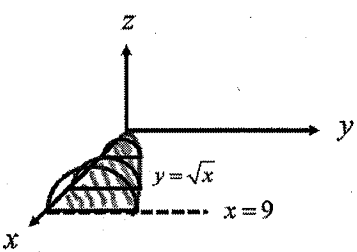
3) Determine a formula for the cross-sectional area (usually a function of the summing variable)

4) Sum these cross-sectional areas with an integral to find the volume.

ex. Base bounded by $y=x^{1/2}$, $x=9$, x-axis. Sections perpendicular to the x-axis are semicircles. Find the volume.

1) Sketch the 3-D solid:

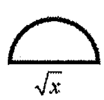




2) Summing along what variable? (perpendicular to cross-sections)

here, cross-sections are in y-z, so summing along x: *dx*

3) Determine a formula for the cross-sectional area (usually a function of the summing variable)

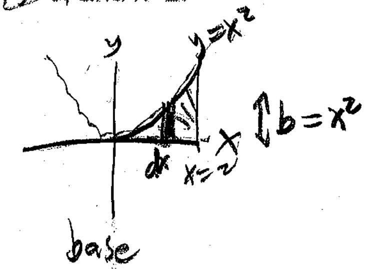


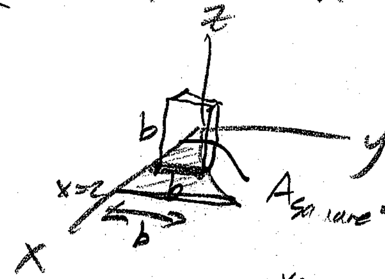
$$A_{cross-section}=\frac{1}{2}\pi r^2=\frac{1}{2}\pi\left(\frac{\sqrt{x}}{2}\right)^2=\frac{\pi}{8}x$$

4) Sum these cross-sectional areas with an integral to find the volume.

$$V=\int_a^b A_{cross-section}\,dw=\int_0^9 \frac{\pi}{8}x\,dx=\frac{81\pi}{16}$$

Find the volume of the solid formed by cross-sections which are perpendicular to the x-axis and form squares. The base of the shape is in the x-y plane, defined by $y=x^2$, $y=0$, and $x=2$.





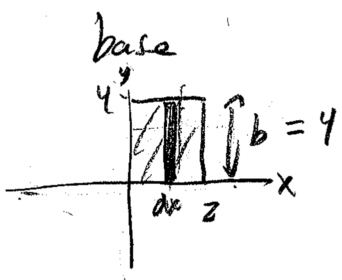
$A_{square} = b^2 = (x^2)^2 = x^4$

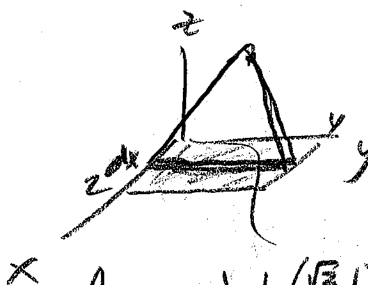
$V = \int_{x_1}^{x_2} A_{cross}\,dx$

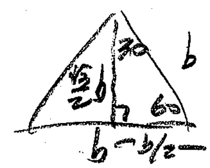
$= \boxed{\int_0^2 x^4\,dx =} \frac{1}{5}[x^5]_0^2$

$= \frac{1}{5}(2^5 - 0^5) = \boxed{6.4}$

Find the volume of the solid formed by cross-sections which are perpendicular to the x-axis and form equilaterial triangles where the base is one of the congruent sides. The base of the shape is in the x-y plane, defined by a rectangle which is 2 x 4.







$A_\triangle = \frac{1}{2}bh$

$= \frac{1}{2}b\left(\frac{\sqrt{3}}{2}b\right)$

$A_{cross} = \frac{1}{2}b\left(\frac{\sqrt{3}}{2}b\right)$

$= \frac{\sqrt{3}}{4}b^2$

$= \frac{\sqrt{3}}{4}(4)^2$

$= 4\sqrt{3}$

$V = \int_{x_1}^{x_2} A_{cross}\,dx = \int_0^2 (4\sqrt{3})\,dx = 4\sqrt{3}\,[x]_0^2$

$= 4\sqrt{3}(2-0)$

$= \boxed{8\sqrt{3}}$

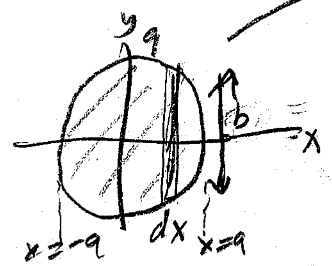
Volumes Examples

The base of S is a circular disk with radius 9. Parallel cross-sections perpendicular to the base are squares.

base

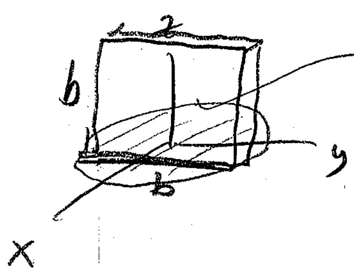
$x^2+y^2=81$

$y=\pm\sqrt{81-x^2}$



$b=+\sqrt{81-x^2}-\left[-\sqrt{81-x^2}\right]$

$b=2\sqrt{81-x^2}$



$A_{cross}=b^2=\left(2\sqrt{81-x^2}\right)^2$

$=4(81-x^2)$

$V=\int_{x_1}^{x_2} A_{cross}\,dx$

$=\int_{-9}^{9} 4(81-x^2)\,dx=\boxed{3888}$