AP Calc BC – Lesson Notes – Unit 6: Intro to Differential Equations

Unit 6-1: Introduction to Differential Equations
(slope fields, verifying solutions)

What is a differential equation?

A differential equation is an equation (contains an equals sign which states the two sides are equal) but where at least one term contains a derivative.

$$\frac{dy}{dx}+5y=e^x \qquad \frac{d^2y}{dx^2}-3\frac{dy}{dx}+5y=6$$

A **solution** to a differential equation **is itself an equation** - an equation which completely satisfies the original differential equation.

Algebra equation: #1. **Verify** $x=-2$ **is a solution to** $x^2+x=2$

$(-2)^2+(-2)\stackrel{?}{=}2$

$4-2\stackrel{?}{=}2$

$2=2$ ✓

yes, $x=-2$ is a solution to $x^2+x=2$

Differential equation: #2. **Verify** $y=x^{-10}+3x^2$ **is a solution to** $x^2y''+9xy'=20y$

$y=x^{-10}+3x^2$

$y'=-10x^{-11}+6x$

$y''=110x^{-12}+6$

$x^2[110x^{-12}+6]+9x[-10x^{-11}+6x]\stackrel{?}{=}20[x^{-10}+3x^2]$

$110x^{-10}+6x^2-90x^{-10}+54x^2\stackrel{?}{=}20x^{-10}+60x^2$

$20x^{-10}+60x^2=20x^{-10}+60x^2$ ✓

yes $y=x^{-10}+3x^2$ is a solution to $x^2y''+9xy'=20y$

Method for find a solution to a DE: Integration

There are many methods for finding the solution functions to differential equations (which is why there is an entire course about this), but we will learn just a few methods for solving DEs this year. The first is useful if we have a DE in which the highest order derivative is a first-derivative, and which can be solved for that derivative (called first-order differential equation). If this case, we can find the solution function by simply integrating the differential equation.

#3. Find the solution(s) for $\frac{dy}{dx} = 0.1x\sqrt{x^2+1}$

$$y = \int 0.1x\sqrt{x^2+1}\,dx$$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}du$

$$y = \frac{0.1}{2}\int u^{1/2}\,du$$

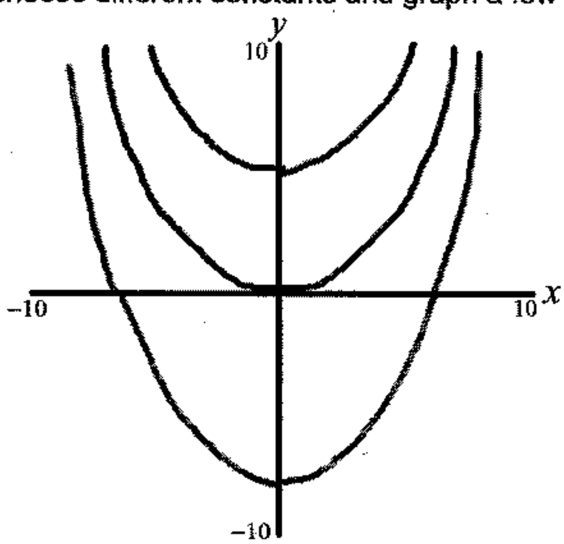
$$y = \frac{0.1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$$\boxed{y = \frac{1}{30}(x^2+1)^{3/2} + C}$$

The general solution and particular solutions

Solving a differential equation always involves undoing a derivative in some way so there is always an integration constant. There are an infinite number of these solutions each with a different integration constant, so this solution is called the **general solution** to the differential equation.

We could choose different constants and graph a few of these solutions:



$y = \frac{1}{30}(x^2+1)^{3/2} + C$ **general solution**

$y = \frac{1}{30}(x^2+1)^{3/2} + 5$

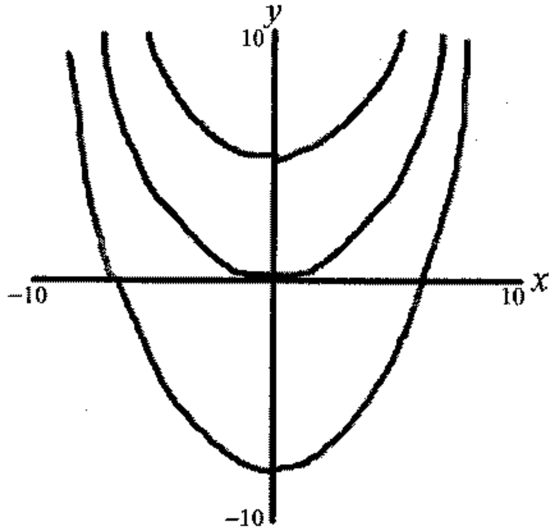
$y = \frac{1}{30}(x^2+1)^{3/2} + 0$

$y = \frac{1}{30}(x^2+1)^{3/2} - 8$

The general solution and particular solutions

Solving a differential equation always involves undoing a derivative in some way so there is always an integration constant. There are an infinite number of these solutions each with a different integration constant, so this solution is called the **general solution** to the differential equation.

Each specific solution with one selected integration constant value is called a **particular solution:**



$$y=\frac{1}{30}\left(x^2+1\right)^{3/2}+C$$ **general solution**

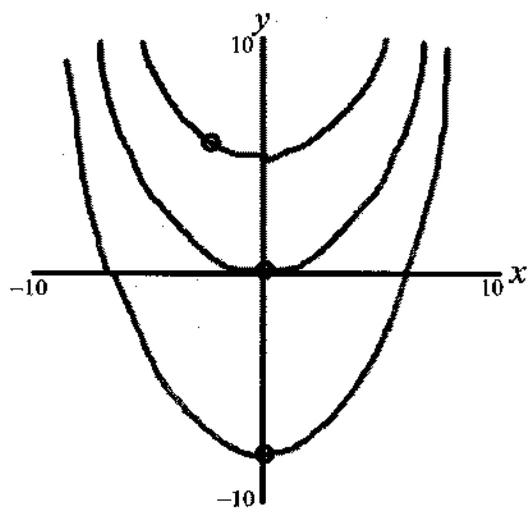
$$y=\frac{1}{30}\left(x^2+1\right)^{3/2}+5$$

$$y=\frac{1}{30}\left(x^2+1\right)^{3/2}+0$$

$$y=\frac{1}{30}\left(x^2+1\right)^{3/2}-8$$

these are particular solutions

In order to find a particular solution from a general solution we have to have one known *x-y* pair...this is called an **initial condition**, that must make the differential equation solution true. By plugging the initial condition into the general solution to the differential equation we can solve for the integration constant which will choose the particular solution that goes through this point.



$$y=\frac{1}{30}\left(x^2+1\right)^{3/2}+C$$ **general solution**

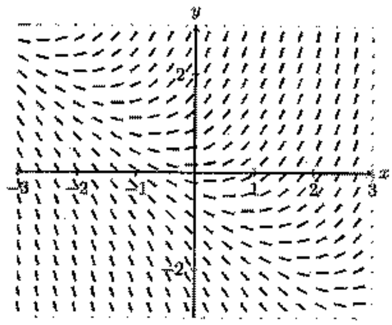
	initial conditions:
$y=\frac{1}{30}\left(x^2+1\right)^{3/2}+5$	$y(-2)=5.373$
$y=\frac{1}{30}\left(x^2+1\right)^{3/2}+0$	$y(0)=\frac{1}{30}$
$y=\frac{1}{30}\left(x^2+1\right)^{3/2}-8$	$y(0)=-7.967$

Slope Fields

For first-order differential equations, because they can be solved for the derivative on the left side, the expression on the right gives the 'slope' of the solution curve at any point *x,y* in the domain:

$$\frac{dy}{dx} = f(x,y)$$

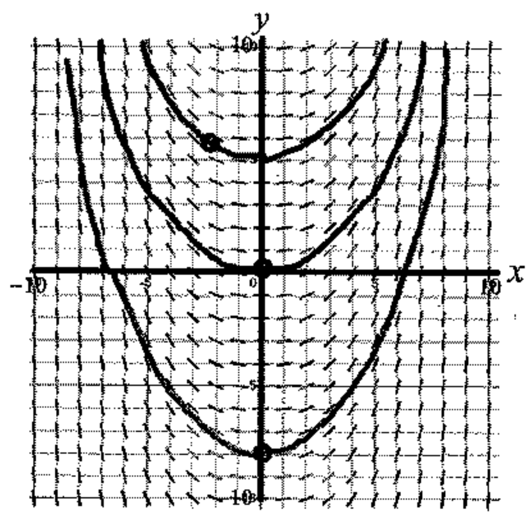
If you were to plug in various *x,y* points to the right side of the DE and evaluate, the result is the slope of the solution function at that point. This could be graphed by including slopes at many points, creating what is called a **<u>slope field (or direction field)</u>**:



...where the little line segments indicate the slope of the solution function curve at that *x,y* value.

The solution curves follow the 'flow' of the slope field line segments

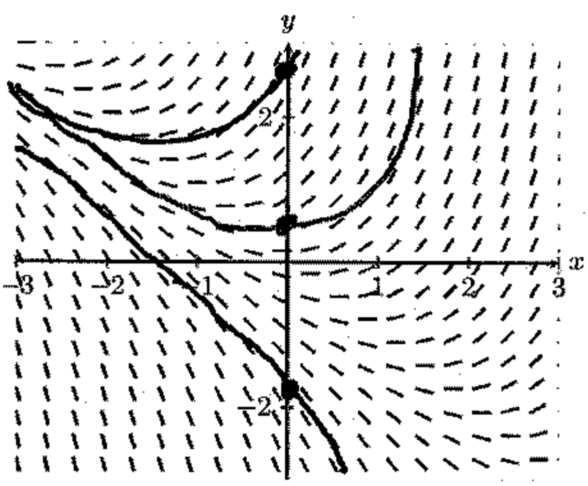
Because the slope field represents the value of the derivative ('slope') of the solution curve at each x,y point in the domain, the solution curves follow the 'flow' of the slope field.



Here is the slope field for the differential equation we've been working with:

$$\frac{dy}{dx} = 0.1x\sqrt{x^2+1}$$

...which has the general solution $y = \frac{1}{30}\left(x^2+1\right)^{3/2} + C$



Here is the slope field for a different differential equation.

Even if we don't know the differential equation or the general solution, we could still sketch solution curves in starting at any given initial condition point.

#4. For the differential equation $xy'' + y' = 0$

a) Verify that $y = C_1 + C_2 \ln(x)$ is the general solution to the differential equation.

b) Find the particular solution by using the initial conditions $y(2)=0,\ \ y'(2)=\frac{1}{2}$

a) $y = C_1 + C_2 \ln(x)$

$y' = C_2 \frac{1}{x} = C_2 x^{-1}$

$y'' = -C_2 x^{-2} = \frac{-C_2}{x^2}$

$x\left[-\frac{C_2}{x^2}\right] + \left[C_2 \frac{1}{x}\right] \stackrel{?}{=} 0$

$-\frac{C_2}{x} + \frac{C_2}{x} \stackrel{?}{=} 0$

$0 = 0$ ✓ $\boxed{\text{yes}}$

b) $y' = \frac{C_2}{x}$

$\frac{1}{2} = \frac{C_2}{2} \rightarrow C_2 = 1$

$y = C_1 + (1)\ln(x)$

$0 = C_1 + \ln(2) \rightarrow C_1 = -\ln(2)$

$\boxed{y = (-\ln(2)) + \ln(x)}$

#5. Given the differential equation $y' = \frac{2}{x}$ and the initial condition $y(1) = 8$

a) Find the general solution

b) Use the initial condition to find the particular solution

c) Use the differential equation to sketch the first quadrant of a slope field for the differential equation. Include lineal elements for every 2 units in x and y from (2,2) to (10,10).

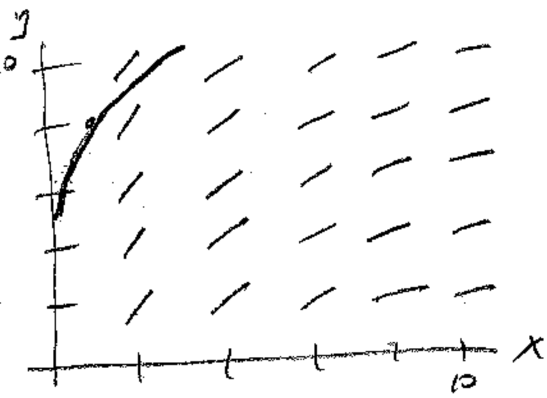
d) Use your calculator to graph your solution curve and add it to your slope field.

a) $y = \int \frac{2}{x}\,dx$

$\boxed{y = 2\ln|x| + C}$

b) $8 = 2\ln(1) + C \rightarrow C = 8$

$\boxed{y = 2\ln|x| + 8}$



c)

(x,y)	$y' = \frac{2}{x}$
(2, any)	$\frac{2}{2} = 1$
(4, any)	$\frac{2}{4} = \frac{1}{2}$
(6, any)	$\frac{2}{6} = \frac{1}{3}$
(8, any)	$\frac{2}{8} = \frac{1}{4}$
(10, any	$\frac{2}{10} = \frac{1}{5}$

d) $y = 2\ln|x| + 8$

Unit 6-2: Approximating a solution curve for a differential equation using Euler's Method

We can't always find the solution function analytically

Not all differential equations will be in the forms we know how to solve (even after taking a full differential equations course). We may still need to know something about the solution, though - need to know the y for given x values in the solution.

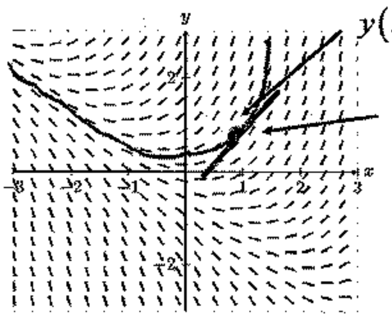
When we can't find a solution to a differential equation analytically, we use **numerical methods** to find the approximate solution - an approximate *y* value for any given value *x* for the solution curve, for a specific solution curve (that is, we need to be given an initial condition).

Euler's Method

In this course we learn one such method, called **Euler's Method**. This method takes advantage of the fact that with a first-order differential equation of the form...

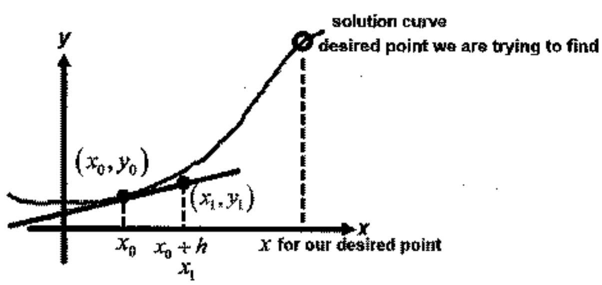
$$\frac{dy}{dx} = f(x,y) \quad \text{...with an initial condition given:} \quad y(x_0) = y_0$$

...we know the initial condition is on the solution curve, and we can plug in this (x_0,y_0) into the differential equation to get the 'slope' of the tangent line to the solution curve at this point:



Given: $\frac{dy}{dx} = f(x,y)$ ...with an initial condition given: $y(x_0) = y_0$

We can move a distance *h* away in x from the initial condition towards the *x* value we wish to know the solution curve *y* value...



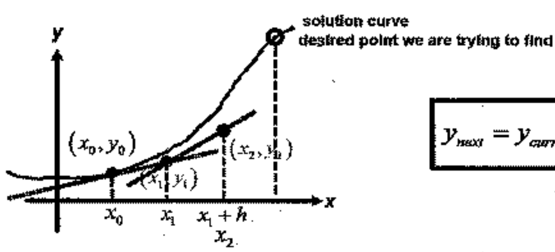
...and use the slope to compute an updated *y* value:

$$\frac{dy}{dx} = f'(x_0,y_0) = 'slope' = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{h}$$

$$y_1 = y_0 + f'(x_0,y_0)\,h$$

Euler's Method

We then use this new point (x_1,y_1) to establish the slope for another estimate moving closer to the desired point by plugging this point into the original DE *f(x,y)* function to establish the next slope:



$$\boxed{y_{next} = y_{current} + f'(x_0,y_0)\,\Delta x}$$

This process continues iteratively until we are at the desired x value, and we then have an estimate of the *(x,y)* on the solution curve at the desired *x* value (and we trace out an approximation of the solution curve along the way).

Easiest to understand this through an example...

#1. Use Euler's method to obtain a four-decimal approximation for $y(1.5)$ on the solution curve for $y' = 0.2xy$ *with* $y(1) = 1$

$$y_{next} = y_{current} + f'(x_0, y_0)\,\Delta x$$

Let's set $h = \Delta x = 0.1$, taking 5 iterations to reach from x=1 to x=1.5:

(x,y)	$y_{next} = y_{current} + f'(x_0, y_0)\,\Delta x$
(1,1)	$y = 1 + [0.2(1)(1)]0.1 = 1.02$
(1.1, 1.02)	$y = 1.02 + [0.2(1.1)(1.02)]0.1 = 1.04244$
(1.2, 1.04244)	$y = 1.04244 + [0.2(1.2)(1.04244)]0.1 = 1.0674...$
(1.3, 1.0674...)	$y = 1.0674... + (0.2(1.3)(1.0674...)]0.1 = 1.0952...$
(1.4, 1.0952...)	$y = 1.0952... + [0.2(1.4)(1.0952...)]0.1 = 1.125878...$
(1.5, 1.125878...)	

$\therefore \boxed{y(1.5) \approx 1.126}$

#2. The table gives values of $f'(x)$, the derivative of a function $f(x)$. If $f(1) = 4$ what is the approximation to $f(2)$ obtained by using Euler's method with a step size of 0.5?

x	$f'(x)$
1	0.2
1.5	0.5
2	0.9

(x,y)	$y_{n+1} = y_n + f'(x)\,\Delta x$
(1,4)	$y = 4 + [0.2]0.5 = 7.1$
(1.5, 7.1)	$y = 7.1 + [0.5]0.5 = 7.35$
(2, 7.35)	

$\therefore \boxed{f(2) \approx 7.35}$

Unit 6-3: Solving DEs using Separation of Variables

Solving a Separable Differential Equation by Direct Integration

Solving a differential equation by direct integration requires that when you solve the DE for the derivative, the function on the RHS contains only the 'x' variable:

#1. $\frac{dy}{dx} = x^5$

$y = \int x^5 dx$

$\boxed{y = \frac{1}{6}x^6 + C}$

Solving a Separable Differential Equation by Separation of Variables

If you can solve the differential equation for the derivative, but the RHS function contains a mix of 'x' and 'y' variables, sometimes, you can still solve. First you must <u>separate the variables</u> - use algebra to rearrange the equation so that each side contains only one variable (including the differential)...

#2. $\frac{dy}{dx} = xy$

$dy = xy\,dx$

$\frac{1}{y}dy = x\,dx$

$\int \frac{1}{y}dy = \int x\,dx$

$\ln|y| + C_1 = \frac{1}{2}x^2 + C_2$

$\ln|y| = \frac{1}{2}x^2 + C_3$ ← implicit, general solution

$e^{\ln|y|} = e^{(1/2x^2 + C_3)}$

$|y| = e^{1/2x^2} e^{C_3}$

$|y| = Ce^{1/2x^2}$

$\boxed{y = Ce^{1/2x^2}}$

Not all differential equations are separable

Separation of Variables works when the differential equation is first-order and when solved for the derivative, the RHS can be factored into two factors, each containing only one of the variables:

$$\frac{dy}{dx} = g(x)\,h(y)$$

...then it is possible to solve by integration.

Separable DE

$$\frac{dy}{dx} = y^2 e^{3x+4y}$$

$$\frac{dy}{dx} = y^2 e^{3x} e^{4y}$$

$$\frac{dy}{dx} = \left(e^{3x}\right)\left(y^2 e^{4y}\right)$$

Non-Separable DE

$$\frac{dy}{dx} = y + \sin x$$

Take Differential Equations - 2nd semester of next year's course - if you want to know more about how to solve other forms of differential equations :)

You may also be given an initial condition and asked to find the particular solution...

#3. $2xy' - \ln(x) = 0 \qquad y(1) = 2$

$2x\frac{dy}{dx} = \ln(x)$

$2dy = \frac{\ln(x)}{x}\,dx$

$\int 2dy = \int \frac{\ln(x)}{x}\,dx$

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}dx$

$\int 2dy = \int u\,du$

$2y = \frac{1}{2}(\ln(x))^2 + c_1$

$y = \frac{1}{4}(\ln(x))^2 + \frac{c_1}{2}$

$y = \frac{1}{4}(\ln(x))^2 + C$

$y = \frac{1}{4}(\ln(x))^2 + C \qquad x=1,\ y=2$

$2 = \frac{1}{4}(\ln(1))^2 + C$

$2 = 0 + C \Rightarrow C = 2$

$\boxed{y = \frac{1}{4}(\ln(x))^2 + 2}$

#4. $\frac{dy}{dx} = \frac{3x^2}{y^2}$

$$y^2 dy = 3x^2 dx$$

$$\int y^2 dy = 3\int x^2 dx$$

$$\frac{1}{3}y^3 = x^3 + C$$

$$y^3 = 3x^3 + C$$

$$\boxed{y = \sqrt[3]{3x^3 + C}}$$

#5. $\sqrt{x} + \sqrt{y}y' = 0 \qquad y(1) = 9$

$$x^{1/2} + y^{1/2}\frac{dy}{dx} = 0$$

$$y^{1/2}\frac{dy}{dx} = -x^{1/2}$$

$$y^{1/2}dy = -x^{1/2}dx$$

$$\int y^{1/2}dy = -\int x^{1/2}dx$$

$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C$ now use $y(1) = 9$

$$\frac{2}{3}(9)^{3/2} = -\frac{2}{3}(1)^{3/2} + C$$

$$\frac{2}{3}(27) = -\frac{2}{3} + C$$

$$18 = -\frac{2}{3} + C$$

$$C = 18 + \frac{2}{3} = \frac{56}{3}$$

$$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + \frac{56}{3}$$

$$y^{3/2} = \frac{3}{2}\left[-\frac{2}{3}x^{3/2} + \frac{56}{3}\right]$$

$$\boxed{y = \left(\frac{3}{2}\left[-\frac{2}{3}x^{3/2} + \frac{56}{3}\right]\right)^{2/3}}$$

Unit 6-4: DE Applications - Growth and Decay

Modeling = finding a (differential) equation to model a scenario

The term 'modeling' is used to mean finding a function or equation (in this chapter, a differential equation) which applies to a given scenario, applying given conditions to solve for any constants and then using the differential equation and its solution to answer questions about the scenario.

Growth/Decay proportion to amount

There are many situations where the rate of change of a quantity (with respect to time) is proportional to the current amount of the quantity.

In problems like these, the appropriate differential equation to model this would be:

$$\frac{dy}{dt} = ky$$ ...where k is a constant of proportionality.

Some specific examples...

Unrestricted Population Growth: $\frac{dP}{dt} = kP$

Continuously Compounded Interest: $\frac{dA}{dt} = kA$

Radioactivity Decay: $\frac{dQ}{dt} = kQ$

Let's take one of these, and solve for the form of the solution to the differential equation:

Continuously Compounded Interest: $\frac{dA}{dt} = kA$

This particular form can be solved using Separation of Variables...

$$\frac{dA}{dt} = kA$$

$$\frac{1}{A}dA = kdt$$

$$\int \frac{1}{A}dA = \int kdt$$

$$\ln|A| = kt + C_1$$

...and solving for A...

$$\ln|A| = kt + C_1$$

$$A = e^{kt+C_1}$$

$$A = e^{kt}e^{C_1}$$

$$A = Ce^{kt}$$

We usually have an initial condition: $A(0) = A_0$

...and can use it to solve for C... $A_0 = Ce^{k(0)},\ C = A_0$

...which gives us the particular solution for the scenario: $\boxed{A = A_0e^{kt}}$

(for continuously compounded interest, the k has a specific meaning: the annual interest rate)

'Proportional to' language

When problems state that a rate of change is proportional to something, we need to distinguish between directly proportional and inversely proportional:

"The rate of change of a population is proportional to the current population" $\frac{dP}{dt} = kP$

"The rate of change of the y is inversely proportional to the cube of time" $\frac{dy}{dt} = \frac{k}{t^3}$

"The rate of change of the population of y is proportional to the current value of y squared, and inversely proportional to the square root of time." $\frac{dy}{dt} = \frac{ky^2}{\sqrt{t}}$

Using the differential equation model

Once we have found a differential equation to model a scenario, we can then find the solution function, and use initial conditions to establish the constants. Then we can use the completed model to answer other questions about the scenario.

#1. The population of bacteria increases at a rate which is proportional to the amount of bacteria. A culture initially has 200 bacteria. At $t = 1$ hr, the population of bacteria has increased to 300 bacteria. If the rate of growth is proportional to the number of bacteria present, determine the time needed for the bacteria population to quadruple.

$$\frac{dP}{dt} = kP$$

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln|P| = kt + C_1$$

$$P = e^{(kt+C_1)} = e^{kt}e^{C_1}$$

$$P = Ce^{kt}$$

$$200 = Ce^{k(0)} = C(1) \rightarrow C = 200$$

$$P = 200e^{kt}$$

$$300 = 200e^{k(1)}$$

$$\frac{300}{200} = \frac{3}{2} = e^k$$

$$e^k = \frac{3}{2}$$

$$k = \ln\left(\frac{3}{2}\right) = .4054651081$$

$$P = 200e^{.4054651081t}$$

t	P
0	200
1	300
?	800

$$800 = 200e^{.4054651081t}$$

$$\frac{800}{200} = 4 = e^{.4054651081t}$$

$$.4054651081t = \ln(4)$$

$$t = \frac{\ln(4)}{.4054651081} = \boxed{3.419 \text{ hrs}}$$

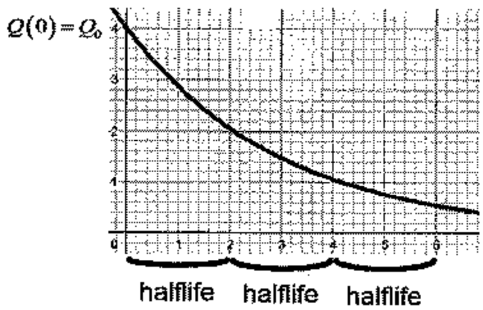
Radioactive Decay

Radioactive substances spontaneously eject particles which results in the amount of material left decreasing over time. Initially, there is much material from which particles can eject so the rate of ejection is high, but as more and more material is removed, there is less material left from which particles can be ejected.

The rate of change of quantity of material with respect to time is proportional to the amount of material currently remaining, similar to population growth...

$\frac{dQ}{dt} = kQ$ ...which has the solution: $Q = Q_0 e^{kt}$

But because quantity is decreasing over time, the k constant will be negative, producing radioactive decay.



"halflife" = the amount of time it takes for half of the quantity to be ejected.

Radioactive Carbon Dating

An interesting application of this is using radioactive carbon dating to establish the age of previously living materials. In 1950, a chemist named Willard Libby found a way to use the ratio of the amount of radioactive carbon-14 to ordinary carbon in a living substance to establish the time when it died.

The action of cosmic radiation on nitrogen in the atmosphere turns some of the regular carbon into radioactive carbon-14, and the ratio of carbon-14 to regular carbon is constant and is absorbed by all living things, so the same ratio appears in the living tissues. But when an organism dies, the absorption of the carbon-14 by either breathing or eating stops. The regular carbon remains in the tissues, but the radioactive carbon-14 decays over time according to a radioactive decay model, and it is known that the 'half-life' of carbon-14 is 5,600 years (the time when only half of the initial carbon-14 remains).

Here is a specific example:

Ex) A fossilized bone is found to contain one-thousandth of the C-14 level found in living matter. Estimate the age of the fossil.

State an appropriate form DE: $\frac{dQ}{dt} = kQ$ Solve the DE: by separation of variables, solution is: $Q = Q_0 e^{kt}$

Use conditions to establish constants: For carbon-14, half-life=5600 yrs, so $Q(5600) = \frac{1}{2} Q_0$

$$\frac{1}{2} Q_0 = Q_0 e^{k(5600)}$$

$$e^{k(5600)} = \frac{1}{2}$$

$$5600k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5600} = -1.23776283 \cdot 10^{-4}$$

$$\boxed{Q = Q_0 e^{(-1.23776283 \cdot 10^{-4})t}}$$

Now answer the question: age when quantity is one-thousandth

$$Q(t_{age}) = \frac{1}{1000} Q_0 \qquad \frac{1}{1000} Q_0 = Q_0 e^{(-1.23776283 \cdot 10^{-4})t}$$

$$e^{(-1.23776283 \cdot 10^{-4})t} = \frac{1}{1000}$$

$$(-1.23776283 \cdot 10^{-4})t = \ln\left(\frac{1}{1000}\right)$$

$$t = \frac{\ln\left(\frac{1}{1000}\right)}{-1.23776283 \cdot 10^{-4}} = \boxed{55808 \text{ years}}$$

Newton's Law of Cooling/Warming

A slightly different form DE is found in scenarios where objects starting at one temperature are immersed in a medium at a different temperature. Assuming the medium is large (the object isn't big enough to heat or cool the medium) than the rate of change (over time) of the temperature of the object is proportional to the difference between its temperature and the medium:

$$\frac{dT}{dt} = k(T - T_m) \quad \textit{where } T_m \textit{ is the temperature of the medium}$$

We can solve this using Separation of Variables:

$$\frac{dT}{T - T_m} = k\,dt$$

$$\int \frac{1}{T - T_m}\,dT = \int k\,dt$$

$$\ln|T - T_m| = kt + C_1$$

$$T - T_m = e^{kt + C_1}$$

$$T - T_m = Ce^{kt}$$

$$\boxed{T = T_m + Ce^{kt}}$$

#2. A murder victim's body is found by detectives who wish to establish the time of death. When the victim was alive, their body temperature was 98.6 °F, which as soon as death occurs, the body begins cooling towards the ambient temperature. This victim was found in a building with air conditioning which maintained the ambient temperature at a constant 78 °F. Detectives arrived on scene at 6:00am and found the core temperature of the body to be 84 °F. Core temperature was measured again at 6:30am and found to be 83 °F. What was the time of death?

We will use Newton's Law of Cooling which has solution: $T = T_m + Ce^{kt}$

Let's define 6:00am at t=0 and use hours, to 6:30am is t=0.5. Then time of death will be some negative time value.

$T(0) = 84$

$T(0.5) = 83$

$T(t_{death}) = 98.6$

$T = 78 + Ce^{kt}$

$84 = 78 + Ce^{k(0)}$

$84 = 78 + C \rightarrow C = 6$

$T = 78 + 6e^{kt}$

$83 = 78 + 6e^{k(0.5)}$

$5 = 6e^{0.5k}$

$e^{0.5k} = \frac{5}{6}$

$0.5k = \ln(\frac{5}{6})$, $k = \frac{\ln(\frac{5}{6})}{.5} = 2\ln(\frac{5}{6})$

$T = 78 + 6e^{[2\ln(\frac{5}{6})]t}$

$T = 78 + 6e^{[2\ln(\frac{5}{6})]t}$

$98.6 = 78 + 6e^{[2\ln(\frac{5}{6})]t}$

$20.6 = 6e^{[2\ln(\frac{5}{6})]t}$

$e^{[2\ln(\frac{5}{6})]t} = \frac{20.6}{6}$

$2\ln(\frac{5}{6})t = \ln(\frac{20.6}{6})$

$t = \frac{\ln(\frac{20.6}{6})}{2\ln(\frac{5}{6})} = -3.383$ hr

time of death was 3.383 hrs before 6:00AM (2:37AM)

Unit 6-5: DE Applications - The Logistic Growth Model

The Logistic Equation models population growth in an environment which can only support a defined maximum population

We know that, in general, rate of population growth is proportional to the amount of population...

Unrestricted Population Growth: $\frac{dP}{dt} = kP$

...which produces this solution: $P = Ce^{kt}$

But more realistically, an environmental system has factors which limit the total population, things like availability of food and water. In these cases, the most common way to model population growth is with the following differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

The constant L is called the **carrying capacity** and represents the maximum population this system can support. By including this factor (which goes to zero as P approaches L) we are forcing the growth rate to zero as we approach this maximum population level.

Solution: The Logistic Equation

This is the logistic equation: $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$

This is the solution function: $P = \frac{L}{1 + Ce^{-kt}}$

L is the carrying capacity (maximum population)

C is a constant which we need an initial condition to establish (population at time=0)

k is a 2nd constant which we need a second condition to establish

We can solve this using Separation of Variables...

$$\frac{dP}{dt} = kP\left(1-\frac{P}{L}\right)$$

$$\frac{1}{P\left(1-\frac{P}{L}\right)}dP = kdt$$

$$\int \frac{1}{P\left(1-\frac{P}{L}\right)}dP = \int kdt$$

$$\int \left[\frac{1}{P}+\frac{1}{L-P}\right]dP = \int kdt$$

$$\int \frac{1}{P}dP + \int \frac{1}{L-P}dP = \int kdt$$

$$u = L-P$$

$$du = -dP$$

$$\int \frac{1}{P}dP - \int \frac{1}{u}du = \int kdt$$

$$\ln|P| - \ln|u| = kt + C_1$$

$$\ln|P| - \ln|L-P| = kt + C_1$$

$$\ln\left|\frac{P}{L-P}\right| = kt + C_1$$

$$\frac{P}{L-P} = e^{kt+C_1} = C_2 e^{kt}$$

$$P = C_2 e^{kt}(L-P)$$

$$P = LC_2e^{kt} - PC_2e^{kt}$$

$$P + PC_2e^{kt} = LC_2e^{kt}$$

$$P\left(1+C_2e^{kt}\right) = LC_2e^{kt}$$

$$P = \frac{LC_2e^{kt}}{1+C_2e^{kt}}$$

$$P = \frac{L}{\frac{1}{C_2e^{kt}}+1}$$

$$P = \frac{L}{1+\frac{e^{-kt}}{C_2}}$$

$$\boxed{P = \frac{L}{1+Ce^{-kt}}}$$

to proceed on the left, we employ Partial Fractions:

$$\frac{1}{P\left(1-\frac{P}{L}\right)} = \frac{A}{P} + \frac{B}{1-\frac{P}{L}}$$

$$A\left(1-\frac{P}{L}\right) + BP = 1$$

$$A - \frac{A}{L}P + BP = 1$$

$$\left(B-\frac{A}{L}\right)P + A = 0P + 1$$

$system:$

$$\begin{cases} B - \frac{A}{L} = 0 \\ A = 1 \end{cases}$$

$$B - \frac{1}{L} = 0,\ B = \frac{1}{L}$$

$$\frac{1}{P\left(1-\frac{P}{L}\right)} = \frac{A}{P} + \frac{B}{1-\frac{P}{L}} = \frac{1}{P} + \frac{\frac{1}{L}}{1-\frac{P}{L}}$$

$$\frac{1}{P\left(1-\frac{P}{L}\right)} = \frac{1}{P} + \frac{1}{L-P}$$

Comparing logistic and unrestricted growth

#1. The wolf population has unrestricted growth in a forest. There are 20 wolves at t = 0 months, and 40 wolves at t = 10 months. What will the wolf population be at time t = 40 months?

Differential Equation: $\frac{dP}{dt} = kP$ Solution: $P = Ce^{kt}$

t	P
0	20
10	40
40	?

$P = Ce^{kt}$

$20 = Ce^{k(0)}$ $\quad 20 = Ce^{0} = C$

$P = 20e^{kt}$

$40 = 20e^{k(10)}$ $\quad 20e^{10k} = 40$

$e^{10k} = \frac{40}{20} = 2$

$10k = \ln(2)$

$k = \frac{\ln(2)}{10} = .069315$

$\underline{P = 20e^{.069315t}}$

complete model

use the model:

$P(40) = 20e^{.069315(40)}$

$= \boxed{92{,}674 \text{ wolves}}$

#2. The wolf population in a forest grows in a forest which can only support a maximum of 100 wolves. There are 20 wolves at t = 0 months, and 40 wolves at t = 10 months. What will the wolf population be at time t = 40 months?

Differential Equation: $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$ Solution: $P = \frac{L}{1 + Ce^{-kt}}$

t	P
0	20
10	40
40	?

$P = \frac{L}{1 + Ce^{-kt}}$ $\quad L = 100$

$P = \frac{100}{1 + Ce^{-kt}}$

$20 = \frac{100}{1 + Ce^{-k(0)}} = \frac{100}{1 + C}$, $\quad 20(1 + C) = 100$

$1 + C = \frac{100}{20} = 5$

$C = 4$

$P = \frac{100}{1 + 4e^{-kt}}$

$40 = \frac{100}{1 + 4e^{-k(10)}}$ $\quad 40(1 + 4e^{-10k}) = 100$

$1 + 4e^{-10k} = \frac{100}{40} = \frac{5}{2}$

$4e^{-10k} = \frac{3}{2}$

$e^{-10k} = \frac{3}{8}$

$-10k = \ln\left(\frac{3}{8}\right)$

$k = \frac{\ln\left(\frac{3}{8}\right)}{-10} = .09808$

$P = \frac{100}{1 + 4e^{-.09808t}}$

complete model

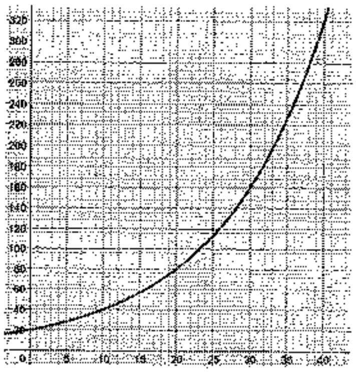
use the model:

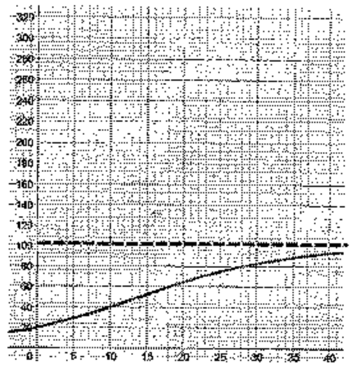
$P(40) = \frac{100}{1 + 4e^{-.09808(40)}}$

$= \boxed{320.004 \text{ wolves}}$

Comparing logistic and unrestricted growth

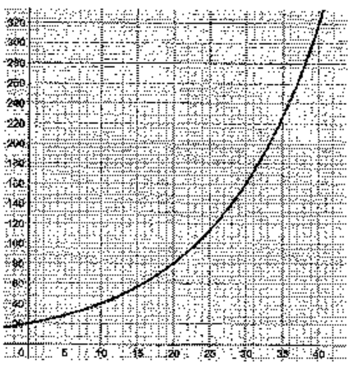
unrestricted growth		logistic growth
$\dfrac{dP}{dt} = kP$	Differential Equation	$\dfrac{dP}{dt} = kP\left(1-\dfrac{P}{L}\right)$
$P = Ce^{kt}$	Solution	$P = \dfrac{L}{1+Ce^{-kt}}$
$P = 20e^{.069315t}$		$P = \dfrac{100}{1+4e^{-.09808t}}$



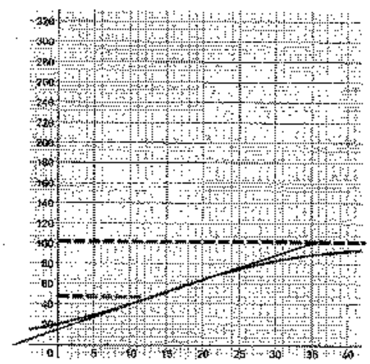


Maximum rate of growth

unrestricted growth



logistic growth



With unrestricted growth, the rate of growth keeps getting larger and larger so there is no time where there is a 'maximum rate of growth'.

However, with the logistic growth model, there is a time when the derivative (instantaneous rate of change) is largest. This occurs when the population is at <u>half of the carrying capacity</u>.

$$\frac{dP}{dt} = kP\left(1-\frac{P}{L}\right)$$

Max for $\dfrac{dP}{dt}$ occurs when $\dfrac{d^2P}{dt^2} = 0$. Since P is a function of t, we must use Chain Rule for the derivative:

$$kP\left(-\frac{1}{L}\right) + \left(1-\frac{P}{L}\right)k = 0$$

$$\left(1-\frac{P}{L}\right)k = kP\frac{1}{L}$$ since k is non-zero, can divide it out

$$1-\frac{P}{L} = \frac{P}{L}$$

$$L - P = P$$

$$L = 2P$$

$$P = \frac{1}{2}L$$

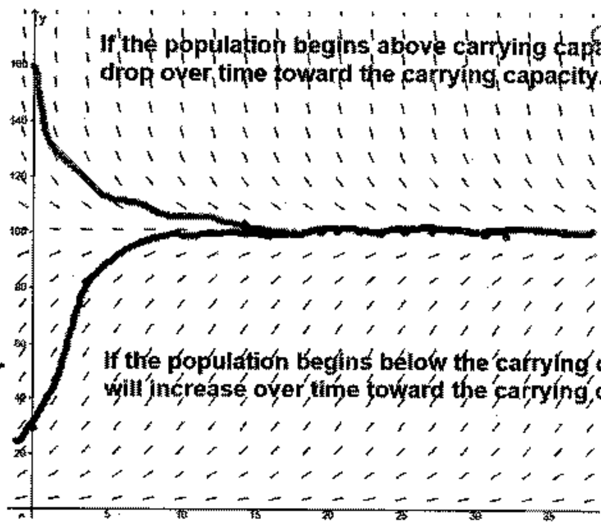
Slope Field for a Logistic Equation

A slope field for a logistic model differential equation will show curves with increasing population over time approaching, but not reaching the carrying capacity population.

concave up

concave down

concave up



In this system, there is only one stable 'state'...when the population is at the carrying capacity. Any population other than that will approach this stable value over time.

The Logistic Growth Model

Logistic model differential equation:	$\frac{dP}{dt} = kP\left(1-\frac{P}{L}\right)$	L is the carrying capacity (maximum population)
Logistic model solution:	$P = \frac{L}{1+Ce^{-kt}}$	*Maximum rate of growth occurs when* $P=\frac{1}{2}L$

Memorize all of this

Logistic problems on the AP Exam are easy to solve if you can recognize that the problem is about logistic growth:

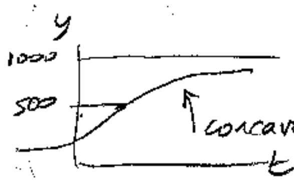
#3. A population y changes at a rate modeled by the differential equation $\frac{dy}{dt} = 0.2y(1000-y)$ $= 200y - 0.2y^2$ where t is measured in years. What are all values of y for which the population is increasing at a decreasing rate? (= concave down)

$\frac{dy}{dt} = 0.2y(1000-y)$

$\frac{dy}{dt} = 0.2y\,1000\left(1-\frac{y}{1000}\right)$

$\frac{dy}{dt} = 200y\left(1-\frac{y}{1000}\right)$

$\frac{dy}{dt} = ky\left(1-\frac{y}{L}\right)$ $\quad L = 1000$



concave down for $\boxed{500 < y < 1000}$

#4. At time t = 0 a bacterial culture weighs 1 gram. Two hours later, the culture weights 4 grams. The maximum weight of the culture is 20 grams.

(a) Write a logistic differential equation that models the weight of the bacterial culture.

(b) Solve the differential equation.

(c) Find the culture's weight after 5 hours.

(d) When will the culture's weight reach 18 grams?

(e) After how many hours is the culture's weight increasing most rapidly?

(a) $L = 20g$ so $\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{20}\right)}$

(b) $P = \frac{20}{1+Ce^{-kt}}$

t	P
0	1
2	4

$1 = \frac{20}{1+Ce^{0}} = \frac{20}{1+C}$

$1 + C = 20$

$C = 19$

$P = \frac{20}{1+19e^{-kt}}$

$4 = \frac{20}{1+19e^{-k(2)}}$

$1+19e^{-2k} = \frac{20}{4} = 5$

$19e^{-2k} = 4$

$e^{-2k} = \frac{4}{19}$

$-2k = \ln\left(\frac{4}{19}\right)$

$k = \frac{\ln\left(\frac{4}{19}\right)}{-2} = .77907$

$\boxed{P(t) = \frac{20}{1+19e^{-.77907t}}}$

(c) $P(5) = \frac{20}{1+19e^{-.77907(5)}} = \boxed{14.426 \text{ grams}}$

(d) $18 = \frac{20}{1+19e^{-.77907t}}$

$1+19e^{-.77907t} = \frac{20}{18} = \frac{10}{9}$

$19e^{-.77907t} = \frac{1}{9}$

$e^{-.77907t} = \frac{1}{171}$

$-.77907t = \ln\left(\frac{1}{171}\right)$

$t = \frac{\ln\left(\frac{1}{171}\right)}{-.77907} = \boxed{6.600 \text{ hours}}$

(e) when at half $L = 10$

$10 = \frac{20}{1+19e^{-.77907t}}$

$1+19e^{-.77907t} = \frac{20}{10} = 2$

$19e^{-.77907t} = 1$

$e^{-.77907t} = \frac{1}{19}$

$-.77907t = \ln\left(\frac{1}{19}\right)$

$t = \frac{\ln\left(\frac{1}{19}\right)}{-.77907} = \boxed{3.778 \text{ hours}}$