AP Calc BC – Lesson Notes – Unit 5: Integration

Unit 5-1: Antiderivatives, Differential Equations, Applications
Larsen: 4.1 (Stewart: 4.9)

Antiderivatives

If we are given a function, we can find the derivative:

$$f(x) = 3x^4 \longrightarrow f'(x) = 12x^3$$

If we are given a derivative function, could we find the function from which it came:

$$f'(x) = 12x^3 \longrightarrow$$

'Reversing' the process of finding the derivative is called finding the **antiderivative**.

Symbols for antiderivatives

If $f(x) = 2x$

$F(x) = x^2$ is an antiderivative of $f(x)$

But the following are all also antiderivatives of $f(x)$:

$$F(x) = x^2 + 1$$
$$F(x) = x^2 - 22$$
$$F(x) = x^2 + 15{,}432{,}167$$
$$F(x) = x^2 + \frac{2}{7}$$

If $f(x) = 2x$

All the antiderivatives of $f(x)$ are of the form:

$$F(x) = x^2 + C$$

The process of taking an antiderivative of $f(x)$ is represented with the **integral sign** like this:

$$\int f(x)\,dx = F(x) + C$$

'integration constant'

Compare notation to taking a derivative...

$$\frac{d}{dx}\left[f(x)\right] = f'(x) \qquad \int f(x)\,dx = F(x) + C$$

$\frac{d}{dx}[\quad]$ = 'find derivative of' $\qquad \int(\quad)\,dx$ = 'find antiderivative of'

Antiderivative Shortcuts

For every basic derivative shortcut, we have an antiderivative shortcut...

$\frac{d}{dx}[C]=0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx]=k$	$\int k\,dx = kx + C$
$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1}+C \quad (n\neq -1)$
$\frac{d}{dx}\left[e^x\right]=e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}\left[a^x\right]=(\ln a)\,a^x$	$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$
$\frac{d}{dx}[\ln x]=\frac{1}{x} \quad (x>0)$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}[\sin x]=\cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x]=-\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}[\tan x]=\sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x]=\sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + C$
$\frac{d}{dx}[\csc x]=-\csc x\tan x$	$\int \csc x\cot x\,dx = -\csc x + C$
$\frac{d}{dx}[\cot x]=-\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\arcsin x]=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
$\frac{d}{dx}[\arccos x]=\frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + C$
$\frac{d}{dx}[\arctan x]=\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan x + C$
$\frac{d}{dx}[arc\sec x]=\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \mathrm{arc}\sec x + C$
$\frac{d}{dx}[arc\csc x]=\frac{-1}{x\sqrt{x^2-1}}$	$\int \frac{-1}{x\sqrt{x^2-1}}\,dx = \mathrm{arc}\csc x + C$
$\frac{d}{dx}[arc\cot x]=\frac{-1}{1+x^2}$	$\int \frac{-1}{1+x^2}\,dx = \mathrm{arc}\cot x + C$

Antiderivative Properties

Some of the properties for derivatives are similar for antiderivatives...

$$\frac{d}{dx}\left[k\,f(x)\right] = k\,f'(x) \qquad \int k\,f(x)\,dx = k\int f(x)\,dx$$

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x) \qquad \int \left(f(x) \pm g(x)\right)dx = \int f(x)\,dx + \int g(x)\,dx$$

...others are different (more about these later)...

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x) + g(x)\cdot f'(x) \longrightarrow$$ (Integration by Parts)

(Product Rule)

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)\cdot g'(x) \longrightarrow$$ (Integration by Substitution)

(Chain Rule)

Examples

Find the most general antiderivative of the function.
(Check your answer by differentiation.)

$$f(x) = 3e^x + 7\sec^2 x$$

Find f.

$$f''(x) = 1 + x^{4/5}$$

Find f. $\quad f'(x) = 3\sqrt{x} - \frac{1}{\sqrt{x}}, \quad f(1) = 2$

Examples

Find *f*. $f''(x) = 12x^2 - 6x + 2, \quad f(0) = 1, \quad f(2) = 11$

50. a) Use a graphing device to graph $f(x) = 2x - 3\sqrt{x}$.
    b) Starting with the graph in part a), sketch a rough graph of the antiderivative *F* that satisfies $F(0) = 1$.
    c) Use the rules of this section to find an expression for $F(x)$.
    d) Graph *F* using the expression in part c). Compare with your sketch in part b).

What is a differential equation?

A differential equation is an equation (contains an equals sign which states the two sides are equal) but where at least one term contains a derivative.

$$\frac{dy}{dx} + 5y = e^x \qquad \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 5y = 6$$

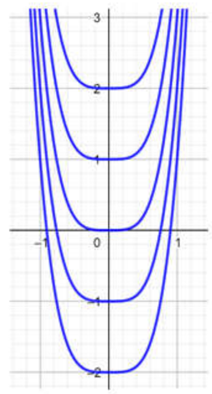
A **solution** to a differential equation is itself an equation - an equation which completely satisfies the original differential equation.

To verify if an equation is a solution to given differential equation, you first take the required derivatives, and then plug everything in:

Ex) Verify that $y = 100e^{.05t} + 50$
is a solution to the differential equation $\frac{dy}{dt} = 5e^{.05t}$

General and Particular Solutions of Differential Equations

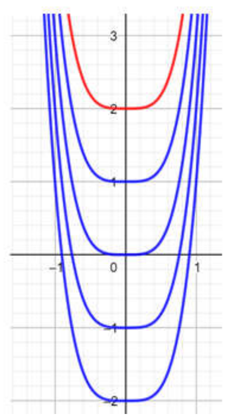
Differential Equation: $\frac{dy}{dx} = 12x^3$



General Solution: $y = 3x^4 + C$

'initial condition': $(0, 2)$

$$(2) = 3(0)^4 + C$$
$$C = 2$$



Particular Solution: $y = 3x^4 + 2$

Application: Physics

Distance (displacement) $s(t)$

$$\downarrow \frac{d}{dx} \qquad \uparrow \int dx$$

Velocity (magnitude = speed) $v(t) = s'(t)$

$$\downarrow \frac{d}{dx} \qquad \uparrow \int dx$$

Acceleration $a(t) = v'(t) = s''(t)$

In Earth's gravity, acceleration is a constant...

$$a(t) = -32 \, {}^{ft}\!/_{s^2} \qquad a(t) = -9.81 \, {}^{m}\!/_{s^2}$$

If an object is only being acted upon by gravity (called 'ballistic motion'), then we start with acceleration and use antiderivatives to find functions for velocity, then displacement.

65. A stone is dropped from the upper observation deck (the Space Deck) of the CN Tower, 450 m above the ground.
    a) Find the distance of the stone above the ground level at time t.
    b) How long does it take the stone to reach the ground?
    c) With what velocity does it strike the ground?
    d) If the stone is thrown downward with a speed of 5 m/s, how long does it take to reach the ground?

Unit 5-2: Sigma Notation, Area of a Plane Region
Larsen: 4.2 (Stewart: 5.1)

Sigma Notation

In various parts of the course, we will need to 'sum' things (add them, accumulate) and one way to compactly represent summation is with **Sigma Notation**:

$$\sum_{i=c}^{d} a_i = a_c + a_{c+1} + a_{c+2} + \dots + a_d$$

A couple of examples...

$$\sum_{k=0}^{3}(2k+3) = (2(0)+3)+(2(1)+3)+(2(2)+3)+(2(3)+3)$$
$$=(3)+(5)+(7)+(9)=24$$

$$\sum_{i=2}^{5}(i^2) = (2^2)+(3^2)+(4^2)+(5^2)$$
$$=(4)+(9)+(16)+(25)=54$$

Sigma Notation Theorems and Properties

The following can be shown to be true for summations in Sigma Notation:

$$\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i \qquad \sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

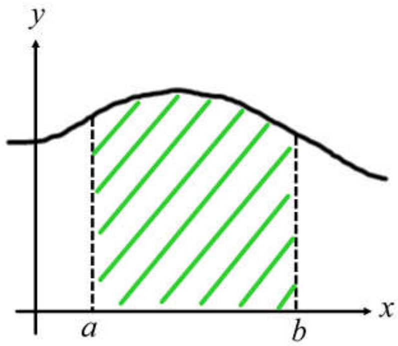
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Example

Evaluate the sum: $\sum_{i=1}^{25}(i^3 - 2i)$

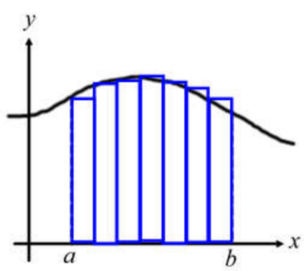
Finding Area under a function curve using a Summation

Imagine we wanted to find the approximate area under a function curve between two x-values....

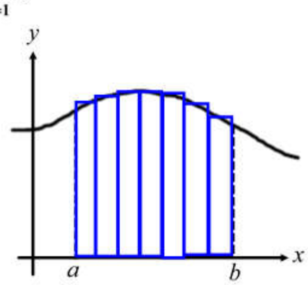


We could use a summation to add the areas of rectangles which approximately fill the area. We can use equal-width rectangles, and let the function curve's value establish the height for each rectangle...

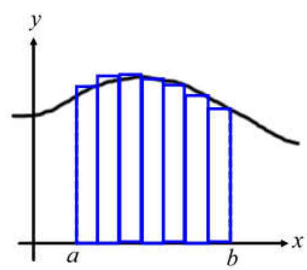
$$Area = \sum_{i=1}^{n} \left(individual\ rectangle\ areas \right)$$



...using the lower edge of the rectangle's x to establish height (called a lower sum)



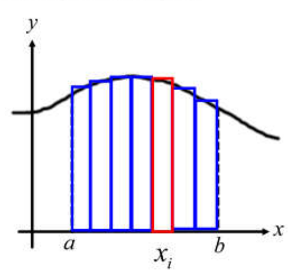
...using the middle of the rectangle's x to establish height (called 'midpoint rule')



...using the upper edge of the rectangle's x to establish height (called an upper sum)

(whether these estimates underestimate or overestimate the area depends upon the shape of the curve)

Whichever rule we use to choose the x-value, for each rectangle, there is chosen x-value which we can plug into the function to establish that rectangle's height.



If we divide the x-distance between *a* and *b* into *n* rectangle, the width of each rectangle will be: $width = \frac{b-a}{n}$

To find the *x* value for the i^{th} rectangle, we start at *a* and add *i* widths: $x_i = a + i\left(\frac{b-a}{n}\right)$

The height of this rectangle is then... $height = f\left(x_i\right)$

...and this rectangle's area is: $area = height \bullet width = f\left(x_i\right) \bullet \left(\frac{b-a}{n}\right)$

We then sum the individual rectangle areas: $Area = \sum_{i=1}^{n} f\left(x_i\right) \bullet \left(\frac{b-a}{n}\right)$

Finding Area under a function curve using a Summation

An example with a fixed number of rectangles:

Use the Midpoint Rule with n = 8 to approximate the area of the region bounded by the x-axis and the graph of the function over the given interval.

$$f(x) = x^2 + 4x, \quad [0,4]$$

Finding Area under a function curve using the Limit Definition

Of course, this approximation will be more accurate if we include more rectangles!

So we consider including an infinite number of rectangles and do this by using the previous summation structure, but then taking the limit as the number of rectangles, *n*, approaches infinity:

$$Area = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \left(\frac{b-a}{n} \right)$$

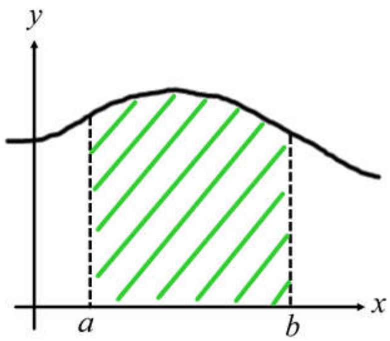
Example: Use the limit process to find the area of the region bounded by the graph of the function and the x-axis over the given interval:

$$f(x) = x^2 + 4x, \quad [0,4]$$

Unit 5-3: Riemann Sums and the Definite Integral
Larsen: 4.3 (Stewart: 5.1/5.2)

Riemann Sums

In the last section, we used summation to find the approximate area under a function curve. This made sense if the value of the function was always positive in the region we were considering:

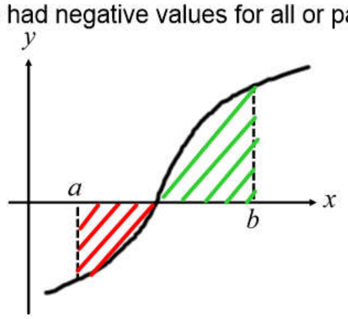


$$Area = \sum_{i=1}^{n} f(x_i) \cdot \left(\frac{b-a}{n}\right)$$

or for infinitely many rectangles:

$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \left(\frac{b-a}{n}\right)$$

But what if the function curve had negative values for all or part of the region of interest?



In that case, $f(x)$ would be negative so we would have a negative contribution to the summation from these parts of the region...

$$? = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \left(\frac{b-a}{n}\right)$$

...so this expression wouldn't necessarily represent 'area' any longer. It could, though, as long as we kept the positive and negative regions separate and negated anything that was negative before summing it into the summation.

A **Riemann Sum** is a summation defined in a more general way. First, the function curve can be either positive or negative for different regions in the interval of interest. Second, the shapes used to find the areas of the subregions no longer have to have equal widths, in fact, they don't even have to be rectangles (they can be other shapes, such as trapezoids).

Let f be defined on the closed interval $[a,b]$,
and let Δ be a partition of $[a,b]$ given by

$$a = x_0 < x_1 < x_2 .... < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval : $[x_{i-1}, x_i]$
If c_i is any point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a Riemann Sum *of f for the partition Δ.*

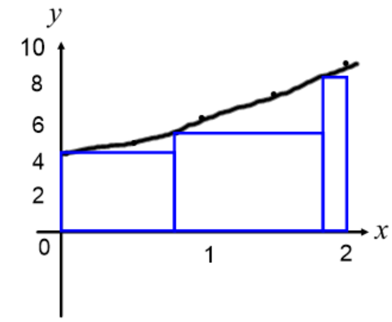
(The area summations we found in the last section are Riemann Sums, but there are other, more general forms of Riemann Sums as well.)

Riemann Sums

Example: The values of a function are shown in a table for specific x-values. Evaluate a Riemann Sum using rectangular partitions, left endpoints, and the specific subintervals given:

x	0.00	0.50	0.75	1.00	1.50	1.75	2.00
y	4.32	4.58	5.79	6.14	7.64	8.08	8.14

subintervals: [0.00, 0.75] [0.75, 1.75] [1.75, 2.00]

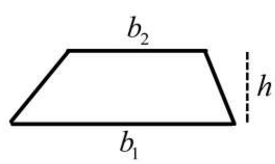


Riemann Sums using the Trapezoidal Rule

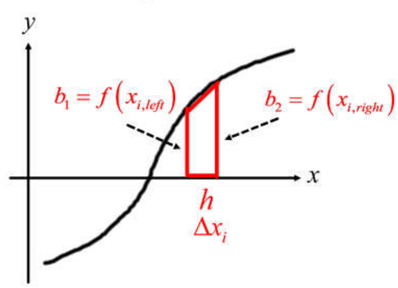
Riemann Sums allow us to use any reasonable shape to estimate the area of a subinterval, and one that is common and may appear on the AP Calculus Exam is called the **Trapezoidal Rule**.

The area of a trapezoid is given by the formula

$$area_{trapezoid} = \frac{1}{2}(b_1 + b_2)h$$

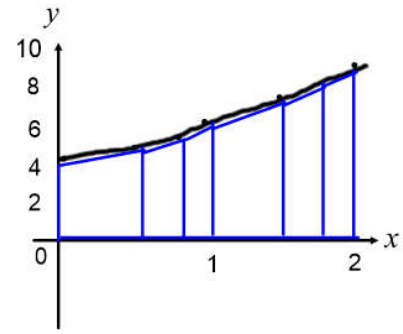


To use a trapezoid as an area element, we turn it on it side, so that the 'height' of the trapezoid is the width of the subinterval, and the 'bases' are the left and right endpoint function values:



Example: The values of a function are shown in a table for specific x-values. Evaluate a Riemann Sum using the Trapezoidal Rule for all the partitions included in the table.

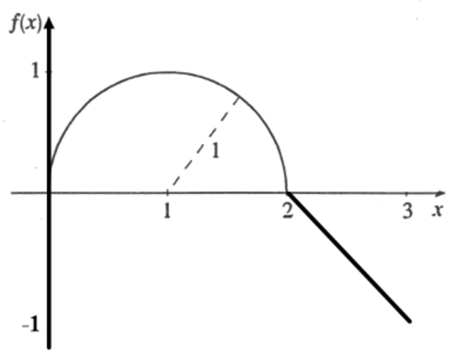
x	0.00	0.50	0.75	1.00	1.50	1.75	2.00
y	4.32	4.58	5.79	6.14	7.64	8.08	8.14



Riemann Sums using the other Geometric Shapes

You can also use other geometric shapes to estimate areas in computing Riemann Sums:

Ex) Compute a Riemann Sum for the function given by the graph over the interval [0, 3]



The Definite Integral

Riemann Sums allow any region with negative $f\left(x\right)$ to add negative contributions to the accumulation. This turns out to be extremely useful, as we will explore in later sections.

For now, we will also use a notation called the **Definite Integral** to represent a Riemann Sum with infinitely many subintervals:

$$\int_a^b f\left(x\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(c_i\right)\Delta x_i$$

Definite Integral

$$\int_a^b f\left(x\right)dx$$

...means to compute a value for the summation of the area between the function curve with negative contributions for negative *f(x)* regions.

Result = a number

Indefinite Integral

$$\int f\left(x\right)dx$$

...means to find the antiderivative of the function *f(x)*, including the integration constant.

Result = a family of functions

Evaluate the definite integral $\int_0^2 \left(x^3 - 1\right)dx$ for n = 4, using right endpoints.

Evaluate the definite integral $\int_0^2 \left(x^3 - 1\right)dx$ for n = 40, using right endpoints.

The Definite Integral

Evaluate the definite integral $\int_0^2 \left(x^3 - 1\right) dx$ using the limit process (for infinite rectangles).

Riemann Sum to Definite Integral

Let's write out the Riemann Sum used to evaluate the integral $\int_1^3 x^3\,dx$ using the limit process:

width $\Delta x = \frac{UB - LB}{n} = \frac{3-1}{n} = \frac{2}{n}$

$x_i = LB + i\Delta x = 1 + i\frac{2}{n}$

$f(x) = x^3, \quad f(x_i) = \left(1 + i\frac{2}{n}\right)^3$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+i\frac{2}{n}\right)^3\frac{2}{n}$$

$$\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(1+i\frac{2}{n}\right)^3$$

$$\lim_{n\to\infty}\frac{2}{n}\left[\left(1+1\frac{2}{n}\right)^3+\left(1+2\frac{2}{n}\right)^3+\left(1+3\frac{2}{n}\right)^3+\ldots.+\left(1+n\frac{2}{n}\right)^3\right]$$

$$\lim_{n\to\infty}\frac{2}{n}\left[\left(1+\frac{2}{n}\right)^3+\left(1+\frac{4}{n}\right)^3+\left(1+\frac{6}{n}\right)^3+\ldots.+\left(1+\frac{2n}{n}\right)^3\right]$$

Sometimes the AP Exam will give you a Riemann Sum written out like this, and ask you to figure out what definite integral it represents.

$$\int_1^3 x^3\,dx$$

width $\Delta x = \frac{UB - LB}{n} = \frac{3-1}{n} = \frac{2}{n}$

$x_i = LB + i\Delta x = 1 + i\frac{2}{n}$

$f(x) = x^3, \quad f(x_i) = \left(1 + i\frac{2}{n}\right)^3$

Here's what you need to find...

$$\lim_{n\to\infty}\frac{2}{n}\left[\left(1+\frac{2}{n}\right)^3+\left(1+\frac{4}{n}\right)^3+\left(1+\frac{6}{n}\right)^3+\ldots.+\left(1+\frac{2n}{n}\right)^3\right]$$

This is the lower limit of integration (LB)

This is the width $\Delta x = \frac{UB - LB}{n}$

The parentheses represents the integrand function...here, it is cubed, so

$f(x) = x^3$

Let's try one...find the definite integral which the following Riemann Sum evaluates:

$$\lim_{n\to\infty}\frac{1}{n}\left[\left(3+\frac{1}{n}\right)^2+\left(3+\frac{2}{n}\right)^2+\left(3+\frac{3}{n}\right)^2+\ldots.+\left(3+\frac{n}{n}\right)^2\right]$$

Properties of the Definite Integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
where c is between a and b

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$\int_a^b [f(x)dx \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\textit{If } f(x) \le g(x) \textit{ on } [a,b], \textit{ then } \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

43. Write the given sum or difference as a single integral in the form $\int_a^b f(x)dx$.

$$\int_1^3 f(x)dx + \int_3^6 f(x)dx + \int_6^{12} f(x)dx$$

17. Express the limit as a definite integral on the given interval.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[2\left(x_i^*\right)^2 - 5x_i^*\right]\Delta x, \quad [0,1]$$

21. Use the form of the definition of the integral given in Equation

3: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

to evaluate the integral: $\int_0^2 \left(2 - x^2\right) dx$

Unit 5-4 day 1: The Fundamental Theorem of Calculus and examples
Larsen: 4.4 (Stewart: 5.3/5.4)

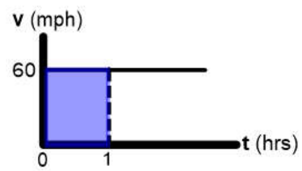
The Fundamental Theorem of Calculus

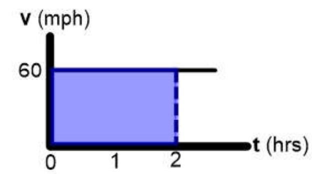
Start with two seemingly different ideas (that use the same notation)...

$$\int f(x)\,dx \qquad\qquad \int_a^b f(x)\,dx$$

Antiderivative of $f(x)$, $F(x)$ — Area under $f(x)$ curve from x=a to x=b

Consider a car traveling at a constant 60 mph:





$$\begin{aligned} \textit{area of rect} &= v(t)\cdot\Delta t \\ &= 60\,\frac{miles}{hr}\cdot 1\ hr \\ &= 60\ miles \\ &= \textit{distance traveled} \\ &\quad \textit{from } 0-1\ hrs \end{aligned}$$

$$\begin{aligned} \textit{area of rect} &= v(t)\cdot\Delta t \\ &= 60\,\frac{miles}{hr}\cdot 2\ hr \\ &= 120\ miles \\ &= \textit{distance traveled} \\ &\quad \textit{from } 0-2\ hrs \end{aligned}$$

Area under the velocity curve = the total (accumulated) distance traveled

But we also know that the velocity function is the derivative of the distance (displacement) function... $v(t)=s'(t)$

...and therefore the distance function is the antiderivative of the velocity function.

$$s(t)=\int v(t)\,dt$$

$$s(t)=V(t)$$ ← antiderivative of the velocity function

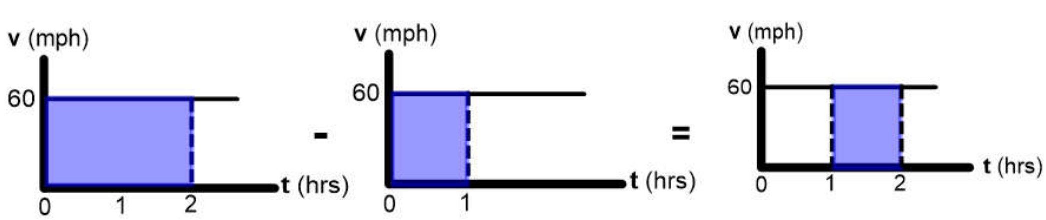
the total distance traveled = antiderivative of velocity

Since the total distance traveled = area under the velocity curve
and the total distance traveled = antiderivative of velocity

Area under the velocity curve = Antiderivative of velocity

$$\int_0^t v(t)\,dt = V(t)$$

And if we wanted to find the distance traveled between time t=1 and t=2
we could subtract one area from the other:

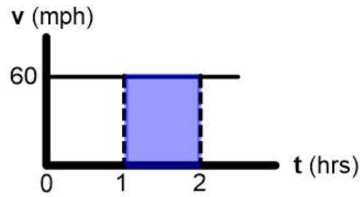


$$\int_0^2 v(t)\,dt = V(2) \qquad \int_0^1 v(t)\,dt = V(1) \qquad \int_1^2 v(t)\,dt = V(2)-V(1)$$

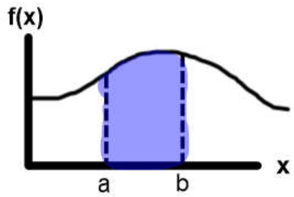
The accumulated distance depends upon finding the values of the antiderivative of velocity (distance) but only at the ends of the interval.

The Fundamental Theorem of Calculus

This idea occurred to two people: Issac Newton and Gottfried Leibniz who further showed that this idea, that the area under a function curve over an x-interval is equal to the antiderivative evaluated at the endpoints, is generalizable to all functions, not just constant functions...



$$\int_1^2 v(t)\,dt = V(2) - V(1)$$



$$\int_a^b f(x)\,dx = F(b) - F(a)$$

...and is called the ***Fundamental Theorem of Calculus (part 2)***.

A quick example

Evaluate: $\int_2^4 \left(x^2 + 2x\right) dx$

Proof of The Fundamental Theorem of Calculus (Part 2)

If a function f is continuous on the closed interval [a,b] and F is an antiderivative of f on the interval [a,b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof:

Let Δ be any partition of [a,b]: $a = x_0 < x_1 < x_2 \ldots. < x_{n-1} < x_n = b$

The right hand side can be replaced with the endpoint values, along with the values between these endpoints in the interval, each one added and subtracted back off...

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + F(x_{n-2}) \ldots - F(x_1) + F(x_1) - F(x_0)$$

Regrouping into pairs in order...

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \cdots + F(x_1) - F(x_0)$$

This can now be expressed as a summation...

$$F(b) - F(a) = \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1}) \right]$$

By the Mean Value Theorem, we know that there exists a number c_i in the i^{th} subinterval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

therefore $F(x_i) - F(x_{i-1}) = F'(c_i) \bullet (x_i - x_{i-1})$

Because F is the antiderivative of f, $F'(c_i) = f(c_i)$ and you define $\Delta x_i = x_i - x_{i-1}$, then...

$$F(x_i) - F(x_{i-1}) = f(c_i)\,\Delta x_i$$

From earlier: $F(b) - F(a) = \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1}) \right]$

$$F(b) - F(a) = \sum_{i=1}^{n} \left[f(c_i)\,\Delta x_i \right]$$

This result tells us that by repeatedly applying the Mean Value Theorem, you can always find a collection of c_i values such that the *constant F(b) - F(a)* is a Riemann Sum of *f* on [a,b] for any partition.

Now taking a limit to apply infinitely many subintervals:

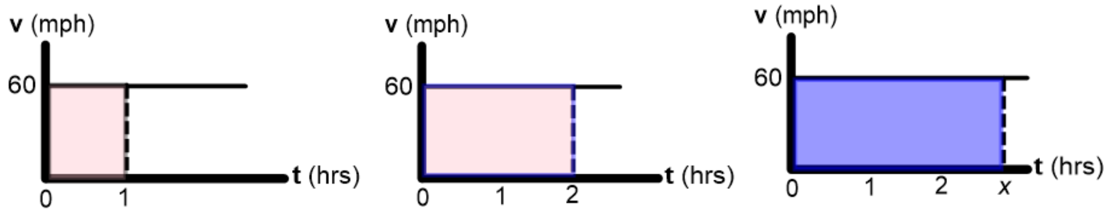
$$F(b) - F(a) = \lim_{n \to \infty} \sum_{i=1}^{n} \left[f(c_i)\,\Delta x_i \right]$$

$$\boxed{F(b) - F(a) = \int_a^b f(x)\,dx}$$

The Fundamental Theorem of Calculus (part 1)

So if the previous theorem is known as The Fundamental Theorem of Calculus Part 2, which is Part 1? To see, let's return to our driving analogy...

We said if we had a function representing velocity (derivative of distance) the area under the curve represents the accumulated distance traveled from time t = 0:



If we leave the ending value as a variable, *x*, then we are producing a <u>function</u> of time (with variable *x* allowing us to set the stop time later) and this function gives us the accumulated distance from zero to that time, *x*.

$$\int_0^1 v(x)\,dx = V(1)-V(0) = s(1)-s(0)$$

a number representing distance travelled

$$\int_0^2 v(x)\,dx = V(2)-V(0) = s(2)-s(0)$$

a number representing distance travelled

$$\int_0^x v(t)\,dt = V(x)-V(0) = s(x)-s(0)$$

a <u>function</u> which gives us distance travelled for any x

The resulting function is sometimes called an **<u>accumulation function</u>**.

Although we've shown this for a constant velocity function, this can be shown to be true for any function. Conceptually, this means that if you have a function which represents the <u>derivative</u> of a quantity (how it is changing, for example, over time), then integrating that function from a constant to *x* will give a function for how that quantity <u>accumulates</u>.

For example, if you have a water tank which is leaking water and rate at which water is leaking is changing over time: $R(t)$

...then you could create a function for the total accumulated amount of water leaked out by time *x* using:

$$accumulated\ leaked\ water(x) = \int_a^x R(t)\,dt$$

One final point about the FTC part 1: because the function we are integrating must represent the derivative of the quantity we are accumulating, and doing so produces a function of the original quantity, this means that the integral and the derivative are operating as 'inverse operations' - undoing each other's effects:

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

...and this is the way that the Fundamental Theorem of Calculus Part 1 is often written.

The Fundamental Theorem of Calculus Summary

The Fundamental Theorem of Calculus Part 1

If a function f is continuous on an open interval I containing *a*, then, for every *x* in the interval

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

how it is used...

$$\int_a^x Q'(t)\,dt = q(x) - q(a)$$

Given a function for the derivative of a quantity, produce a function for the accumulation of that quantity (result is a function).

The Fundamental Theorem of Calculus Part 2

If a function f is continuous on the closed interval [a,b] and F is an antiderivative of f on the interval [a,b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Given a definite integral of a function, use the antiderivative of that function at the endpoints to evaluate the integral, instead of Riemann Sums (results is a number).

The Net Change Theorem

If $F'(x)$ is the rate of change of a quantity $F(x)$, then the definite integral of $F'(x)$ from *a* to *b* gives the total change, or **net change**, of $F(x)$ on the interval [*a*,*b*]:

$$\int_a^b F'(x)\,dx = F(b) - F(a) \quad \text{'net change of } F\text{'}$$

Examples

$$\int_{-1}^{3} x^5\,dx$$

$$\int_{-1}^{1} \frac{3}{t^4}\,dt$$

Examples

Evaluate the integral and interpret it as a difference of areas.
Illustrate with a sketch.

$$\int_{\pi/4}^{5\pi/2} \sin x \, dx$$

Evaluate...

$$g(x) = \int_{2}^{x} t^2 \, dt$$

$$g(x) = \int_{3}^{x^2} 3t^3 \, dt$$

Examples

Find the derivative of...

$$F(x) = \int_{\pi/2}^{x^3} \cos t \, dt \qquad F(x) = \int_{0}^{x^2} 5 \ln t \, dt$$

$$\frac{d}{dx}\int_{-\pi}^{x} \cos t \, dt \qquad \frac{d}{dx}\int_{0}^{x} \frac{1}{1+t^2} \, dt \qquad \frac{d}{dx}\int_{x}^{0} t^3 \, dt$$

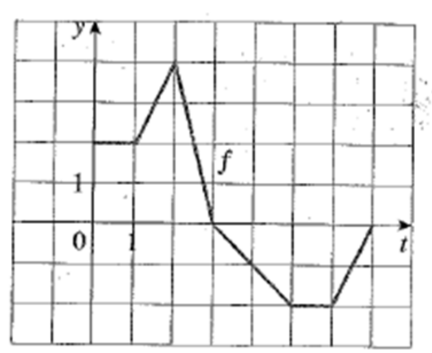
Let $g(x) = \int_0^x f(t)\,dt$ where f is the function whose graph is shown.

a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.

b) On what interval is g increasing?

c) Where does g have a maximum value?

d) Sketch a rough graph of g.



Examples

Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

$$g(x) = \int_0^x \sqrt{1+2t}\,dt \qquad\qquad y = \int_{e^x}^0 \sin^3 t\,dt$$

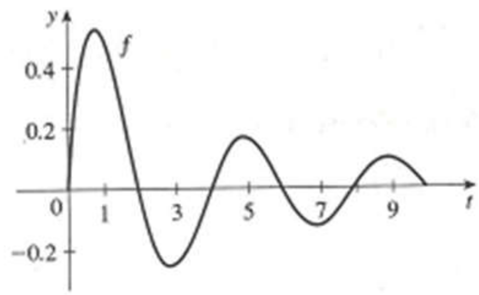
Find the interval on which the curve is concave upward.

$$y = \int_0^x \frac{1}{1+t+t^2}\,dt$$

Examples

Let $g(x)=\int_0^x f(t)\,dt$, where f is the function whose graph is shown.

a) At what values of x do the local maximum and minimum values of g occur?



b) Where does g attain its absolute maximum value?

c) On what intervals is g concave downward?

d) Sketch the graph of g.

$$\text{Let}\quad f(x)=\begin{cases}0 & \text{if } x<0\\ x & \text{if } 0\le x\le 1\\ 2-x & \text{if } 1<x\le 2\\ 0 & x>2\end{cases}$$

$$\text{and}\qquad g(x)=\int_0^x f(t)\,dt$$

a) Find an expression for $g(x)$ similar to the one for $f(x)$.

b) Sketch the graphs of f and g.

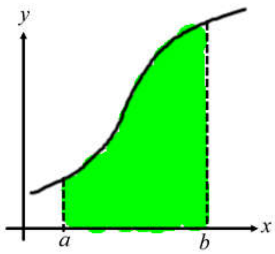
c) Where is f differentiable? Where is g differentiable?

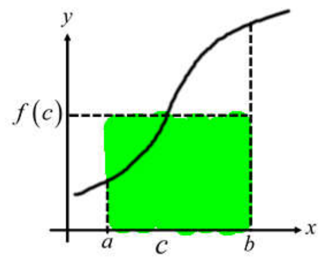
Unit 5-4 day 2: Net Change Theorem, Average Value of a Function
Larsen: 4.4 (Stewart: 5.3/5.4)

The Mean Value Theorem for Integrals

If f is continuous on the closed interval [*a*,*b*], then there exists a number c in the closed interval [*a*,*b*] such that

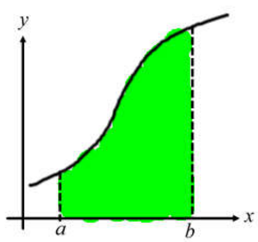
$$\int_a^b f(x)\,dx = f(c)(b-a)$$

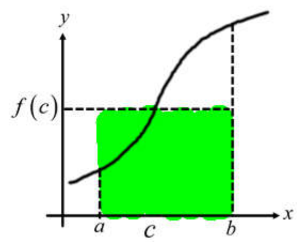




For any definite integral over an interval, there is some x-value in the interval where a rectangle of height f(c) times interval width has the same area as the actual area under the curve in the interval.

Average Value of a Function





From the previous theorem, at this value *c*, the height of the rectangle with the same area as the area under the function curve is *f(c)*. We define *f(c)* as the **average value of the function over this interval**, and can compute it as follows:

If f is integrable on the closed interval [*a*,*b*], then the **average value** of f on the interval is

$$average\ value = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Examples

A liquid flows into a storage tank at a rate of $(180 + 3t)$ liters per minute. Find the amount of liquid that flows into the tank during the first 20 minutes.

Examples

Find the average value of $f(x) = x^3 - 3x^2$ on the interval [1,4]

The velocity function, in feet per second, is $v(t) = t^2 - t - 12 \quad 1 \le t \le 5$
for a particle moving along a straight line.

(a) Find the displacement over the interval.

(b) Find the total distance that the particle travels over the given interval.

Unit 5-5: Integration by Substitution
Larsen: 4.5 (Stewart: 5.5)

Integration by Substitution

Some integrals cannot be evaluated by using the basic integration formulas, so we need other integration techniques. One of these is **integration by substitution** which is based on the Chain Rule.

Derivative using Chain Rule

$$y = \left(x^2 + 5\right)^4$$

useful when one function is 'inside' of another

$u = x^2 + 5 \qquad y = u^4$

$\dfrac{du}{dx} = 2x \qquad \dfrac{dy}{du} = 4u^3$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = 4u^3(2x)$$

$$\frac{dy}{dx} = 4\left(x^2+5\right)^3(2x)$$

Integration by Substitution

$$\int 4\left(x^2+5\right)^3 2x\, dx$$

$u = x^2 + 5$

$\dfrac{du}{dx} = 2x$

$du = 2x\, dx$

$4\int u^3 du$

$4\left[\dfrac{1}{4}u^4\right] + C$

$u^4 + C$

$\left(x^2+5\right)^4 + C$

1) define the 'inside' function to be u.

2) Find du/dx and solve for du to get a 'toolkit' with du and u.

3) Substitution all expressions with x and dx in the original integral to obtain a new integral using u as the variable.

4) Integrate, then resubstitute u to use the original x variable.

One of the most common u-substitutions involves quantities in the integrand that are raised to a power (as in this example). This is given the special name 'General Power Rule for Integration'. More generally, this procedures is knows as 'Integration by Substitution', 'u-substitution', or 'the Substitution Rule for Integration'.

Examples

$$\int x^3\left(1-x^4\right)^5 dx$$

$$\int \frac{x^2}{\sqrt{1-x}}\, dx$$

Using Integration by Substitution to evaluate Definite Integrals

You can also use Integration by Substitution when evaluating definite integrals. There are two variations on how to handle the limits of integration (both a valid):

$$\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

$$\int_0^{\pi/3} \frac{\sin\theta}{(\cos\theta)^2}\,d\theta$$

$$\int_0^{\pi/3} u^{-2}\sin\theta\,d\theta$$

$$-\int_{\theta=0}^{\theta=\pi/3} u^{-2}\,du$$

$$-\left[(-1)u^{-1}\right]_{\theta=0}^{\theta=\pi/3}$$

$$\left[\frac{1}{u}\right]_{\theta=0}^{\theta=\pi/3}$$

$$\left[\frac{1}{\cos\theta}\right]_{\theta=0}^{\theta=\pi/3}$$

$$\left[\frac{1}{\cos\left(\frac{\pi}{3}\right)}\right]-\left[\frac{1}{\cos(0)}\right]$$

$$\left[\frac{1}{\left(\frac{1}{2}\right)}\right]-\left[\frac{1}{1}\right]$$

$$2-1=1$$

$$u=\cos\theta$$

$$\frac{du}{d\theta}=-\sin\theta$$

$$du=-\sin\theta\,d\theta$$

$$\sin\theta\,d\theta=-du$$

You can convert the limits of integration into u values as well, and stay with u for the rest of the problem...

...or you can ignore the limits of integration at first, complete the integration and resubstitute into terms of x, then substitute in the original x limits of integration.

$$\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

$$\int_0^{\pi/3} \frac{\sin\theta}{(\cos\theta)^2}\,d\theta$$

$$\int_0^{\pi/3} u^{-2}\sin\theta\,d\theta$$

$$\theta=0 \rightarrow u=\cos(0)=1$$

$$\theta=\pi/3 \rightarrow u=\cos(\pi/3)=1/2$$

$$-\int_1^{1/2} u^{-2}\,du$$

$$-\left[(-1)u^{-1}\right]_1^{1/2}$$

$$\left[\frac{1}{u}\right]_1^{1/2}$$

$$\left[\frac{1}{\left(\frac{1}{2}\right)}\right]-\left[\frac{1}{1}\right]$$

$$2-1=1$$

Examples

$$\int \frac{\sin x}{1+\cos^2 x}\,dx$$

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$$

Evaluate $\int_{-2}^{2} (x+3)\sqrt{4-x^2}\,dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

$$\int \cos x \cos(\sin x)\, dx$$

$$\int \sin^3 x \cos x \, dx$$

$$\int_1^4 \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\, dx$$

$$\int \sec x \tan x \sqrt{1+\sec x}\, dx$$

Unit 5-6: Other Integration Techniques
Larsen: 4.7

Other Integration Techniques

Some integrals don't resolve using basic antiderivative shortcuts or integration by substitution immediately, but you can do some additional work to put them into a form where these techniques will work. Here, we examine a few such techniques.

Inverse Trig forms with constants

We've memorized this form: $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

But what if we need to integrate this: $\int \frac{1}{4+x^2}\,dx$

It can be shown that the following are true...

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$$

(We don't usually use the cosine, cotangent, or cosecant forms because they are just the negatives of these).

We can update our table of antiderivative shortcuts to memorize...

Updated antiderivative list

$\frac{d}{dx}[C]=0$	$\int 0\,dx=C$
$\frac{d}{dx}[kx]=k$	$\int k\,dx=kx+C$
$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$	$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C \quad (n\neq -1)$
$\frac{d}{dx}\left[e^x\right]=e^x$	$\int e^x\,dx=e^x+C$
$\frac{d}{dx}\left[a^x\right]=(\ln a)\,a^x$	$\int a^x\,dx=\left(\frac{1}{\ln a}\right)a^x+C$
$\frac{d}{dx}[\ln x]=\frac{1}{x} \quad (x>0)$	$\int \frac{1}{x}\,dx=\ln\lvert x\rvert+C$
$\frac{d}{dx}[\sin x]=\cos x$	$\int \cos x\,dx=\sin x+C$
$\frac{d}{dx}[\cos x]=-\sin x$	$\int \sin x\,dx=-\cos x+C$
$\frac{d}{dx}[\tan x]=\sec^2 x$	$\int \sec^2 x\,dx=\tan x+C$
$\frac{d}{dx}[\sec x]=\sec x\tan x$	$\int \sec x\tan x\,dx=\sec x+C$
$\frac{d}{dx}[\csc x]=-\csc x\tan x$	$\int \csc x\cot x\,dx=-\csc x+C$
$\frac{d}{dx}[\cot x]=-\csc^2 x$	$\int \csc^2 x\,dx=-\cot x+C$

$$\frac{d}{dx}[\arcsin x]=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arccos x]=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan x]=\frac{1}{1+x^2}$$

$$\frac{d}{dx}[arc\sec x]=\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}[arc\csc x]=\frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}[arc\cot x]=\frac{-1}{1+x^2}$$

$$\int \frac{du}{\sqrt{a^2-u^2}}=\arcsin\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{a^2+u^2}=\frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}\operatorname{arcsec}\left(\frac{\lvert u\rvert}{a}\right)+C$$

Splitting into multiple integrals

Sometimes, u-sub won't work as initially stated, but we can split the integral into multiple integrals:

$$\int \frac{x+2}{\sqrt{4-x^2}}\,dx$$

Completing the Square

An old algebra technique is also sometimes useful...completing the square.

If you have a quadratic which is not factorable, like... x^2-4x+7

...you can perform a 'complete the square' procedure to make a portion a binomial squared:

This can sometimes help with integration: $\int \frac{1}{x^2-4x+7}\,dx$

Trig function with an argument other than 'x'

Sometimes using simple trig identities can help with integration...

$\int \tan(2x)\,dx$

$\int x^3 \tan(x^4)\,dx$

One last example from the textbook: $\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$