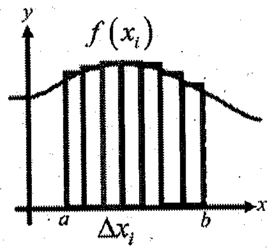
Unit 5-1: Area between curves

Integrals are 'summing machines'

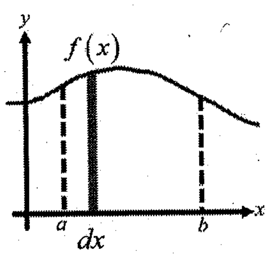
One of our definitions using the integral symbol nicely illustrates the idea that, in general, what an integral does is 'sum up' or 'accumulate' things.

When we first looked at area under a function curve, we did so by adding up subinterval rectangle areas with a Riemann Sum:



$$area = \sum_{i=1}^{n} (rectangle\ area)_i$$
$$= \sum_{i=1}^{n} (height)_i \cdot (height)_i$$
$$= \sum_{i=1}^{n} f(x_i) \cdot \Delta x_i$$

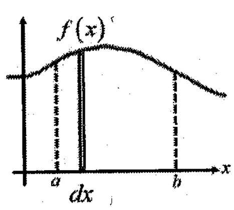
Then we got the definite integral by using an infinite number of infinitely narrow rectangles:



$$area = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x_i$$

$$area = \int_a^b f(x)\, dx$$

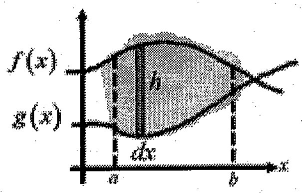
It turns out that integrals can be used to sum up an infinite number of a variety of different things, not just rectangles to produce an area. The only requirement is that one element of the 'thing' being summed must be an infinitely small quantity which can be represented by a differential.



$$area = \int_a^b f(x)\, dx$$

Area between two curves

For example, we could find the area between two function curves:



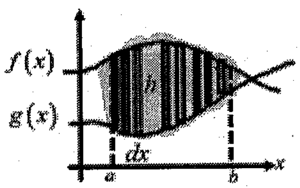
$$area = \int height \cdot width$$

$$= \int_a^b \left(f(x) - g(x)\right) dx$$

most positive curve — most negative curve

Some things to note:

- We want to find a positive area, so the height must be positive, which means we need to be careful to take the 'top' (most positive) curve and subtract the 'bottom' (most negative) curve to form the height.



- We need to imagine that we are 'sliding' the summing rectangle across different x-values to fill the area. This means that each rectangle is at its own unique x-value position, so:
  - Limits of integration are x-values: the lowest and highest x-value we need to 'cover' the region.
  - The function curves must be expressed as functions of the x-variable.

We call this '<u>integrating with respect to x</u>'.

#1. Find the area enclosed by $y = -x + 1$ and $y = -x^2 + 3x + 1$

<u>intersections:</u> $\begin{cases} y = -x+1 \\ y = -x^2+3x+1 \end{cases}$

$-x+1 = -x^2+3x+1$

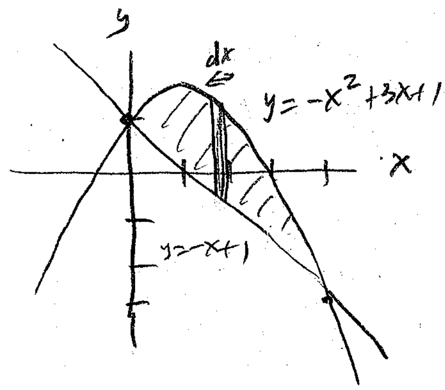
$x^2 - 4x = 0$

$x(x-4) = 0$

$x=0, \; x=4$

$y=1, \; y=-3$

$(0,1) \quad (4,-3)$



$A = \int_a^b (f_{top} - f_{bottom})\,dx$

$\int_0^4 [(-x^2+3x+1) - (-x+1)]\,dx$

$\int_0^4 (-x^2+4x)\,dx$

$\left[-\frac{1}{3}x^3 + 2x^2\right]_0^4$

$\boxed{\left(-\frac{1}{3}(4)^3 + 2(4)^2\right) - \left(-\frac{1}{3}(0)^3 + 2(0)^2\right)}$ $\left(= \frac{32}{3}\right)$

#2. Find the area enclosed by $x=-y$ and $x=-y^2+2y$

Intersections: $\begin{cases} x=-y \\ x=-y^2+2y \end{cases}$

$-y=-y^2+2y$

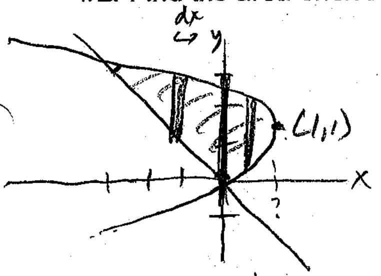
$y^2-3y=0$

$y(y-3)=0$

$y=0,\quad y=3$

$x=0,\quad x=-3$

$(0,0),\quad (-3,3)$



$y=-x$

$y^2-2y=-x$

$y^2-2y+1=-x+1$

$(y-1)^2=-x+1$

$y-1=\pm\sqrt{-x+1}$

$y=1\pm\sqrt{-x+1}$

? at max x: $\frac{dx}{dy}=-2y+2=0$

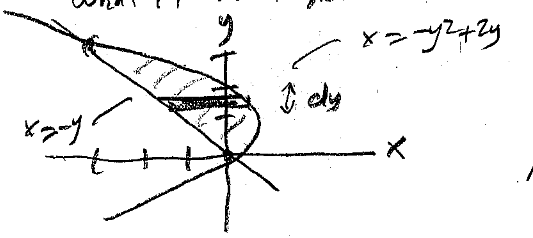
when $y=1$, $x=-(1)^2+2(1)=1$

$(1,1)$

$$A=\int_{-3}^{0}\left[(1+\sqrt{-x+1})-(-x)\right]dx+\int_{0}^{1}\left[1+\sqrt{-x+1}\right]-\left[1-\sqrt{-x+1}\right]dx$$

wow! there must be an easier way...

what if rectangles were horizontal?



$$A=\int_a^b (x_{top}-x_{bottom})\,dy$$

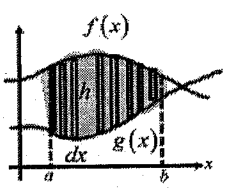
$$A=\int_0^3\left[(-y^2+2y)-(-y)\right]dy$$

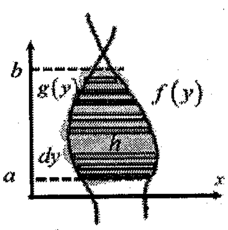
$$\int_0^3\left[-y^2+3y\right]dy$$

$$\left[-\tfrac{1}{3}y^3+\tfrac{3}{2}y^2\right]_0^3$$

$$\boxed{\left[-\tfrac{1}{3}(3)^3+\tfrac{3}{2}(3)^2\right]-\left(\tfrac{1}{3}(0)^3+\tfrac{3}{2}(0)^2\right)}\ \left(=\tfrac{9}{2}\right)$$

Select integrating with respect to x or y depending upon which has fewer regions





$$area=\int_a^b\big(f(x)-g(x)\big)\,dx \qquad area=\int_a^b\big(f(y)-g(y)\big)\,dy$$

__Remember:__ Everything in the integral is either all x or all y (the variables and the limits of integration)

Use a graph to determine which function is more positive (may be different in different regions)

#3. Find the area enclosed by the functions: $f(x)=\sqrt[3]{x-1}$

$g(x)=x-1$

intersections: $\begin{cases} y=\sqrt[3]{x-1} \\ y=x-1 \end{cases}$

$\sqrt[3]{x-1}=x-1$

$x-1=(x-1)^3=(x-1)(x-1)(x-1)$

$x-1=(x-1)(x^2-2x+1)$

$x-1=x^3-2x^2+x-x^2+2x-1$

$x^3-3x^2+2x=0$

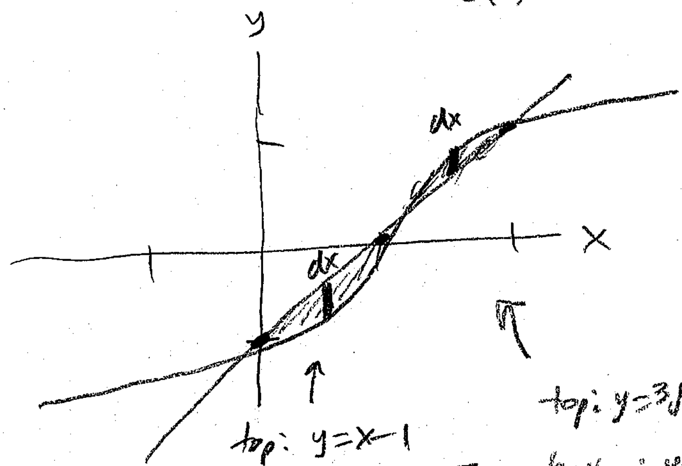
$x(x^2-3x+2)=0$

$x(x-1)(x-2)=0$

$x=0$, $x=1$, $x=2$

$y=-1$ $y=0$ $y=1$

$(0,-1)$ $(1,0)$ $(2,1)$



top: $y=x-1$
bottom $y=\sqrt[3]{x-1}$

top: $y=\sqrt[3]{x-1}$
bottom: $y=x-1$

$$A=\int_0^1[(x-1)-\sqrt[3]{x-1}]dx+\int_1^2[(\sqrt[3]{x-1}-(x-1))]dx$$

$$=\int_0^1(x-1)dx-\int_0^1(x-1)^{1/3}dx+\int_1^2(x-1)^{1/3}dx-\int_1^2(x-1)dx$$

$$=\left[\tfrac{1}{2}x^2-x\right]_0^1-\left[\tfrac{3}{4}(x-1)^{4/3}\right]_0^1+\left[\tfrac{3}{4}(x-1)^{4/3}\right]_1^2-\left[\tfrac{1}{2}x^2-x\right]_1^2$$

$$=\boxed{[(\tfrac{1}{2}(1)^2-(1))-(0)]-[(0)-\tfrac{3}{4}(-1)^{4/3}]+[\tfrac{3}{4}(1)^{4/3}-(0)]-[(\tfrac{1}{2}(2)^2-2)-(\tfrac{1}{2}(1)^2-(1))]}$$

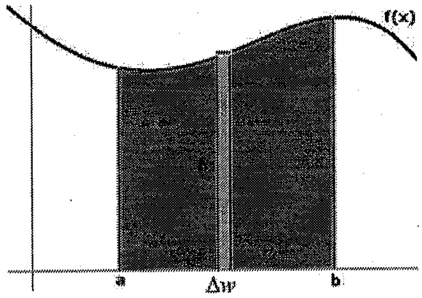
$$=-\tfrac{1}{2}+\tfrac{3}{4}+\tfrac{3}{4}-\tfrac{1}{2}$$

$$=\boxed{\tfrac{1}{2}}$$

Unit 5-2: Volumes of Solids of Revolution (Disc/Washer)

Integrals are 'summing machines'

We can use an integral to find the area under a function curve (between the curve and the x-axis):



The area is a summation of an infinite number of small rectangles:

$$A = \sum (\textit{area of rectangle})$$

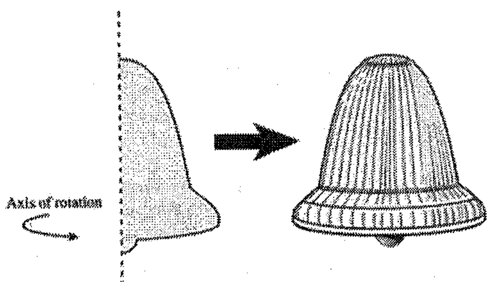
$$A = \sum \textit{height} \bullet \textit{width}$$

$$A = \int_a^b h \bullet \Delta w$$

$$A = \int_a^b f(x)\,dx$$

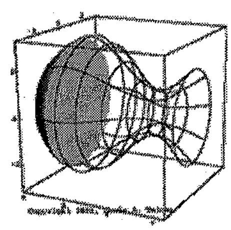
Instead of area elements, we can sum volume elements

Have you ever seen one of those tissue paper accordion-style decorations? They start out as a two-dimensional cardboard shape, but as you open them, you get a three-dimensional shape:

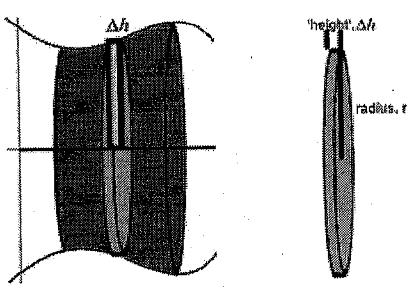


Volumes of Solids of Revolution - Disc Method

When we rotate a 2-D area around an axis, it forms a 3-D **solid of revolution**:

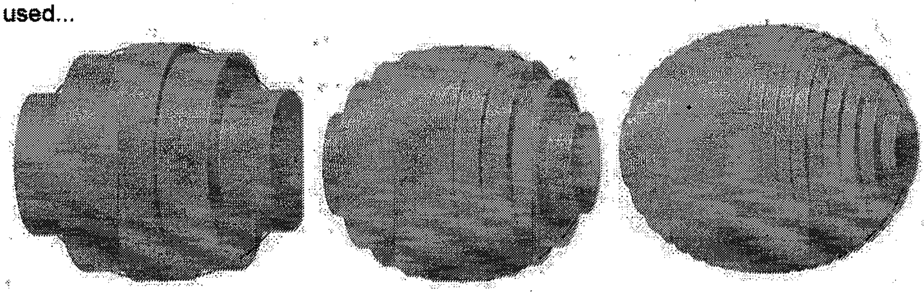


We can then calculate the area by summing a series of infinitely thin 'discs' (cylinders):



Volumes of Solids of Revolution - Disc Method

Of course, the volume becomes closer to the actual volume when more cylinders are used...

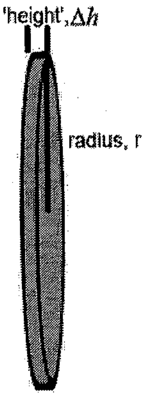


(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

...so we will sum an infinite number of infinitely thin cylinders using an integral.

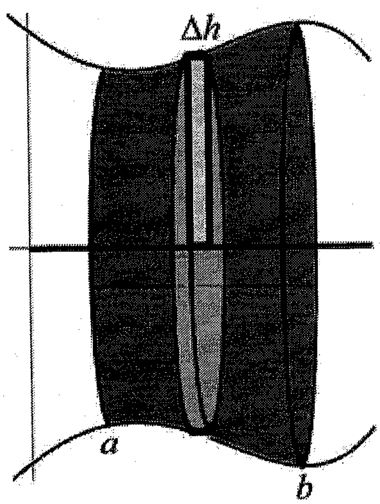
Volumes of Solids of Revolution - Disc Method

From geometry, the volume of a right circular cylinder is $V_{cylinder} = \pi r^2 h$ so the volume in our small (infinitely thin) cylinder is:



$$V_{thin\ cylinder} = \pi r^2 \Delta h$$

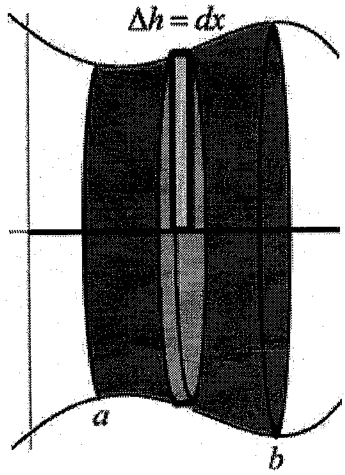
We can therefore use an integral to find the summation of a series of these cylinder volumes to find the volume of the solid of revolution:



$$V = \sum (volume\ of\ cylinder)$$
$$V = \sum \pi r^2 \bullet height$$
$$V = \int_a^b \pi r^2 \Delta h$$

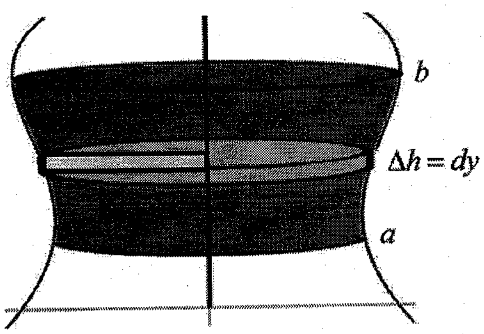
Volumes of Solids of Revolution - Disc Method

For this solid of revolution, the radius, r, is also 'y' which is f(x), and the small height, Δh, is a small change in 'x', which we would write as 'dx':



$$V = \sum (volume\ of\ cylinder)$$
$$V = \sum \pi r^2 \cdot height$$
$$V = \int_a^b \pi r^2 \Delta h$$
$$V = \int_a^b \pi y^2 dx$$
$$\boxed{V = \int_a^b \pi [f(x)]^2 dx}$$

Instead of the function y=f(x), we could have a function x=f(y) and the solid could be revolving around the y-axis. In that case, the radius would be an 'x' value, and the Δh would be a change in y, dy:



$$V = \sum (volume\ of\ cylinder)$$
$$V = \sum \pi r^2 \cdot height$$
$$V = \int_a^b \pi r^2 \Delta h$$
$$V = \int_a^b \pi x^2 dy$$
$$\boxed{V = \int_a^b \pi [f(y)]^2 dy}$$

Volumes of Solids of Revolution - Disc Method

This suggests a procedure we could use to find the volume of a solid of revolution:

1) Draw a sketch (and show 3-D rotation into a solid).
2) Select method ('disc').
3) Draw the rectangle (disc: rectangle is $\Delta h \perp$ to axis of rotation).
4) Determine if this Δh is *dx* or *dy*.
5) Rotate the rectangle to make the cylinder shape and use geometry to write *r* in terms of *x* or *y*.
6) Build an integral for the volume by summing cylinder areas:

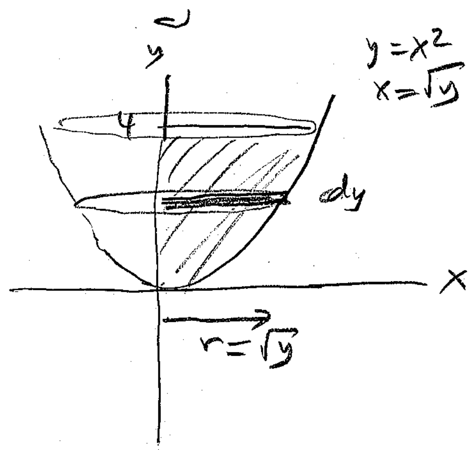
   disc method: $V = \int_a^b \pi r^2 dh$

7) Substitute to get everything in terms of the integration variable (set by *dx* or *dy*).
8) Evaluate the integral.

Volume Examples (Disc Method)

#1. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

$y=x^2$, y-axis, $y=4$, in the first quadrant; about the $y-axis$



$y=x^2$

$x=\sqrt{y}$

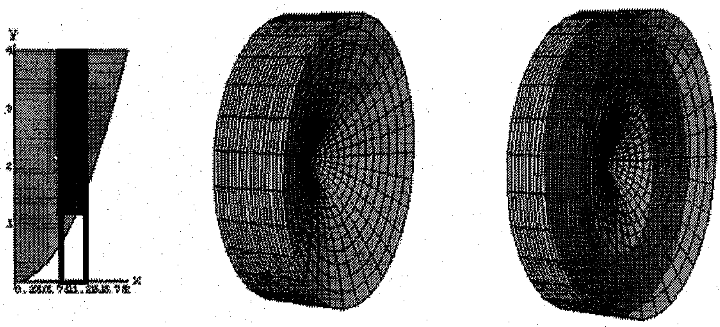
$V = \int \pi r^2 dh$

$= \boxed{\int_0^4 \pi (\sqrt{y})^2 dy}$

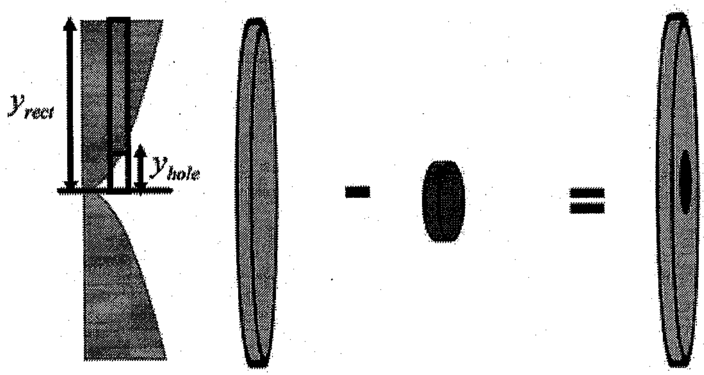
$(= 25.133)$

Volumes of Solids of Revolution - Washer Method

$y=x^2$, y-axis, $y=4$, in the first quadrant;
about the x-axis



We need the Washer Method when only part of the rectangle is filled with material.



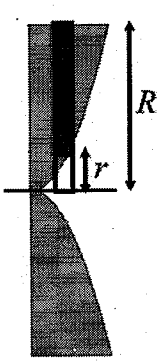
We find the volume of the solid 'washer' by finding the volume of the entire disc, and then subtracting the volume in the hole.

These volumes have the same 'width' but different 'heights'.

$$V = V_{entire\ cylinder} - V_{hole}$$

$$V = \int_a^b \pi y_{rect}^2 dx - \int_a^b \pi y_{hole}^2 dx$$

A better and more flexible way to think of this is to build a single integral for the volume but for the height use the height of just the portion of the rectangle which is filled with material, defining two radii: *R* for the larger 'outer' radius, and *r* for the smaller 'inner' radius:



$$V = \int_a^b \left[\pi R^2 dh - \pi r^2 dh \right]$$

Volumes of Solids of Revolution - Disc/Washer Methods

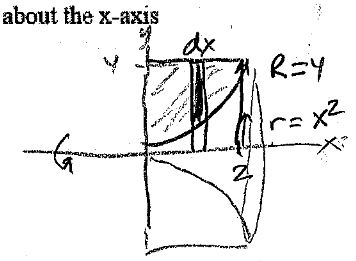
Here is an updated procedure covering both cases:

1) Draw a sketch (and show 3-D rotation into a solid).
2) Draw the rectangle (disc: rectangle is $\Delta h \perp$ to axis of rotation).
3) Select method: disc if rectangle is completely filled, washer if partially filled.
4) Determine if this Δh is *dx* or *dy*.
5) Rotate the rectangle to make the cylinder shape and use geometry to write *r* in terms of *x* or *y*.
6) Build an integral for the volume by summing cylinder areas:

disc method: $V = \int_a^b \pi r^2 dh$ washer method: $V = \int_a^b \pi \left[R^2 - r^2\right] dh$

7) Substitute to get everything in terms of the integration variable (set by *dx* or *dy*).
8) Evaluate the integral.

#2. $y = x^2$, y-axis, $y = 4$, in the first quadrant; about the x-axis

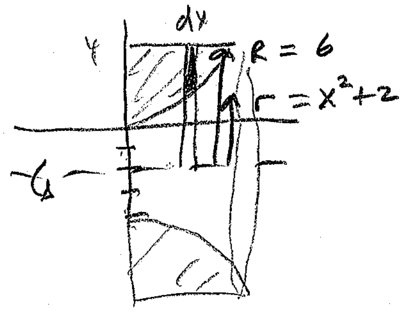


$$V = \int \pi R^2 dh - \int \pi r^2 dh$$

$$\boxed{V = \int_0^2 \pi(4)^2 dx - \int_0^2 \pi(x^2)^2 dx}$$

$= 80.425$

#3. $y = x^2$, y-axis, $y = 4$, in the first quadrant; about $y = -2$



$$V = \int \pi R^2 dh - \int \pi r^2 dh$$

$$\boxed{V = \int_0^2 \pi(6)^2 dx - \int_0^2 \pi(x^2+2)^2 dx}$$

$= 147.445$

Volumes Examples

#4. $y = x^2$, $x = y^2$; about $x = -1$

$x = \sqrt{y}$

Intersections: $\begin{cases} y = x^2 \\ x = y^2 \end{cases}$

$y = (y^2)^2 = y^4$

$y - y^4 = 0$

$y(1 - y^3) = 0$

$y = 0, \ y = 1$

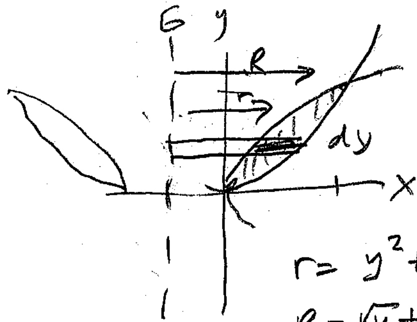
$x = 0, \ x = 1$

$(0,0) \ (1,1)$

$V = \int \pi R^2 dh - \int \pi r^2 dh$

$$\boxed{V = \int_0^1 \pi(\sqrt{y}+1)^2 dy - \int_0^1 \pi (y^2+1)\, dy}$$

$= 3.0369$

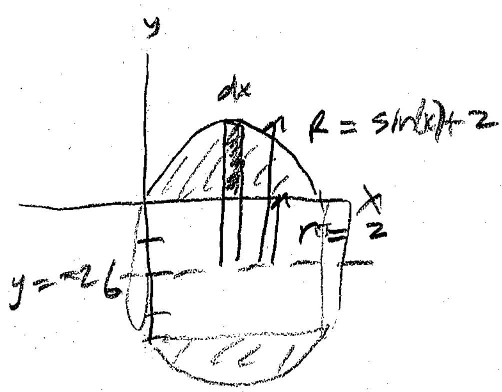


$r = y^2 + 1$

$R = \sqrt{y} + 1$

#5. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

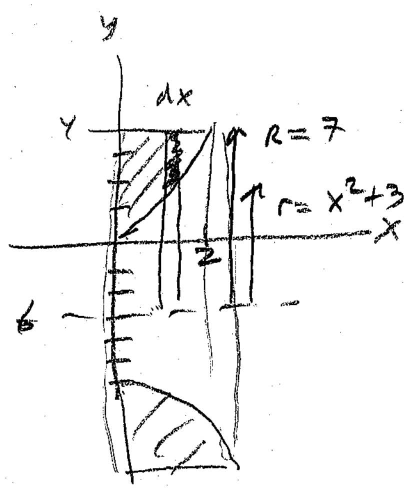
$y=0, \quad y=\sin x, \quad 0 \le x \le \pi; \quad$ about $y=-2$



$V = \int \pi R^2 dh - \int \pi r^2 dh$

$$\boxed{V = \int_0^{\pi} \pi(\sin x+2)^2 dx - \int_0^{\pi} \pi (2)^2 dx}$$

#6. $y=x^2$, y-axis, $y=4$, in the first quadrant; about y = - 3



$V = \int \pi R^2 dh - \int \pi r^2 dh$

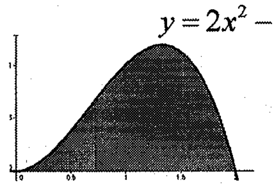
$$\boxed{V = \int_0^{2} \pi (7)^2 dx - \int_0^{2} \pi (x^2+3)^2 dx}$$

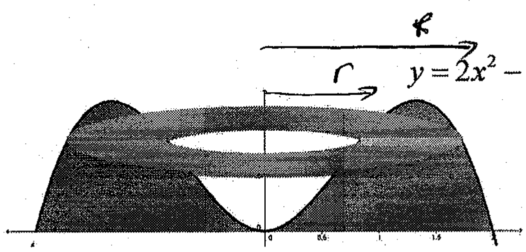
$= 180.956$

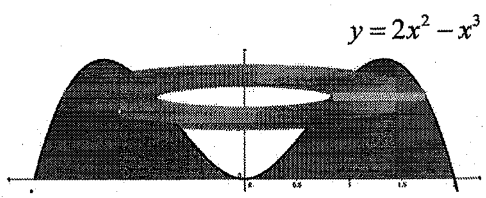
Some problems are difficult to solve in the Disc/Washer method:

Find volume of solid obtained by rotating about the y-axis

r & R are both 'x' distances but we can't solve $y = 2x^2 - x^3$ for x ☹

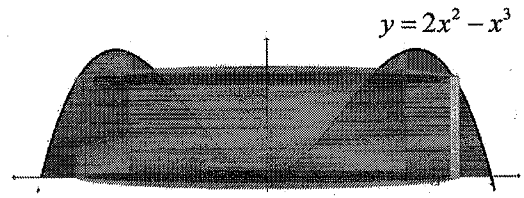






'Disc/Washer Method'

rectangle $\perp$ *axis of rotation*

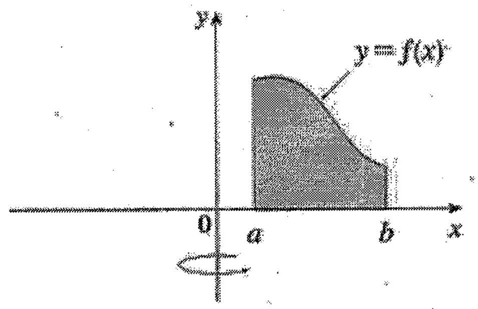


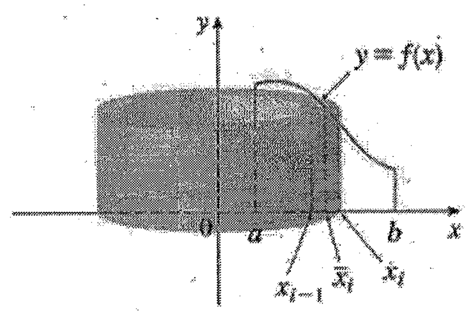
'Shell Method'

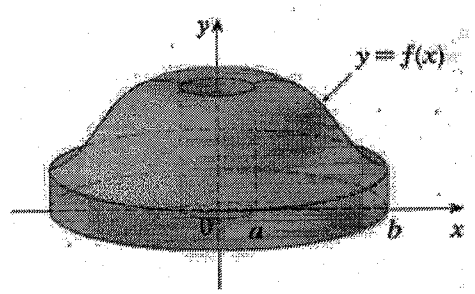
rectangle $\|$ *axis of rotation*

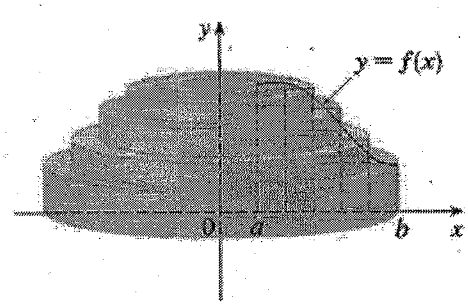
Volumes - Shell Method

Rotating a rectangle parallel to axis of rotation around creates a 'shell' and these shells can be nested to fill a rotational solid:

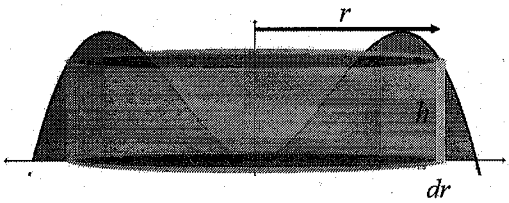


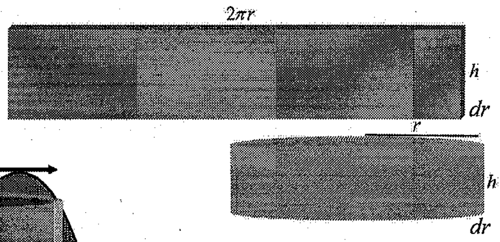






What is the volume of the shell element? A thin-walled cylindrical shell 'folds out' to become a rectangular box:





$$V_{shell} = 2\pi rhdr$$

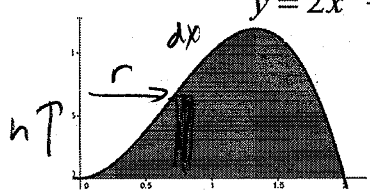
Volumes - Shell Method Example

#1. Find volume of solid obtained by rotating about the y-axis

$V = \int 2\pi rhdr$

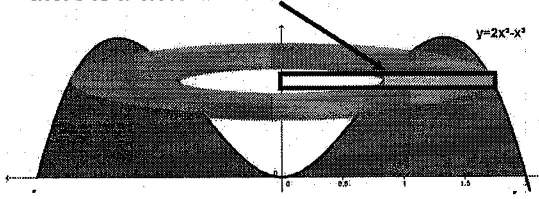
$$\boxed{V = \int_0^2 2\pi(x)(2x^2 - x^3)\,dx} = 10{,}053$$

$y = 2x^2 - x^3$



Shell is better when there is 'no hole' or when you can't solve for an expression

Rectangle doesn't go all the way to axis there is a 'hole' so we need two radii

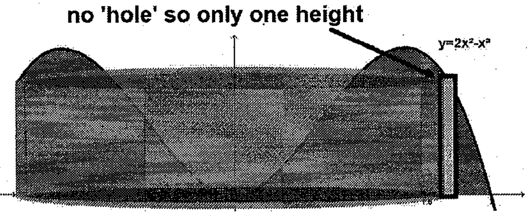


'Disc/Washer Method'

rectangle $\perp$ *axis of rotation*

$$V = \int \pi\left[R^2 - r^2\right] dh$$

Rectangle goes all the way to axis no 'hole' so only one height

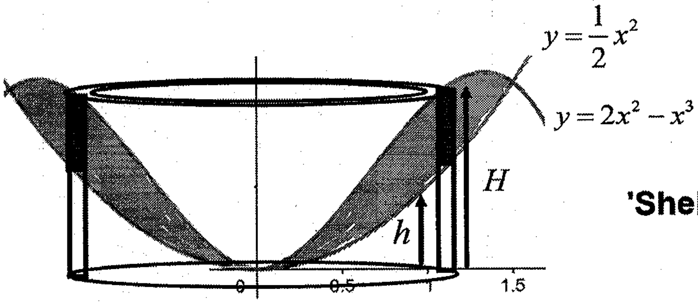


'Shell Method'

rectangle $\|$ *axis of rotation*

$$V = \int 2\pi rhdr$$

...although you can need two heights for shell in some circumstances...

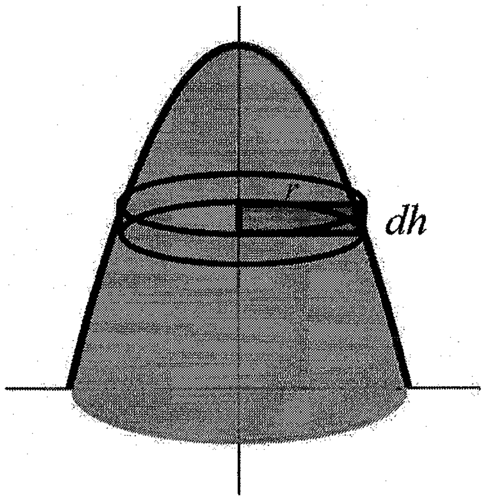


'Shell Method with two curves'

$$V = \int 2\pi r(H - h)\,dr$$

Summary of Methods for Volume of Solids of Rotation

'Disc/Washer Method'



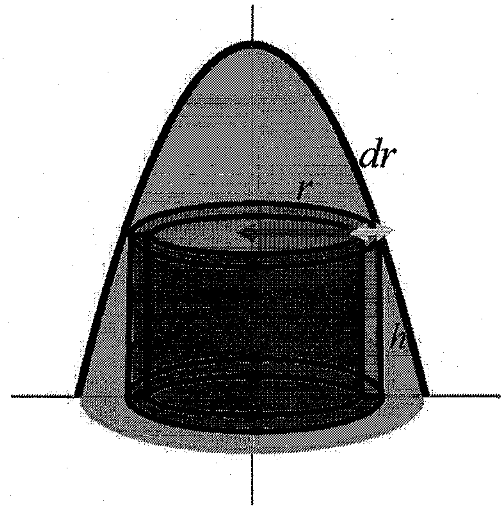
rectangle $\perp$ *axis of rotation*

"perpendiscular"

rectangle filled: $V = \int \pi r^2 dh$

rectangle not filled: $V = \int \pi \left[R^2 - r^2 \right] dh$

'Shell Method'



rectangle $\|$ *axis of rotation*

"parashell"

rectangle filled: $V = \int 2\pi r h dr$

rectangle not filled: $V = \int 2\pi r \left(H - h \right) dr$

Volumes of Solids of Revolution - Disc/Washer/Shell Methods

1) Draw a sketch (and show 3-D rotation into a solid).
2) Select method: disc/washer or shell based upon writing expressions or which has only 1 radii or height.
3) Draw the rectangle (disc: rectangle $\perp$, shell: rectangle $\|$ to axis of rotation).
4) Determine if the rectangle width is *dx* or *dy* (this establishes variable of integration).
5) Rotate the rectangle to make the cylinder shape and use geometry to write radii and heights in terms of *x* or *y*.
6) Build an integral for the volume by summing cylinder areas...

disc/washer method:

rectangle filled: $V = \int \pi r^2 dh$

rectangle not filled: $V = \int \pi \left[R^2 - r^2 \right] dh$

shell method:

rectangle filled: $V = \int 2\pi r h dr$

rectangle not filled: $V = \int 2\pi r \left(H - h \right) dr$

7) Substitute to get everything in terms of the integration variable (set by *dx* or *dy*).
8) Evaluate the integral.

Volumes Example

#2. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis. Sketch the region and a typical shell.

$$y = x^2 - 6x + 10, \quad y = -x^2 + 6x - 6$$

Intersections:

$x^2 - 6x + 10 = -x^2 + 6x - 6$

$2x^2 - 12x + 16 = 0$

$2(x^2 - 6x + 8) = 0$

$2(x-2)(x-4) = 0$

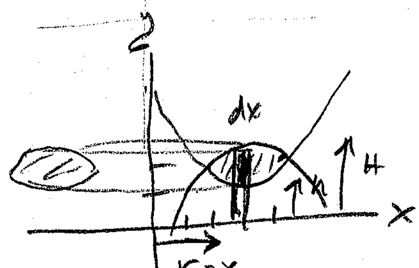
$x = 2, \quad x = 4 \quad (2,2)\ (4,2)$

$y = 2 \quad y = 2$

$$V = \int 2\pi r H\,dr - \int 2\pi r h\,dr$$

$$\boxed{V = \int_2^4 2\pi(x)(-x^2+6x-6)\,dx - \int_2^4 2\pi(x)(x^2-6x+10)\,dx}$$

$= 50{,}265$



$r = x$

$h = x^2 - 6x + 10$

$H = -x^2 + 6x - 6$

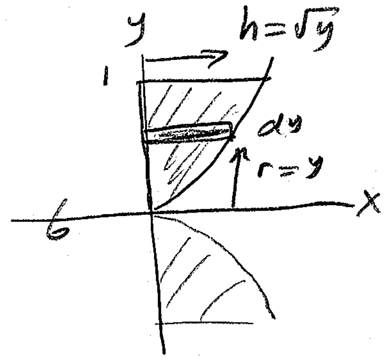
#3. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the x-axis. Sketch the region and a typical shell.

$$x = \sqrt{y}, \quad x = 0, \quad y = 1$$

$y = x^2$

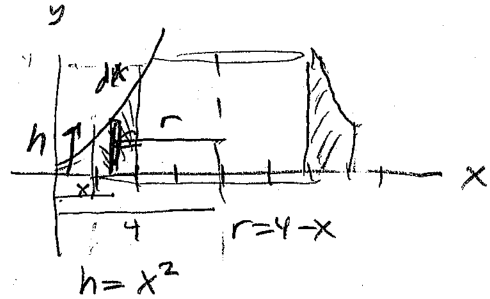
$$H = \int 2\pi r h\,dr$$

$$\boxed{H = \int_0^1 2\pi(y)(\sqrt{y})\,dy} = 2{,}513$$



#4. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis. Sketch the region and a typical shell.

$y = x^2, \quad y = 0, \quad x = 1, \quad x = 2; \quad$ about $x = 4$

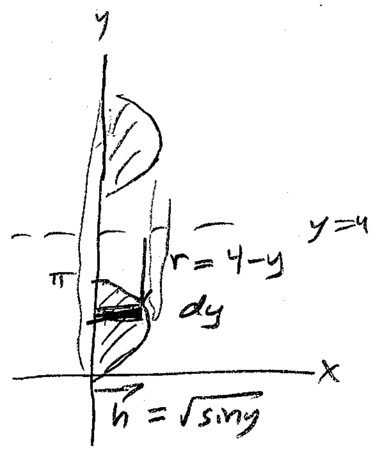


$V = \int 2\pi r h \, dr$

$$\boxed{V = \int_1^2 2\pi(4-x)(x^2)\,dx} = 35.081$$

#5. Set up, but **do not evaluate**, an integral....

$x = \sqrt{\sin y}, \quad 0 \le y \le \pi, \quad x = 0; \quad$ about $y = 4$



$V = \int 2\pi r h \, dr$

$$\boxed{V = \int_0^{\pi} 2\pi(4-y)(\sqrt{\sin y})\,dy}$$

#6. Use a graph to estimate the x-coordinates of the points of intersection of the given curves. Then use this information to estimate the volume of the solid obtained by rotating about the y-axis the region enclosed by these curves.

$$y = x^4, \quad y = 3x - x^3$$

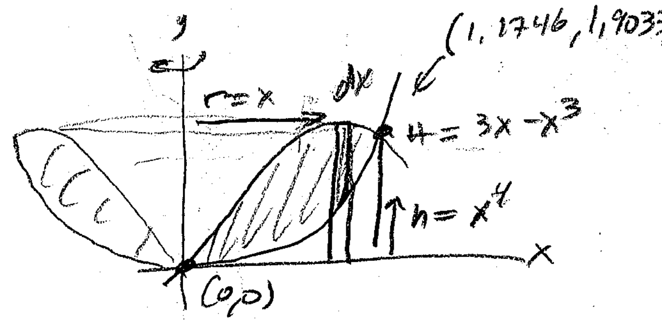
Intersections : $\begin{cases} y = x^4 \\ y = 3x - x^3 \end{cases}$

$x^4 = 3x - x^3$ (calculator)

$$V = \int 2\pi r H\,dr - \int 2\pi r h\,dr$$

$$\boxed{V = \int_0^{1.1746} 2\pi(x)(3x-x^3)\,dx - \int_0^{1.1746} 2\pi(x)(x^4)\,dx}$$

$= 4.622$



Unit 5-4: Volumes of Solids - using Cross Sections

Volumes with fixed Cross-Sectional Shapes

We can also use integrals to find volumes of solids whose shape is not rotationally symmetrical about an axis, but where the 'cross-sections' are constant shapes.

Here, we will still use infinitely thin cross-section 'slices' and the volume will be the infinitely thin differential width times a cross-sectional area:

$$V = \int_a^b A_{cross-section}\, dw$$

1) Sketch the 3-D solid.

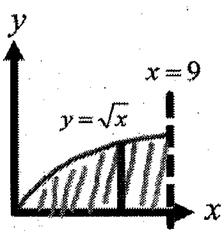
2) Summing along what variable? (perpendicular to cross-sections)

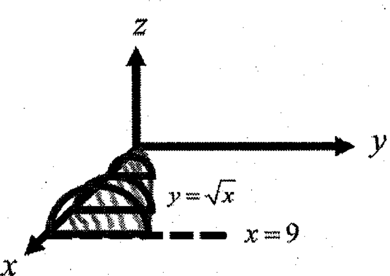
3) Determine a formula for the cross-sectional area (usually a function of the summing variable)

4) Sum these cross-sectional areas with an integral to find the volume.

#1. Base bounded by $y = x^{1/2}$, $x = 9$, x-axis. Sections perpendicular to the x-axis are semicircles. Find the volume.

1) Sketch the 3-D solid:

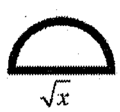




2) Summing along what variable? (perpendicular to cross-sections)

here, cross-sections are in y-z, so summing along x: *dx*

3) Determine a formula for the cross-sectional area (usually a function of the summing variable)

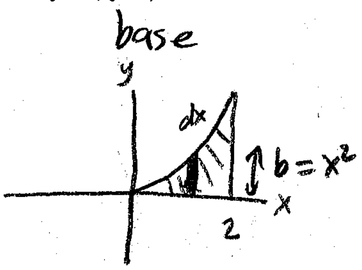


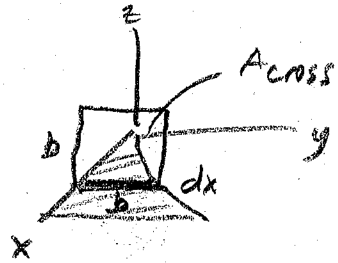
$$A_{cross-section} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{\sqrt{x}}{2}\right)^2 = \frac{\pi}{8}x$$

4) Sum these cross-sectional areas with an integral to find the volume.

$$V = \int_a^b A_{cross-section}\, dw = \int_0^9 \frac{\pi}{8}x\, dx = \frac{81\pi}{16}$$

#2. Find the volume of the solid formed by cross-sections which are perpendicular to the x-axis and form squares. The base of the shape is in the x-y plane, defined by $y=x^2$, $y=0$, and $x=2$.



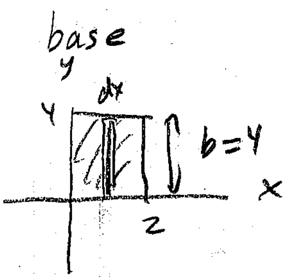


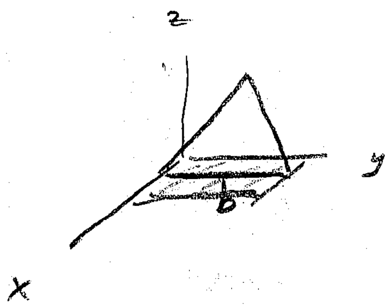
$A_{cross} = b^2 = (x^2)^2$

$$V = \int_a^b A_{cross}\, dx$$

$$\boxed{V = \int_0^2 (x^2)^2 dx} = 6.4$$

#3. Find the volume of the solid formed by cross-sections which are perpendicular to the x-axis and form equilateral triangles where the base is one of the congruent sides. The base of the shape is in the x-y plane, defined by a rectangle which is 2 x 4.





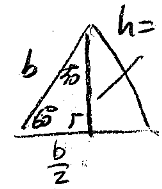
$A_{cross} = \frac{1}{2}bh$

$= \frac{1}{2}b\frac{\sqrt{3}}{2}b$

$= \frac{\sqrt{3}}{4}b^2$

$= \frac{\sqrt{3}}{4}(4)^2$

$= 4\sqrt{3}$

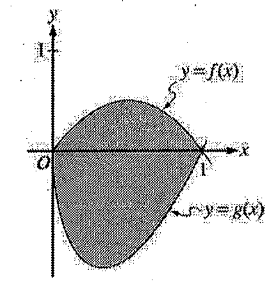


$h = \frac{b}{2}\sqrt{3}$

$$V = \int A_{cross}\, dx$$

$$\boxed{V = \int_0^2 4\sqrt{3}\, dx} = 8\sqrt{3} = 13.856$$

#4.



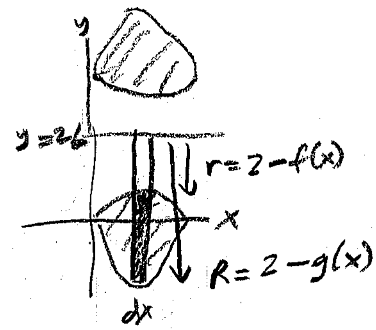
Let f and g be the functions given by $f(x) = 2x(1-x)$ and $g(x) = 3(x-1)\sqrt{x}$ for $0 \le x \le 1$. The graphs of f and g are shown in the figure above.

(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Find the volume of the solid generated when the shaded region enclosed by the graphs of f and g is revolved about the horizontal line $y = 2$.

(c) Let h be the function given by $h(x) = kx(1-x)$ for $0 \le x \le 1$. For each $k > 0$, the region (not shown) enclosed by the graphs of h and g is the base of a solid with square cross sections perpendicular to the x-axis. There is a value of k for which the volume of this solid is equal to 15. Write, but do not solve, an equation involving an integral expression that could be used to find the value of k.

a) $\boxed{A = \int_0^1 (f(x) - g(x))\,dx = 1.133}$

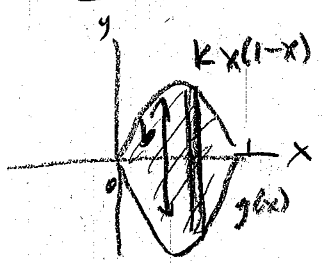


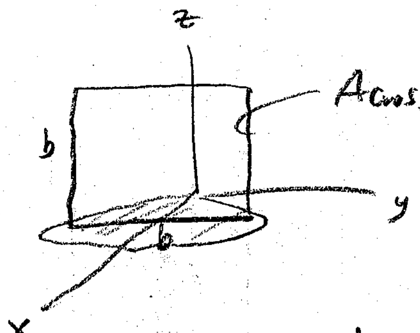
b) $V = \int \pi R^2 dh - \int \pi r^2 dh$

$\boxed{V = \int_0^1 \pi (2 - g(x))^2 dx - \int_0^1 \pi (2 - f(x))^2 dx}$

$V = 24.97566195 - 8.796459 43 = \boxed{16.179}$

c) base





$A_{cross} = b^2$
$= (kx(1-x) - g(x))^2$

$b = kx(1-x) - g(x)$

$V = \int A_{cross}\, dx = \int_0^1 (kx(1-x) - g(x))^2 dx$

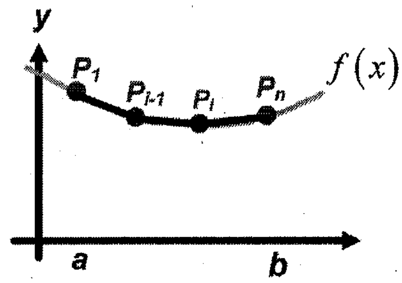
$\boxed{\int_0^1 (kx(1-x) - g(x))^2 dx = 15}$

Unit 5-5: Arc Length, Surface Areas of Revolution

Arc Length

We've been using integrals to add up elements of area and volume, but we can also use an integral to sum small linear segments to find the **arc length** along a function curve path:

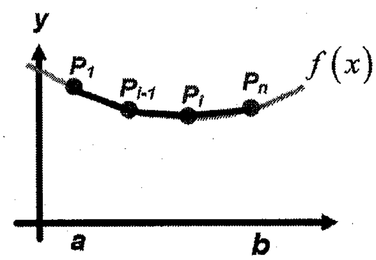
We can first imagine approximating the arc length as a series of small line segments, between points along the function curve from some starting x-value=a at point P_1 and ending a x-value=b at point P_n.



The arc length, L, would then be the summation of the individual line segment lengths. To find each segment length, we use the distance formula. Then, to make this approximation very close to the actual curve, we take the limit to use an infinite number of infinitely small line segments:

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}\left|P_{i-1}P_i\right|$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(\Delta x)^2 + (\Delta y)^2}$$



By the mean value theorem: $f'(x) = \dfrac{\Delta y}{\Delta x}$, $\Delta y = f'(x)\,\Delta x$ so...

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(\Delta x)^2 + \left[f'(x)\,\Delta x\right]^2}$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+\left[f'(x)\right]^2}\sqrt{(\Delta x)^2}$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+\left[f'(x)\right]^2}\,\Delta x$$

Finally, using an integral for the limit of the summation...

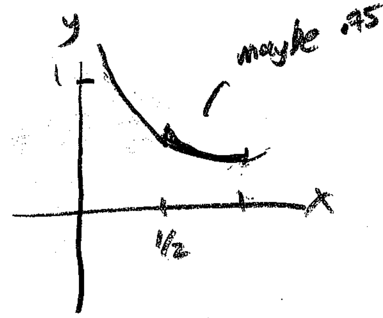
$$\boxed{L = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx}$$ **Arc Length Formula**

#1. Graph the curve and visually estimate its length. Then find its exact length.

$$y=\frac{x^3}{6}+\frac{1}{2x}, \quad \frac{1}{2}\le x\le 1$$

$f(x)=\frac{1}{6}x^3+\frac{1}{2}x^{-1}$

$f'(x)=\frac{1}{2}x^2-\frac{1}{2}x^{-2}$



$L=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$

$L=\int_{1/2}^{1}\sqrt{1+(\frac{1}{2}x^2-\frac{1}{2}x^{-2})^2}\,dx = 0.6458$

#2. Find the length of the curve.

$$y=\frac{x^4}{4}+\frac{1}{8x^2}, \quad 1\le x\le 3$$

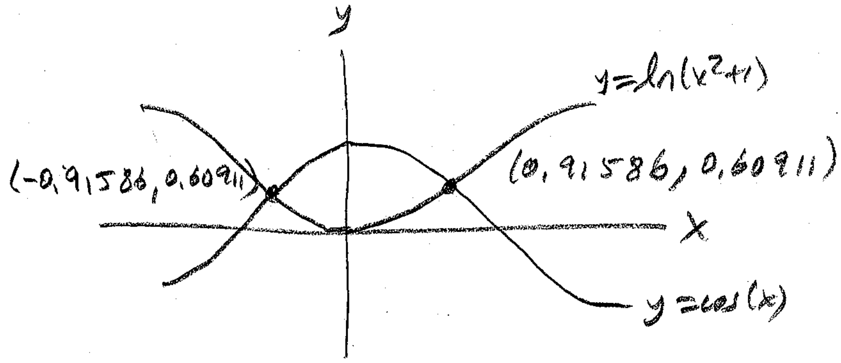
$f(x)=\frac{1}{4}x^4+\frac{1}{8}x^{-2}$

$f'(x)=x^3-\frac{1}{4}x^{-1}$

$$\boxed{L=\int_1^3\sqrt{1+[x^3-\frac{1}{4}x^{-1}]^2}\,dx \doteq 20.111}$$

#3. Let R be the region enclosed by the graphs of $y = \ln(x^2+1)$ and $y = \cos(x)$.

What is the length of the boundary of region R?



$y = \ln(x^2+1)$ $\qquad$ $y = \cos(x)$

$y' = \frac{1}{x^2+1}(2x)$ $\qquad$ $y' = -\sin(x)$

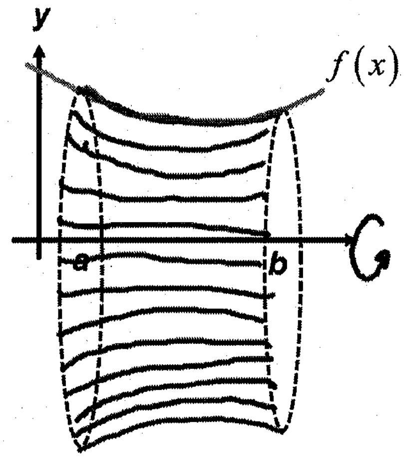
$$\text{Boundary} = \int_{-0,91586}^{0,91586} \sqrt{1+\left[\frac{2x}{x^2+1}\right]^2}\,dx + \int_{-0,91586}^{0,91586} \sqrt{1+[-\sin x]^2}\,dx$$

$$= 2,251762885 + 2,030384607$$

$$= 4,282$$

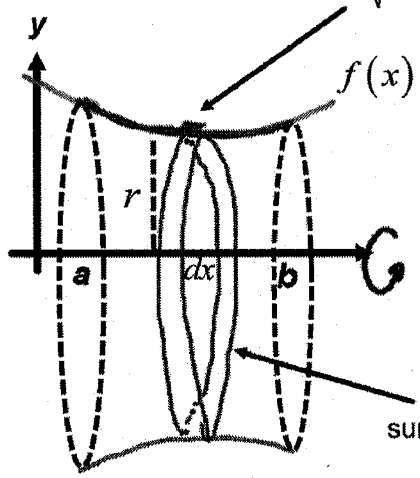
Surface Area of a Surface of Revolution

If you take an arc length and rotate it about an axis it forms a **surface of revolution**:



We can imagine taking a small linear element on the arc length, and rotating this around the axis to form a strip in the form of a ring:

arc length element: $\sqrt{1+\left[f'(x)\right]^2}\;dx$



(circumference of circle is a length) x (width)

surface area element (ring): $2\pi r\sqrt{1+\left[f'(x)\right]^2}\;dx$

Summing these elements with an integral along the path:

$$\boxed{A_{surface}=\int_a^b 2\pi r\sqrt{1+\left[f'(x)\right]^2}\;dx}$$

Surface Area of Surface of Revolution Formula

#4. Find the area of the surface obtained by rotating the curve about the x-axis.

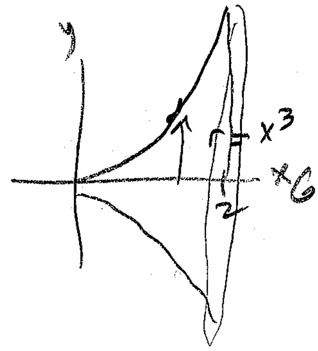
$y = x^3,\ 0 \le x \le 2$

$f(x) = x^3$

$f'(x) = 3x^2$

$$A = \int_a^b 2\pi r\sqrt{1+[f'(x)]^2}\,dx$$

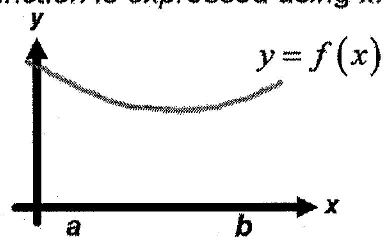
$$\boxed{A = \int_0^2 2\pi(x^3)\sqrt{1+[3x^2]^2}\,dx = 203{,}044}$$



Can use either dx or dy depending upon direction

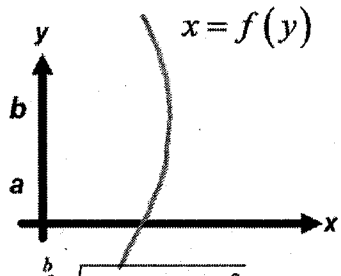
Arc Length

If function is expressed using x:



$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

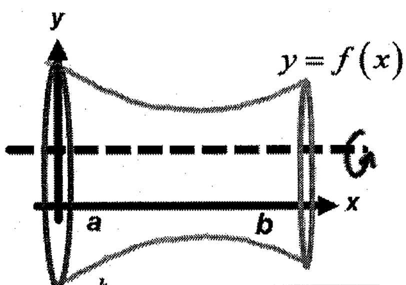
If function is expressed using y:



$$L = \int_a^b \sqrt{1+[f'(y)]^2}\,dy$$

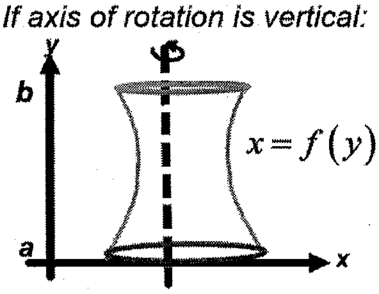
Surface Area of Surface of Revolution

If axis of rotation is horizontal:



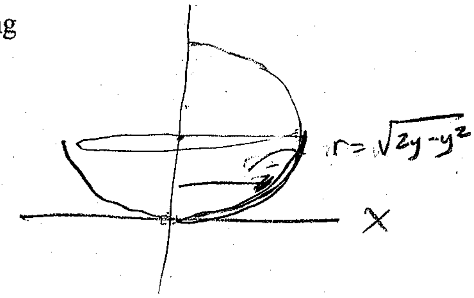
$$A_{surface} = \int_a^b 2\pi r\sqrt{1+[f'(x)]^2}\,dx$$

If axis of rotation is vertical:



$$A_{surface} = \int_a^b 2\pi r\sqrt{1+[f'(y)]^2}\,dy$$

#5. The given curve is rotated about the y-axis. find the area of the resulting surface.



$$x = \sqrt{2y - y^2}, \quad 0 \le y \le 1$$

$$x = (2y-y^2)^{1/2}$$

$$x' = \tfrac{1}{2}(2y-y^2)^{-1/2}(2-2y)$$

$$x = \sqrt{2y-y^2}$$

$$x^2 = 2y - y^2$$

$$x^2 + y^2 - 2y + 1 = 0 \quad +1$$

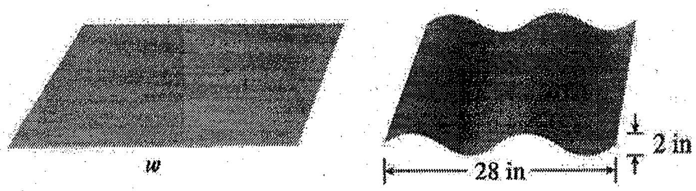
$$x^2 + (y-1)^2 = 1$$

$$A = \int_a^b 2\pi r\sqrt{1+[f'(x)]^2}\,dx$$

$$\boxed{A = \int_0^1 2\pi\left(\sqrt{2y-y^2}\right)\sqrt{1+\left[\tfrac{1}{2}(2y-y^2)^{-1/2}(2-2y)\right]^2}\,dy}$$

$(= 7.1193)$

#6. A manufacturer of corrugated metal roofing wants to produce panels that are 28 in. wide and 2 in. thick by processing flat sheets of metal as shown in the figure. The profile of the roofing takes the shape of a sine wave. Verify that the sine curve has the equation $y = \sin(\pi x/7)$ and find the width w of a flat metal sheet that is needed to make a 28-inch panel.



$$y = \sin\left(\tfrac{\pi}{7}x\right)$$

$$y' = \tfrac{\pi}{7}\cos\left(\tfrac{\pi}{7}x\right)$$

$$\boxed{w = L = \int_0^{28}\sqrt{1+\left[\tfrac{\pi}{7}\cos\left(\tfrac{\pi}{7}x\right)\right]^2}\,dx = 29.361 \text{ in}}$$

Unit 5-6: Average Value of a Function vs Average Rate of Change

Average Value of a Function vs. Average Rate of Change

If you want to find an average of a set of numbers, you add up the numbers and divide by the number of numbers:

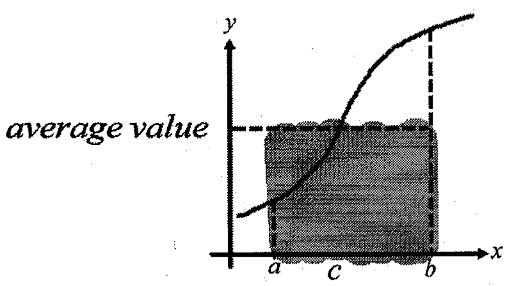
$$\textit{set of numbers}: 2,4,6,8,10$$

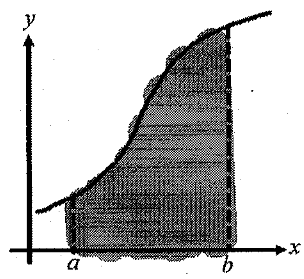
$$\textit{average value} = \frac{2+4+6+8+10}{5} = 6$$

This average value represents a 'typical' value for this set of numbers.

If you have a function that is defined over an interval, we can defind the **average value of the function** as a 'typical' *y*-value over this interval. To compute it, we use an integral to 'sum up the function values' and then to 'divide by the number of numbers' we divide by the width of the interval:

> If f is integrable on the closed interval [*a*,*b*], then the **average value** of f on the interval is
>
> $$\textit{average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$





This average value represents a 'typical' *y*-value for the function over this interval.

(More precisely, it is the *y*-value which has the property that the rectangular area between it and the *x*-axis is the same as the actual function's area under the function curve over the interval.)

Average Value of a Function vs. Average Rate of Change

#1. The number of people who've enrolled in an adult education class over the 40 day enrollment period is given by

$f(t)=\frac{1}{40}t^2+\frac{1}{2}t$ where t is in days and f is the number of people.

Find the average number of people enrolled over the enrollment period $0 \le t \le 40$

average number of people $= \boxed{\frac{1}{40-0}\int_0^{40}\left(\frac{1}{40}t^2+\frac{1}{2}t\right)dt = 23{,}333 \text{ people}}$

Important! The <u>Average Value</u> of a Function is different from the <u>Average Rate of Change</u> of a Function:

#2. The number of people who've enrolled in an adult education class over the 40 day enrollment period is given by

$f(t)=\frac{1}{40}t^2+\frac{1}{2}t$ where t is in days and f is the number of people.

Find the average rate of change in the number of people enrolled over the enrollment period $0 \le t \le 40$

average rate of change in number of people $= \boxed{\frac{f(40)-f(0)}{40-0} = \frac{60-0}{40} = 1{,}5 \frac{\text{people}}{\text{day}}}$

'average' or 'average value'

$$\boxed{\text{average value} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

'average rate of change'

$$\boxed{\text{average rate of change} = \frac{f(t_{end})-f(t_{start})}{t_{end}-t_{start}}}$$

#3. Find the average value of $f(x) = x^3 - 3x^2$ over the interval [1,4]

avg value of $f(x) = \boxed{\frac{1}{4-1}\int_1^4 (x^3-3x^2)\,dx = 0.25}$

#4. Find the average rate of change of $f(x) = x^3 - 3x^2$ over the interval [1,4]

average rate of change of $f(x) = \boxed{\frac{f(4)-f(1)}{4-1} = \frac{16-(-2)}{3} = \frac{18}{3} = 6}$

#5. An object's temperature in °C is changing over time and given by the function

$$T(t) = 25 + \frac{1}{10}t^2 + \cos(0.6t) \quad for \ 0 \le t \le 20$$

where T is in °C and t is in hours.

a) What is the average temperature of the object over $0 \le t \le 20$?

b) What is the average rate of change of the object's temperature over $0 \le t \le 20$?

a) $\boxed{\text{avg temperature} = \frac{1}{20-0}\int_0^{20} T(t)\,dt = 38.289 \ °C}$

b) $\boxed{\text{avg rate of change of } T = \frac{T(20)-T(0)}{20-0} = \frac{65.8438539587 - 26}{20} = 1.992\ °C/hr}$