AP Calc BC – Lesson Notes – Unit 4: Applications of Derivatives

Unit 4-1: Extrema on an Interval
Larsen: 3.1 (Stewart: 4.1)

Terminology of Extrema

Maximum = The highest numerical value.

Minimum = The lowest numerical value.

Extrema = (plural) The set of all maximum and minimum values.

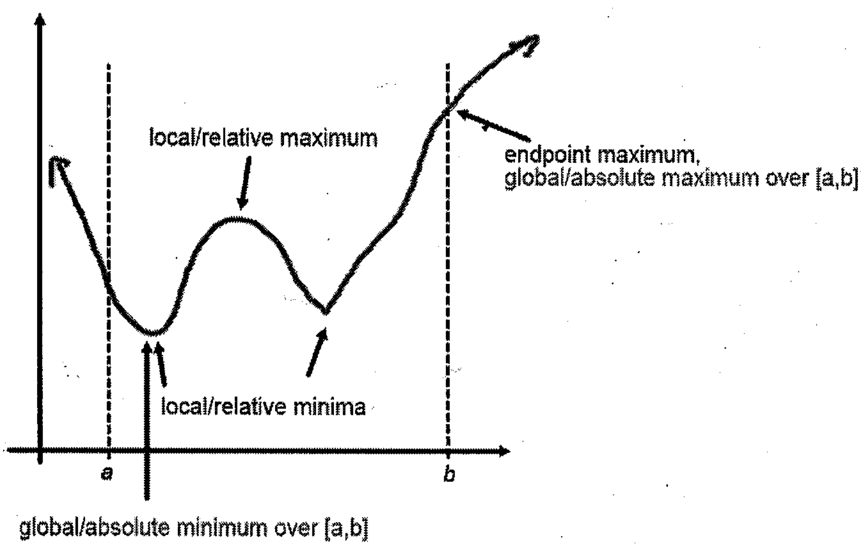
Local/Relative Maximum = The highest numerical value in a local region where a curve changes from increasing to decreasing.

Local/Relative Minimum = The lowest numerical value in a local region where a curve changes from decreasing to increasing.

Endpoint Maximum/Minimum = The highest (or lowest) numerical value over a defined interval if it occurs at a point which exists at the upper or lower end of the interval.

Absolute/Global Maximum/Minimum over an Interval = The highest (or lowest) numerical value over a defined interval regardless of whether it occurs within or at the endpoints of the interval.

Critical Number = An x value where $f'(x)$ is either zero or undefined.



Local/Relative Extrema only occur at critical numbers:

- When first derivative = 0.
- When first derivative is undefined.
- Make a 'slope map'/'wiggle graph' to to verify changing direction.

Absolute/Global Extrema may occur at Local/Relative Extrema or at endpoints:

- Find f(x) at both endpoints.
- Compare endpoint values with local extrema f(x) values to find absolute/global extrema.

Ex) Find the absolute maximum & absolute minimum for:

$$f(x)=\frac{3}{5}x^5-\frac{2}{3}x^3-x+2 \quad \textit{on the interval } [-2,1]$$

$f'(x)=3x^4-2x^2-1$

critical numbers where $f'(x)=0$ or DNE

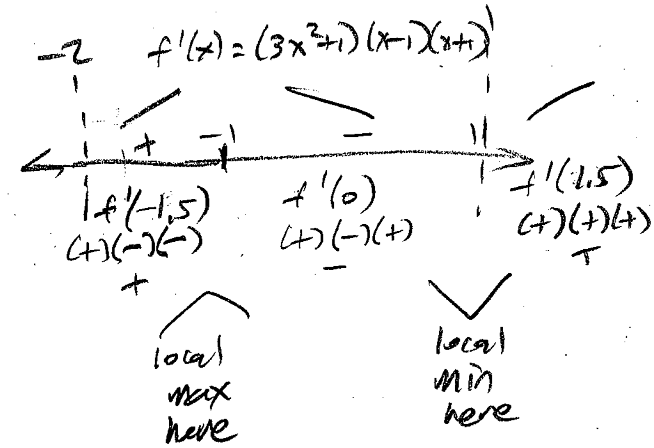
$3x^4-2x^2-1=0$

$\left(\frac{3x^2+1}{1}\right)\left(\frac{3x^2-3}{3}\right)$

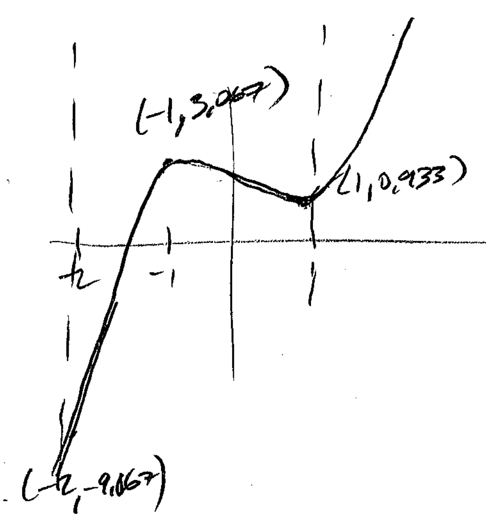
$(3x^2+1)(x^2-1)=0$

$(3x^2+1)(x-1)(x+1)=0$

no zero $\quad x=1 \quad x=-1$



x	$f(x)=\frac{3}{5}x^5-\frac{2}{3}x^3-x+2$
-2	$\frac{3}{5}(-2)^5-\frac{2}{3}(-2)^3-(-2)+2=-\frac{148}{15}\approx-9.867$
-1	$\frac{3}{5}(-1)^5-\frac{2}{3}(-1)^3-(-1)+2=\frac{46}{15}\approx 3.067$
1	$\frac{3}{5}(1)^5-\frac{2}{3}(1)^3-(1)+2=\frac{14}{15}\approx 0.933$



$\left(-2,-\frac{148}{15}\right)$ is left endpoint, absolute minimum

$\left(-1,\frac{46}{15}\right)$ is relative maximum, absolute maximum

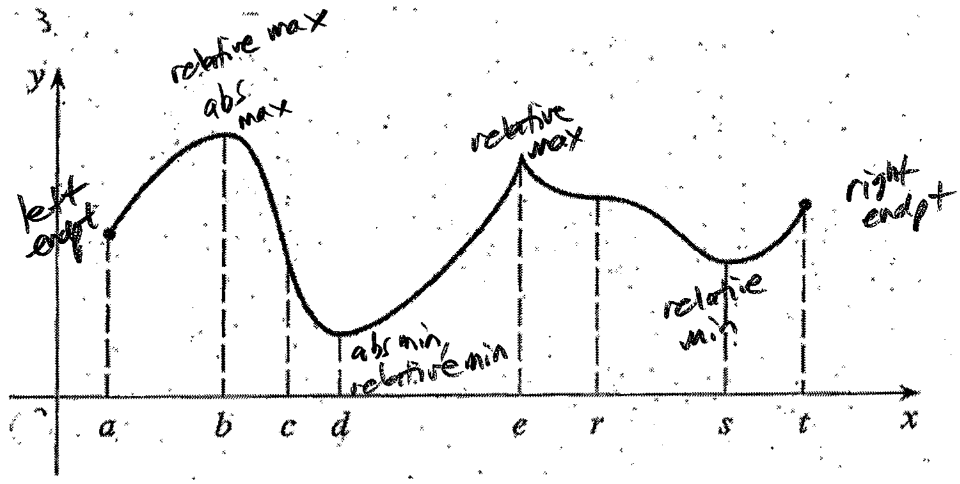
$\left(1,\frac{14}{15}\right)$ is right endpoint, relative minimum

Remember...

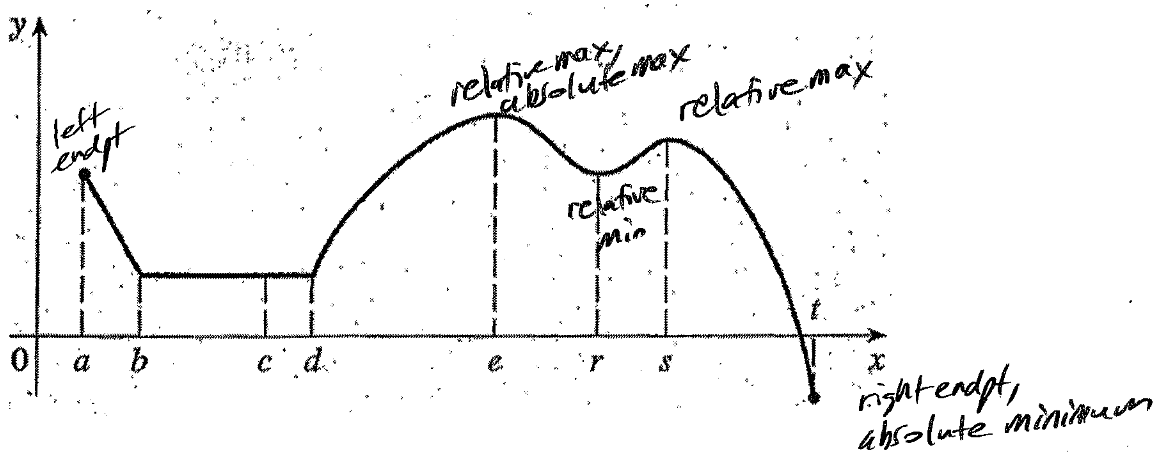
Three places to check for a max or a min:

- **derivative = 0**
- ***f* ' does not exist**
- **endpoints**

Ex) Find all local and absolute extrema over the interval [a, t]

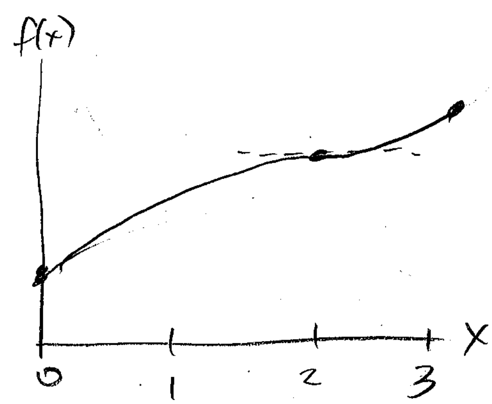


Ex) Find all local and absolute extrema over the interval [a, t]



Ex) Sketch the graph of a function f that is continuous on [0,3] and has the given properties:

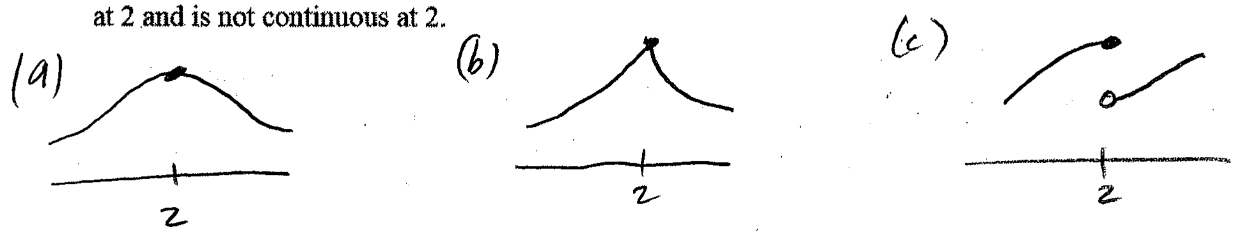
2 is a critical number, but f has no local maximum or minimum



Ex) a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.

b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.

c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.



Ex) Find the absolute and local max and min values of f. Begin by sketching its graph by hand.

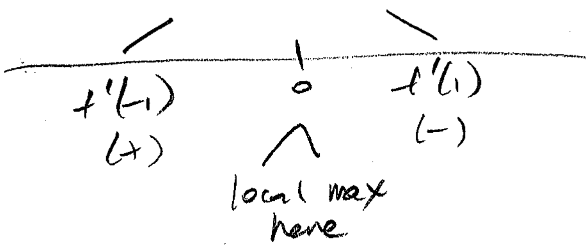
$((\sqrt{2}-(x)^2)$

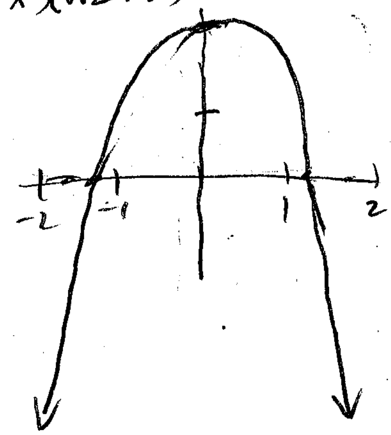
$f(x)=2-x^4 = (\sqrt{2})^2-(x^2)^2 = (\sqrt{2}+x^2)(\sqrt{2}-x^2) = \underset{\text{no zeros}}{(\sqrt{2}+x^2)}(\sqrt{\sqrt{2}}-x)(\sqrt{\sqrt{2}}+x)$

$f(x)$ zeros at $x=\sqrt{\sqrt{2}} \approx 1.1892$

$x=-\sqrt{\sqrt{2}} \approx -1.1892$

$f'(x)=-4x^3=0$ when $x=0$ (only critical value)





x	$f(x)=2-x^4$
0	2

$\boxed{(0,2) \text{ is relative (local) maximum and an absolute maximum}}$

(there are no relative or absolute minima)

Ex) Find the critical numbers of the function:

$$g(t)=\sqrt{t}(1-t) = t^{1/2}(1-t) = t^{1/2} - t^{3/2}$$

$$g'(t) = \tfrac{1}{2}t^{-1/2} - \tfrac{3}{2}t^{1/2} = \frac{1}{2\sqrt{t}} - \frac{3}{2}\sqrt{t} = \frac{1}{2\sqrt{t}} - \frac{3t}{2\sqrt{t}} = \frac{(1-3t)}{2\sqrt{t}}$$

critical numbers at $\boxed{t=0}$

$1-3t=0$

$3t=1$

$\boxed{t=1/3}$

Ex) Find the absolute max and absolute min values of

$f(x)=\dfrac{\ln x}{x}$, on the interval $[1, 3]$

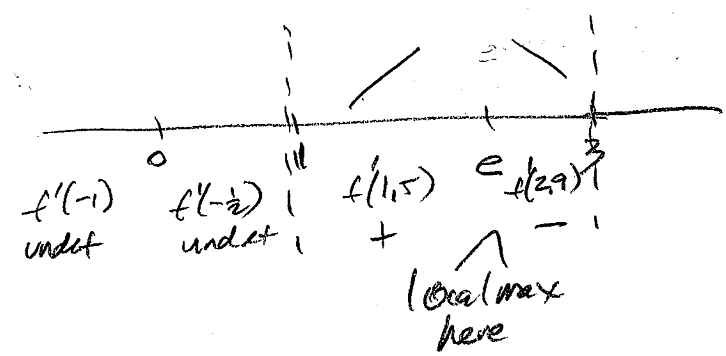
$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x\,(1)}{x^2} = \frac{1-\ln x}{x^2}$$

critical values when $\boxed{\cancel{x=0}}$

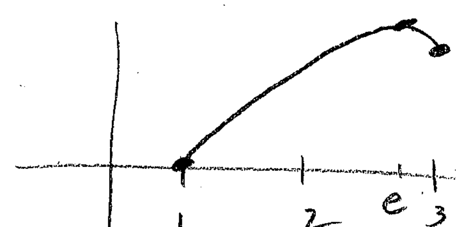
$1-\ln x = 0 \rightarrow \ln x = 1$

$e^{\ln x} = e^1$

$\boxed{x=e}$



x	$f(x)=\frac{\ln x}{x}$
1	$\frac{\ln 1}{1} = \frac{0}{1} = 0$
e	$\frac{\ln(e)}{e} = \frac{1}{e} \approx .3678$
3	$\frac{\ln(3)}{3} \approx 0.3662$



$(1,0)$ is left endpt & absolute minimum

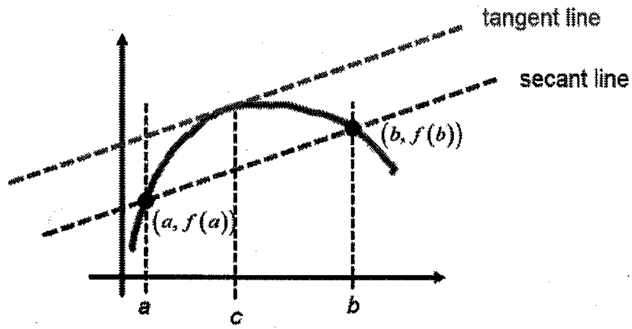
$(e,\frac{1}{e})$ is relative & absolute maximum

$(3,\frac{\ln 3}{3})$ is right endpoint

Mean Value Theorem

> Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a number c in (a,b) such that: $$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

The Mean Value Theorem say that is a curve is continuous and differentiable then if you make a secant line with the points at the ends on an interval there must exist an input value c in the interval where the derivative (slope of the tangent line to the curve at c) is equal to the slope of the secant line. At this point, the tangent line will be parallel to the secant line.



In other words, you can find a mean (average) rate of change across and interval, and there is some input value where the instantaneous rate of change equals the mean rate of change.

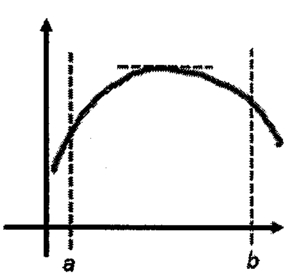
If, on a 2-hour car trip, you average 50 miles per hour, then (according to the Mean Value Theorem) at least once during the trip, your speedometer actually read 50 mph.

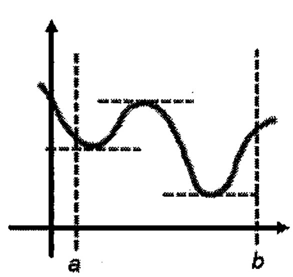
Rolle's Theorem

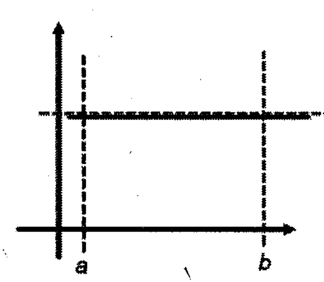
Rolle's Theorem is a specific case of the Mean Value Theorem (with horizontal secant line)...

> Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a)=f(b)$, then there is at least one number c in (a,b) such that $f'(c)=0$.

(as long as a function curve is continuous and differentiable over an interval, there must be at least one place where the first derivative is zero = horizontal tangent)

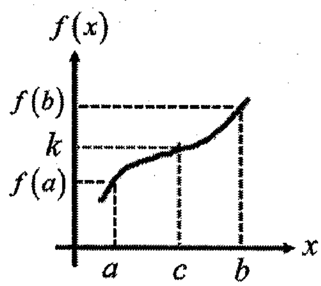


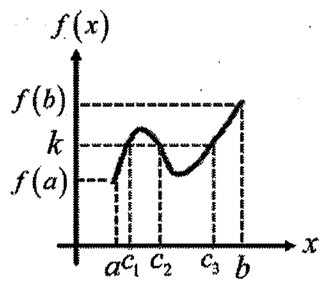




Reminder: Intermediate Value Theorem

> *If f is continuous on the closed interval $[a,b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c) = k$.*

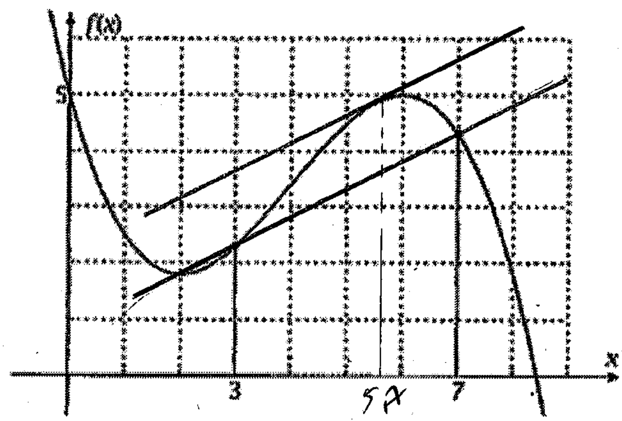




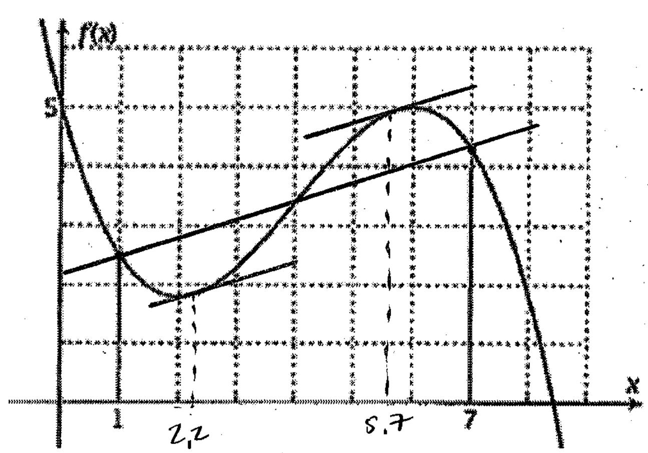
If a function is continuous, to get from one value to another you must go through all the values in-between, and for a value in an x-interval, there must be a value c in the x-interval for which f(c) is between the endpoint y-values.

Examples

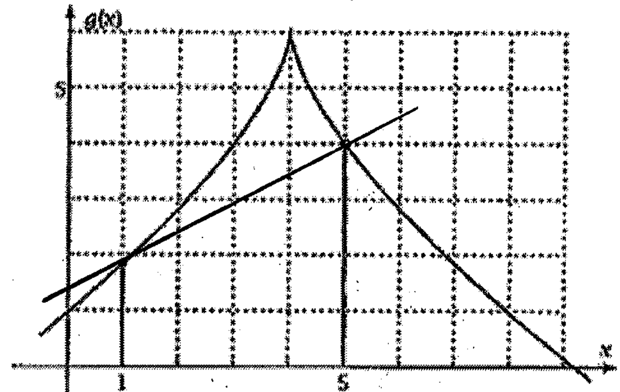
1. For $f(x) = -0.1x^3 + 1.2x^2 - 3.6x + 5$, graphed below, there is a value of $x = c$ between 3 and 7 at which the tangent to the graph is parallel to the secant line through (3, $f(3)$) and (7, $f(7)$).
   - Draw the secant line and the tangent line.
   - From the graph, $c \approx$ 5,7
   - Is f differentiable on (3, 7)? yes
   - Is f continuous on [3, 7]? yes



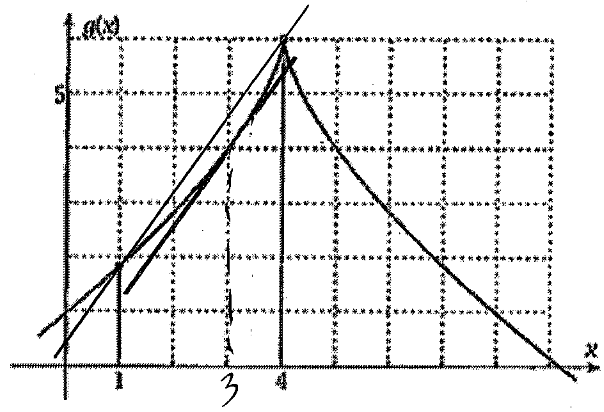
2. Function f from Problem 1 has *two* values of $x = c$ between $x = 1$ and $x = 7$ at which $f'(c)$ equals the slope of the corresponding secant line. (That is, the tangent line parallels the secant line.)
   - Draw the secant and tangents on the graph below.
   - From the graph, $c \approx$ 2,2 and $c \approx$ 5,7
   - Is f differentiable on (1, 7)? yes
   - Is f continuous on [1, 7]? yes



3. For $g(x) = 6 - 2(x - 4)^{2/3}$, graphed below,
   - Draw a secant line through $(1, g(1))$ and $(5, g(5))$.
   - Is g differentiable on $(1, 5)$? No
   - Is g continuous on $[1, 5]$? yes
   - Tell why there is *no* value of $x = c$ between $x = 1$ and $x = 5$ at which $g'(c)$ equals the slope of the secant line. does not satisfy conditions for the Mean Value Theorem (not differentiable at $x = 4$)



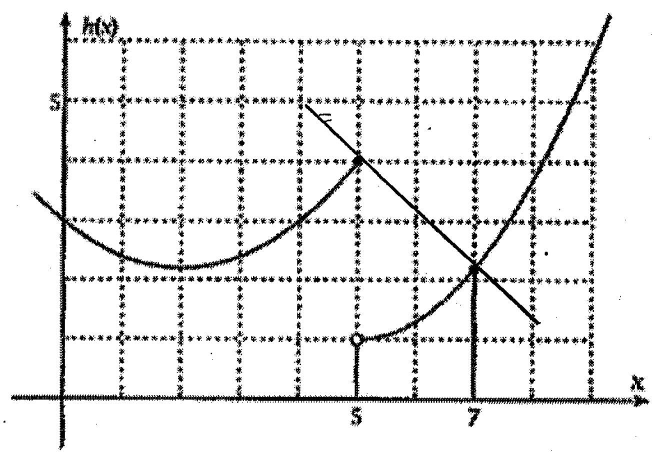
4. Function g from Problem 3 *does* have a value $x = c$ in $(1, 4)$ for which $g'(c)$ equals the slope of the secant line through $(1, g(1))$ and $(4, g(4))$.
   - Draw the secant line and tangent line, below.
   - From the graph, $c \approx$ 3
   - Is g differentiable on $(1, 4)$? yes
   - Is g continuous on $[1, 4]$? yes



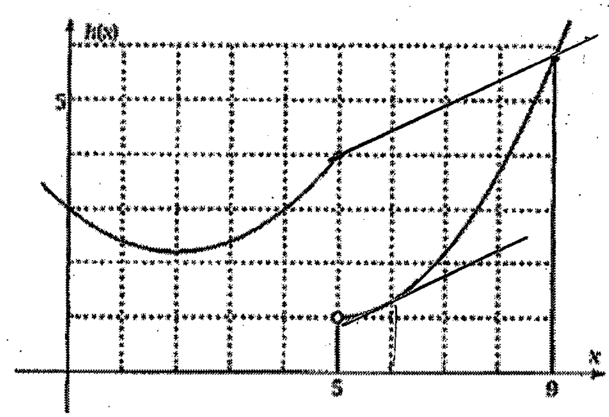
5. Piecewise function h is defined by

$$h(x) = \begin{cases} 0.2(x-2)^2 + 2.2, & \text{if } x \le 5 \\ 0.3(x-5)^2 + 1, & \text{if } x > 5 \end{cases}$$

   - Draw a secant line through $(5, h(5))$ and $(7, h(7))$.
   - Is h differentiable on $(5, 7)$? yes
   - Is h continuous on $[5, 7]$? no
   - Why is there *no* value $x = c$ in $(5, 7)$ for which $h'(c)$ equals the slope of the secant line? does not meet MVT condition (discontinuous at $x = 5$)

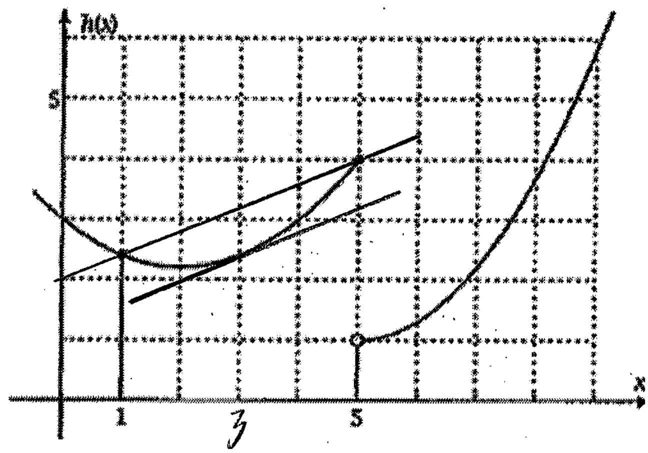


6. The graph below is function h from Problem 5.
   - Draw a secant line through $(5, h(5))$ and $(9, h(9))$.
   - Is h differentiable on $(5, 9)$? yes
   - Is h continuous on $[5, 9]$? no
   - There *is* a point $x = c$ in $(5, 9)$ where $h'(c)$ equals the slope of the secant line. Draw the tangent line. Estimate the value of c. 6



7. The graph below is function h from Problem 5.
   - Draw a secant line through $(1, h(1))$ and $(5, h(5))$.
   - Show that there is a point $x = c$ in $(1, 5)$ where $h'(c)$ equals the slope of the secant line.
   - Is h differentiable on $(1, 5)$? yes
   - Explain why h is continuous on $[1, 5]$, even though there is a step discontinuity at $x = 5$.

   Because the 'hole' is not in [1,5]



8. The mean value theorem states:
   If f is differentiable on (a, b) and
   f is continuous on $[a, b]$,
   then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{i.e., the secant's slope}$$

   For which problem(s) are
   - the hypotheses and conclusion true? 1, 2, 4, 7
   - the hypotheses and conclusion not true? 3, 5
   - the conclusion true, but not the hypotheses? 6

9. The number $x = c$ in the above problems is the "mean" value referred to in the name "mean value theorem." Explain why the hypotheses are sufficient conditions for the conclusion, but *not* necessary conditions.

   If hypotheses (conditions) are true the Mean Value Theorem guarantees a value c in (a,b) with same tangent line slope, but doesn't preclude parallel tangents when conditions aren't met (says nothing about this case).
   So if conditions are not met, there may or may not be a tangent line parallel to the secant line in the interval.

Ex) Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

$f(x) = x^2 - 4x + 1, \quad [0, 4]$

$f(x)$ is a polynomial;
so it is continuous on $[0,4]$ ✓
and differentiable on $(0,4)$ ✓

$f(0) = 0^2 - 4(0) + 1 = 1$ so $f(0) = f(4)$ ✓
$f(4) = 4^2 - 4(4) + 1 = 1$

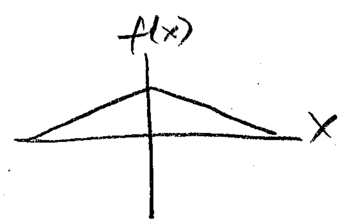
$f'(x) = 2x^2 - 4 = 0$

$m = \frac{f(4) - f(0)}{4 - 0} = \frac{1 - 1}{4} = 0$

when $x = \boxed{c = 2}$

5. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$.

Why does this not contradict Rolle's Theorem?



$f(x) = 1 - \sqrt[3]{x^2}$

$f(x)$ is continuous on $(-1,1)$

$f'(x) = 1 - \frac{2}{3}x^{-1/3} = 1 - \frac{2}{3\sqrt[3]{x}}$

DNE at $x = 0$

$\boxed{f(x) \text{ is not differentiable on } (-1,1)}$

so even though $f(-1) = 1 - \sqrt[3]{(-1)^2} = 0$
and $f(1) = 1 - \sqrt[3]{1^2} = 0$

this doesn't contradict Rolle's because conditions aren't met
so Rolle's Theorem doesn't apply.

11. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f(x)=3x^2+2x+5, \qquad [-1,1]$$

f is a polynomial so continuous on $[-1,1]$ and differentiable on $(-1,1)$.

$f'(x)=6x+2$ $\quad f(-1)=3(-1)^2+2(-1)+5=6$ $\quad m=\frac{10-6}{1-(-1)}=\frac{4}{2}=2$

$f(1)=3(1)^2+2(1)+5=10$

$f'(x)=6x+2=2$

$6x=0$

when $x=\boxed{c=0}$

19. Show that the equation $x^5-6x+c=0$ has at most one root in the interval $[-1,1]$.

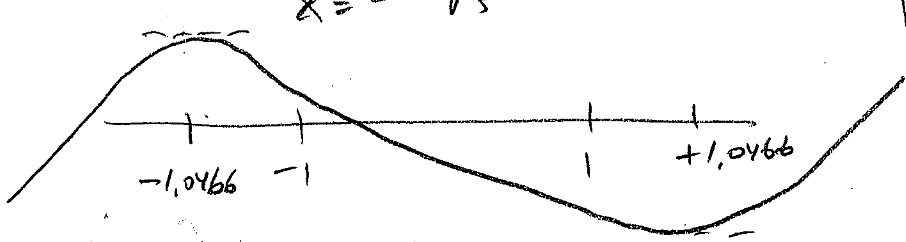
using Rolle's Theorem f continuous on $[-1,1]$, differentiable on $(-1,1)$

then $f'(x)=5x^4-6=0$

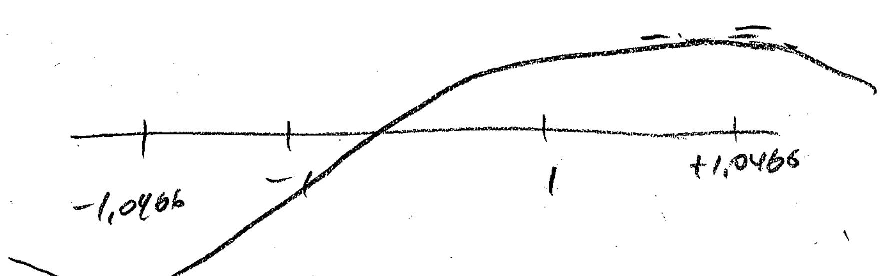
$5x^4=6$

$x^4=\frac{6}{5}$

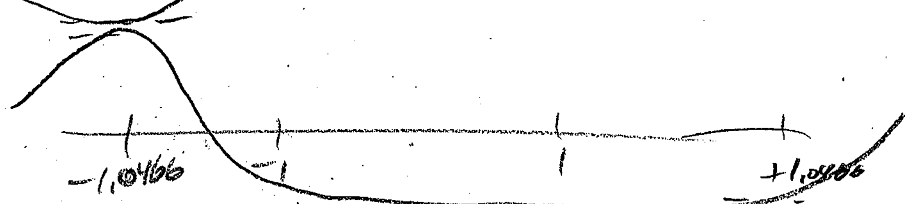
$x=\pm\sqrt[4]{\frac{6}{5}}\approx\pm 1{,}046635$



Because $f'(x)=0$ only outside the interval $[-1,1]$ the function must be either always increasing or always decreasing with $[-1,1]$

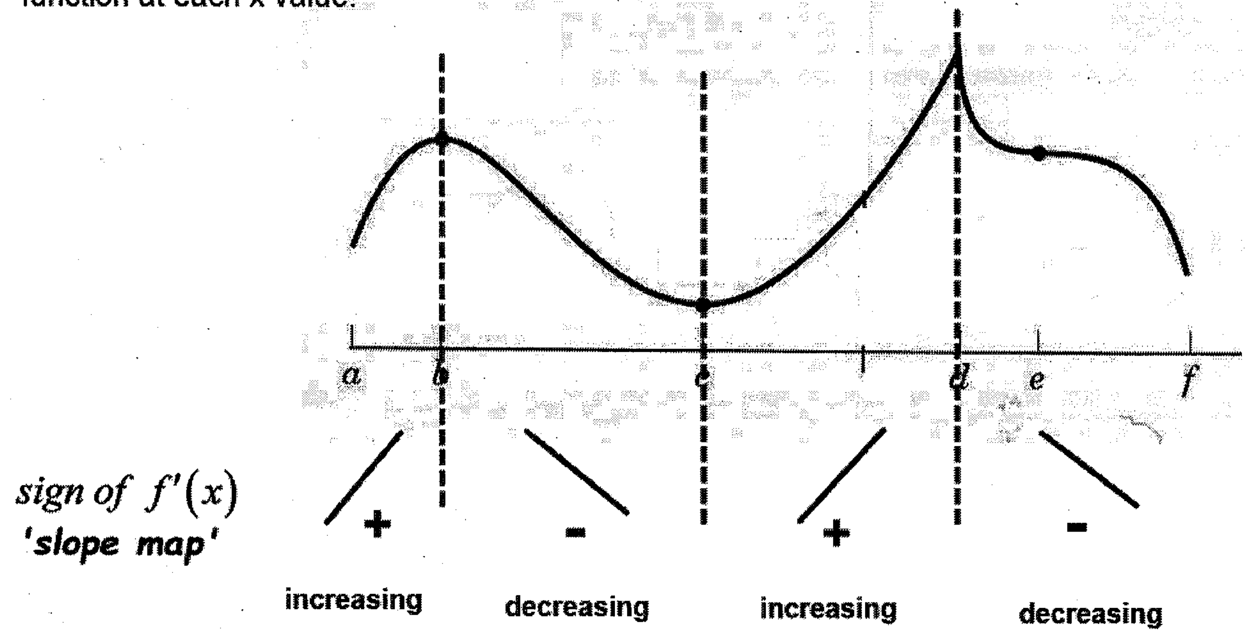


could have either one or no zeros in $[-1,1]$



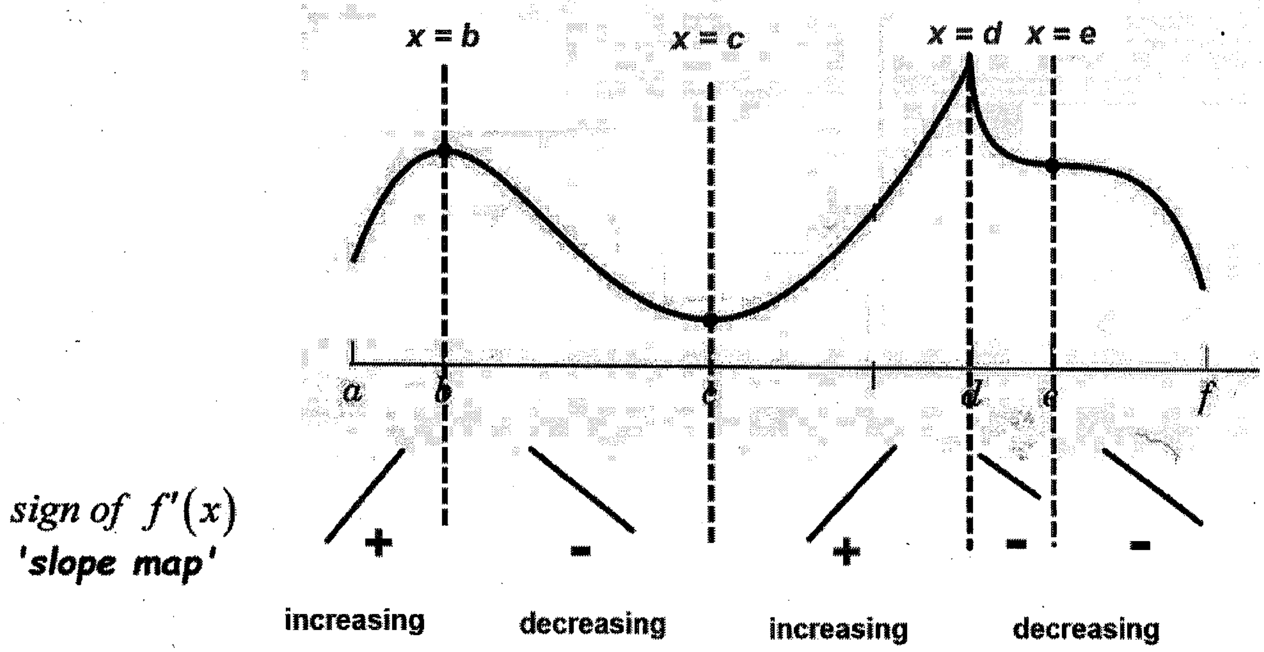
The First Derivative Test

The **first derivative** of a function gives the instantaneous rate of change, or '**slope**', of the function at each x value:



...by checking the *sign of* $f'(x)$ we can determine if the function is increasing or decreasing.

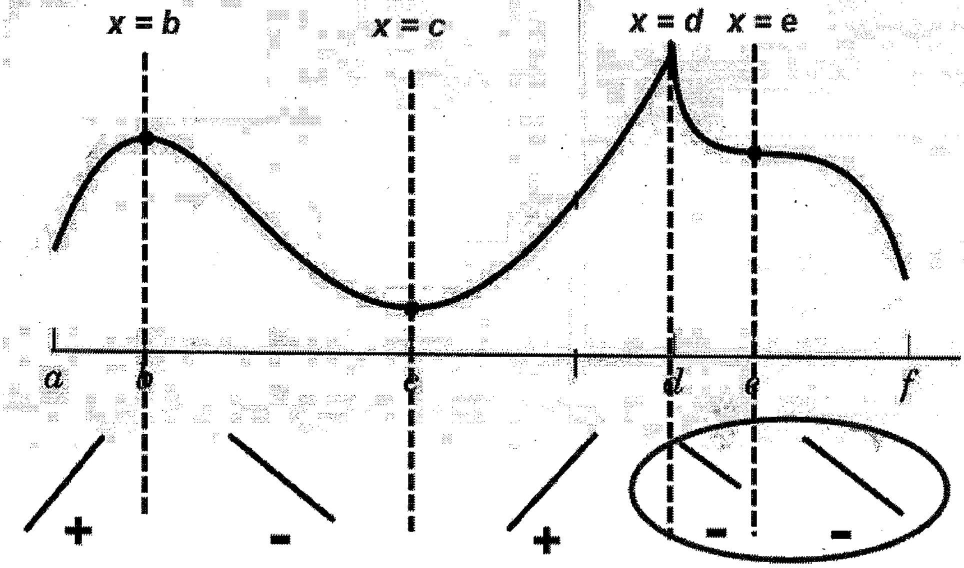
The sign of $f'(x)$ can only change at certain x-values, which are called **critical numbers**



A critical number occurs at any x-value where $f'(x)=0 \quad or \quad f'(x)\ DNE$

The First Derivative Test

Note that the sign of $f'(x)$ usually, but not always, changes as x crosses a critical number...

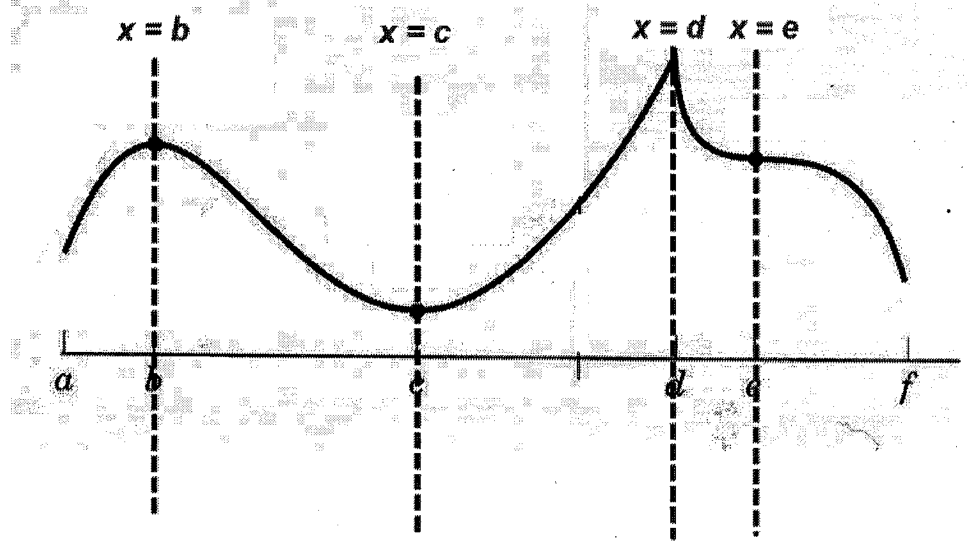


sign of $f'(x)$
'slope map'

increasing **decreasing** **increasing** **decreasing**

To conduct a first derivative test on a function, we first identify all the critical numbers, then we select a single test x-value in each region between critical numbers, and find the sign of the first derivative to determine if the function is increasing or decreasing in this region.

$f'(x)$



sign of $f'(x)$
'slope map'

Where we use the First Derivative Test

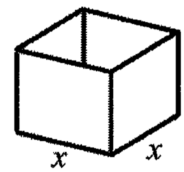
There are two primary reasons we use the First Derivative Test:

1) To quickly determine where the relative maximum or minimum values of a function occur.

2) To sketch a function curve without using a calculator (we also use some precalculus procedures, and sometimes also the Second Derivative test).

Using the First Derivative Test to determine a function's relative extrema

Ex) A manufacturer wants to design an open top, square base, box using 108 sq. in. of material. If x is the side length of the base, the volume of the box is given by: $V(x) = 27x - \frac{1}{4}x^3$
Determine the side length which will give the largest volume.



$V'(x) = 27 - \frac{3}{4}x^2$

critical numbers: $27 - \frac{3}{4}x^2 = 0$

$\frac{3}{4}x^2 = 27$

$x^2 = \frac{27(4)}{3} = 36$

$x = \pm 6$ ($x = -6$ not in domain)

$\boxed{x = 6}$ is the optimum side length

to verify this is a maximum find $v'(x)$ at one test value in each region

'slope'

6

$v'(5)$ | $v'(7)$

$27 - \frac{3}{4}(5)^2$ | $27 - \frac{3}{4}(7)^2$

$27 - \frac{3}{4}(25)$ | $27 - \frac{3}{4}(49)$

$+$ | $-$

relative maximum (volume) at $x = 6$

Using the First Derivative Test to sketch a function curve

Ex) Find the critical numbers of the function: $f(x) = 6x^4 + 12x^3 + 20$
Find the domain and intercepts of the function.
Find the open interval(s) on which the function is increasing or decreasing.
Find any relative extrema, and sketch the function ***without using a calculator***.

precalculus

$\boxed{\text{domain? } (-\infty, \infty)}$ (polynomial)

intercepts?

y int when $x = 0$

$f(0) = 20$

$\boxed{(0,20) \text{ is y int}}$

x int when $y = 0$

$6x^4 + 12x^3 + 20 = 0$

(too hard w/o a calculator, skip for now)

$f'(x)$

$f'(x) = 24x^3 + 36x^2$

critical numbers:

$24x^3 + 36x^2 = 0$

$6x^2(4x + 6) = 0$

$\boxed{x = 0}$ $\quad 4x + 6 = 0$

$x = \frac{-6}{4} = \frac{-3}{2}$

$\boxed{x = -\frac{3}{2}}$

$f'(x) = 6x^2(4x+6)$
$(+)(+)(?)$

'slope'

$-\frac{3}{2}$ $\quad$ 0

$f'(-2)$ $\quad$ $f'(-1)$ $\quad$ $f'(1)$

$(+)(+)(-)$ $\quad$ $(+)(+)(+)$ $\quad$ $(+)(+)(+)$

$-$ $\quad$ $+$ $\quad$ $+$

$\boxed{\begin{array}{l}\text{increasing: } (-\frac{3}{2}, 0) \cup (0, \infty) \\ \text{decreasing: } (-\infty, -\frac{3}{2})\end{array}}$

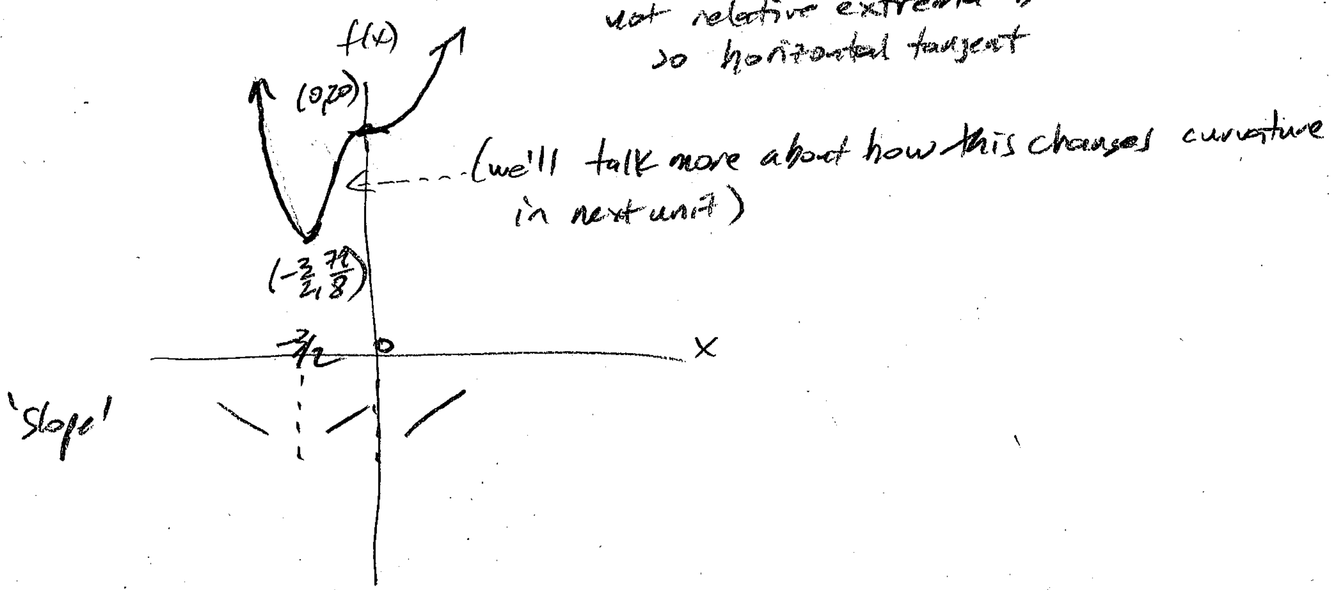
at $x = -\frac{3}{2}$: $f(-\frac{3}{2}) = 6(-\frac{3}{2})^4 + 12(-\frac{3}{2})^3 + 20 = \frac{6(81)}{16} + 12\frac{(-27)}{8} + 20$

$= \frac{3(81)}{8} - \frac{12(27)}{8} + \frac{160}{8}$

$= \frac{79}{8}$

$\boxed{(-\frac{3}{2}, \frac{79}{8}) \text{ is a relative minimum}}$

at $x = 0$: sign doesn't change so not relative extrema but $f'(0) = 0$ so horizontal tangent

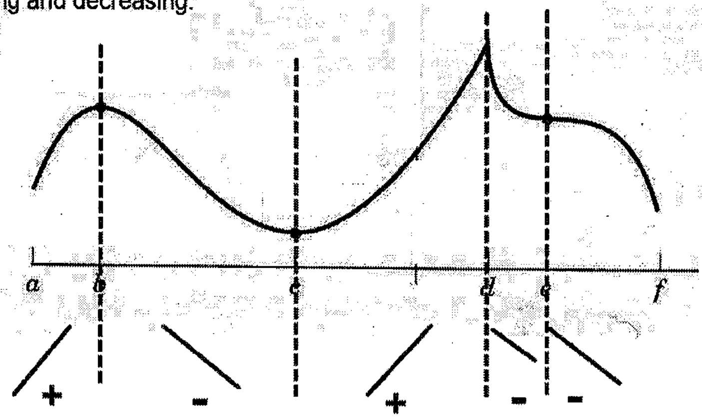


Unit 4-4: Second Derivative Test

Larsen: 3.4 (Stewart: 4.3)

The Second Derivative Test

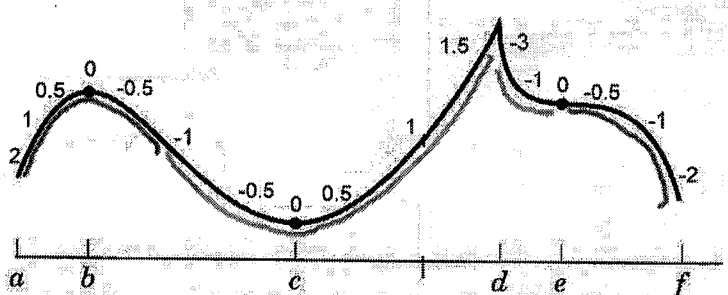
The **first derivative** of a function gives us information on the 'slope' of a curve and where the curve is increasing and decreasing.



sign of $f'(x)$
'slope map'

increasing **decreasing** **increasing** **decreasing**

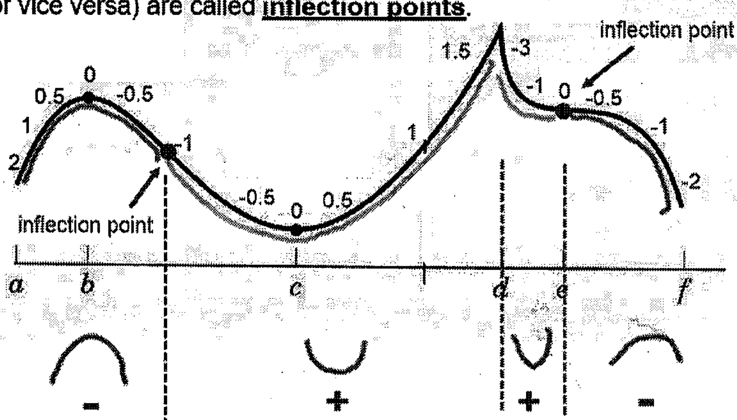
The **second derivative** of a function gives us information on how fast the first derivative of a function is changing. If we estimate the slope of the tangent line at each x...



...in some regions, the first derivative is increasing numerically (which means the second derivative would be positive). This produces a curve that curves upward and is called **concave up**.

...in other regions, the first derivative is decreasing numerically (which means the second derivative would be negative). This produces a curve that curves upward and is called **concave down**.

For the first derivative, the points where the function changed from increasing to decreasing were called critical points. Similarly, the points where the concavity changes from concave up to concave down (or vice versa) are called **inflection points**.



sign of $f''(x)$
'concavity map'

concave down **concave up** **concave up** **concave down**

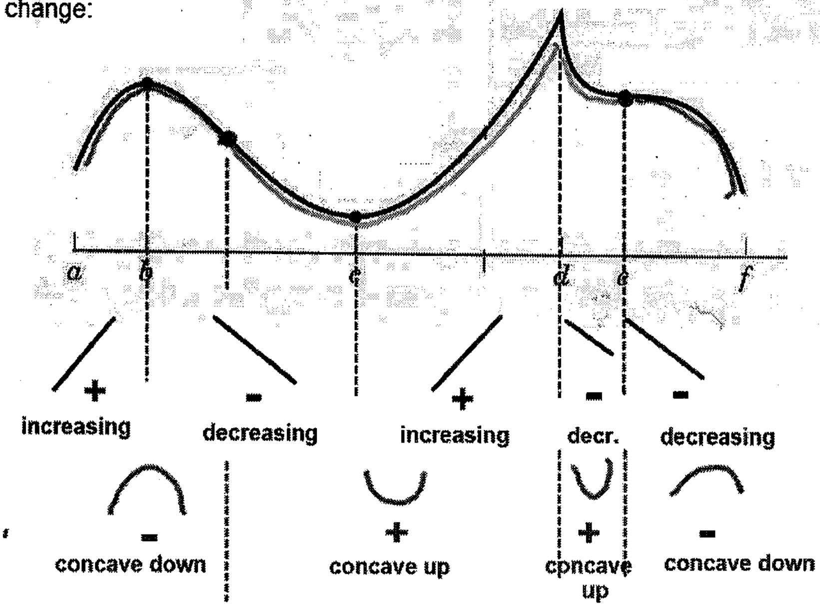
...and these are found by finding the x-values where the second derivative is either zero or undefined.

The Second Derivative Test

Note that the x-values of the critical points and the inflection points are typically not the same (although they can be), and also that you need to check each region because the sign of either the first or second derivative can only change at critical or inflection points, but is not guaranteed to change:

sign of $f'(x)$
'slope map'

sign of $f''(x)$
'concavity map'



Where we use the Second Derivative Test

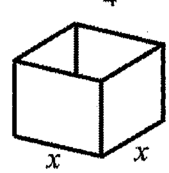
There are two primary reasons we use the Second Derivative Test:

1) When we have used the First Derivative Test to determine where a relative maximum or minimum is located, we can plug that same x-value into the second derivative. If the second derivative is positive, the function is concave up here so this extrema is a relative minimum. If the second derivative is negative, the function is concave down here so this extram is a relative maximum.

2) To sketch a function curve without using a calculator, once we have made a 'slope' map, we can also find the second derivative, find where it is zero or undefined to locate the inflection points, then test values in each region to find the sign of the second derivative to determine where the function's concavity in each region. This gives us more detailed information to sketch the curve shape properly in each region.

Using the Second Derivative Test to determine a function's relative extrema

Ex) A manufacturer wants to design an open top, square base, box using 108 sq. in. of material. If x is the side length of the base, the volume of the box is given by: $V(x)=27x-\frac{1}{4}x^3$
Determine the side length which will give the largest volume.



$V'(x)=27-\frac{3}{4}x^2$

Critical numbers:

$27-\frac{3}{4}x^2=0$

$\frac{3}{4}x^2=27$

$x^2=36$

$x=\pm 6$

$\boxed{x=6}$

now, instead of using a 'slope' map, we take the 2nd derivative...

$V''(x)=-\frac{3}{2}x$

...and plug in $x=6$

$V''(6)=-\frac{3}{2}(6)=-9<0$

So the volume curve is concave down in this region, which means the point at $x=6$ is a maximum volume.

Using the Second Derivative Test to sketch a function curve

Ex) Find the critical numbers of the function: $f(x)=6x^4+12x^3+20$
Find the domain and intercepts of the function.
Find the open interval(s) on which the function is increasing or decreasing.
Find any relative extrema, and sketch the function ***without using a calculator***

<u>precalculus</u>

domain? $(-\infty,\infty)$

yints: $f(0)=20$

$\boxed{(0,20)}$

xints (too hard, skip)

<u>$f'(x)$</u>

$f'(x)=24x^3+36x^2$

critical numbers:

$24x^3+36x^2=0$

$6x^2(4x+6)=0$

$\boxed{x=0}$ $\boxed{x=-\frac{3}{2}}$

$-\frac{3}{2}$ $\quad$ 0

$f'(-2)$ $\quad$ $f'(-1)$ $\quad$ $f'(1)$

$(+)(+)(-)$ $\quad$ $(+)(+)(+)$ $\quad$ $(+)(+)(+)$

$-$ $\quad$ $+$ $\quad$ $+$

$f(-\frac{3}{2})=\frac{79}{8}$

$\boxed{\text{so } (-2, \tfrac{79}{8}) \text{ is a relative minimum}}$

$\boxed{\begin{array}{l}\text{increasing: } (-\frac{3}{2},0)\cup(0,\infty)\\ \text{decreasing: } (-\infty,-\frac{3}{2})\end{array}}$

<u>$f''(x)$</u>

$f''(x)=72x^2+72x$

inflection values

$72x^2+72x=0$

$72x(x+1)=0$

$\boxed{x=0}$ $\boxed{x=-1}$

-1 $\quad$ 0

$f''(-2)$ $\quad$ $f''(-\frac{1}{2})$ $\quad$ $f''(1)$

$(+)(-)(-)$ $\quad$ $(+)(-)(+)$ $\quad$ $(+)(+)(+)$

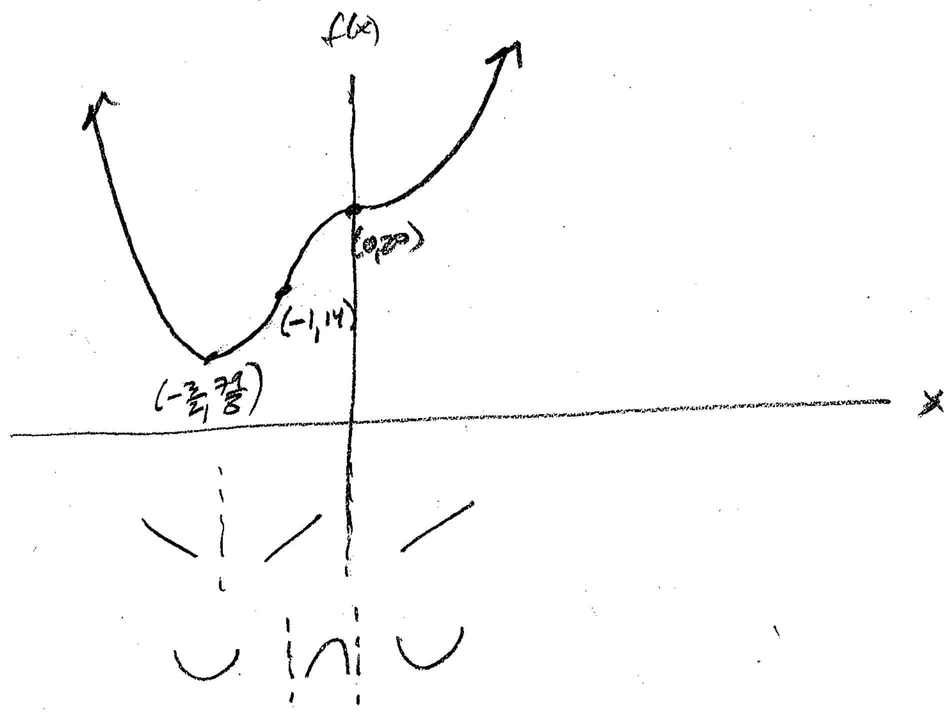
$+$ $\quad$ $-$ $\quad$ $+$

$f(-1)=14$

$f(0)=20$

$\boxed{(-1,14) \text{ and } (0,20) \text{ are inflection points}}$

$\boxed{\begin{array}{l}\text{concave up: } (-\infty,-1)\cup(0,\infty)\\ \text{concave down: } (-1,0)\end{array}}$



'slope'

Concavity

Unit 4-5: Indeterminant Forms, L'Hopital's Rule, Asymptotes
Larsen: 7.7 (Stewart: 4.4)

Curve sketching is assisted by finding any asymptotes

Horizontal asymptotes (possibly) occur if the limit as x becomes large approaches a constant as x approaches infinity.

Vertical asymptotes (possibly) occur if the limit becomes large as x approaches the vertical asymptote (an infinite limit).

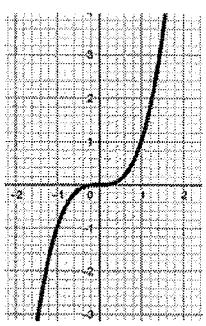
A reminder from earlier in this course...

Infinite Limits

An **<u>infinite limit</u>** is a limit that, when evaluated, is increasing without bound to $+\infty$ or decreasing without bound to $-\infty$.

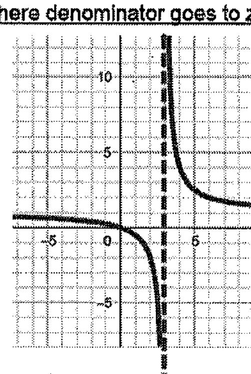
There are three general situations where we see infinite limits:

$x \to \infty$ *or* $x \to -\infty$
<u>in Polynomials</u>



$\lim\limits_{x\to-\infty} x^3 = -\infty \quad \lim\limits_{x\to\infty} x^3 = \infty$

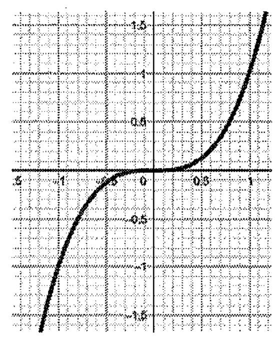
$x \to c$
In some Rational Functions
<u>where denominator goes to zero</u>



$\lim\limits_{x\to3^-} \dfrac{x}{x-3} = -\infty \quad \lim\limits_{x\to3^+} \dfrac{x}{x-3} = \infty$

at a Vertical Asymptote
x = 3

$x \to \infty$ *or* $x \to -\infty$
In Rational Functions
where numerator gets larger
<u>faster than the denominator</u>



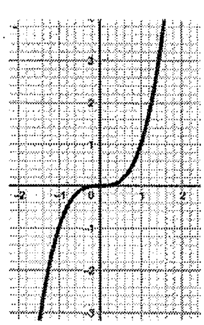
$\lim\limits_{x\to-\infty} \dfrac{x^5}{x^2} = -\infty \quad \lim\limits_{x\to\infty} \dfrac{x^5}{x^2} = \infty$

Limits at Infinity

A **<u>limit at infinity</u>** is a limit where *x* is approaching either $+\infty$ or $-\infty$.

When evaluate limits at infinity by imagining what will happen as *x* gets very large (either in the positive or negative direction). The resulting limit value may be zero, a constant, or an infinite limit, depending upon the function:

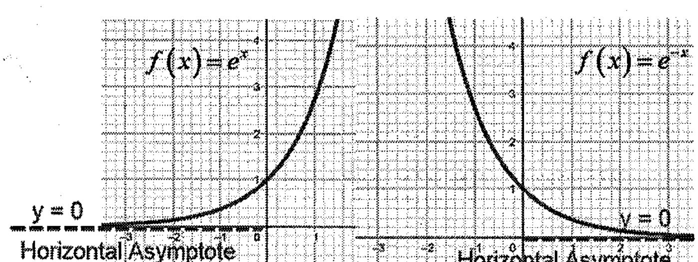
$x \to \infty$ *or* $x \to -\infty$
<u>in Polynomials</u>



$\lim\limits_{x\to-\infty} x^3 = -\infty \quad \lim\limits_{x\to\infty} x^3 = \infty$

you get an infinite limit

$x \to \infty$ *or* $x \to -\infty$
<u>in Exponentials</u>



$\lim\limits_{x\to-\infty} e^x = 0 \quad \lim\limits_{x\to\infty} e^x = \infty \quad \lim\limits_{x\to-\infty} e^{-x} = \infty \quad \lim\limits_{x\to\infty} e^{-x} = 0$

If, as x approaches either positive or negative infinity the value of the limit approaches zero or any other constant, then there is a **<u>horizontal asymptote</u>** at this y value.

Evaluating such limits analytically

To evaluate, first consider what happens as x approaches the target value.

$$\lim_{x\to-\infty} x^2(x-1)$$

Here, two number which are multiplied are both getting very large, but because x is negative and one is squared but the other is not, we have...

$$\infty(-\infty)$$

...so:

$$\lim_{x\to-\infty} x^2(x-1) = -\infty$$

$$\lim_{x\to\infty}\frac{x^5}{e^x}$$

The numerator and denominator are both getting very large:

$\frac{\infty}{\infty}$ this is an **indeterminant form** (the numerator and denominator are 'fighting for control')

...in this case, the exponential increases faster than the power, so the denominator will eventually be much larger than the numerator:

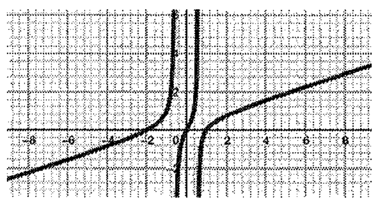
$$\lim_{x\to\infty}\frac{x^5}{e^x} = 0$$

Evaluating Rational Function limits

If you have a limit of a rational function with polynomials for numerator and denominator, three things can happen:

numerator degree is larger:

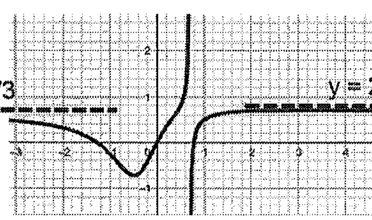
$$\lim_{x\to-\infty}\frac{x^5+x^4-2x^3}{3x^4-x^2}$$



$$\lim_{x\to\infty}\frac{x^5+x^4-2x^3}{3x^4-x^2}\boxed{=\infty}$$

numerator and denominator degrees are the same:

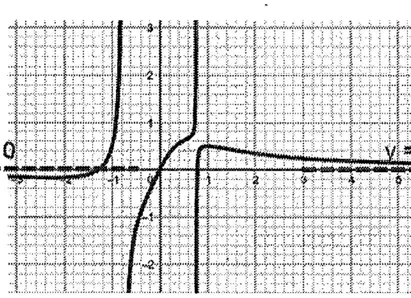
$$\lim_{x\to\infty}\frac{\boxed{2}x^5+x^4-2x^3}{\boxed{3}x^5-x^2}$$



$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^5-x^2}\boxed{=\frac{2}{3}}$$

numerator degree is smaller:

$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^6-x^2}$$



$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^6-x^2}\boxed{=0}$$

Evaluating Rational Function limits without graphing

There is a procedure you can use to evaluate rational function limits without graphing.

- If numerator degree is higher than denominator, then the limit is an infinite limit (be careful of the sign).
- If denominator degree is higher than numerator, then the limit is 0.
- If the degrees of the numerator and denominator are the same, divide every term in both the numerator and denominator by $x^{\text{degree of denominator}}$. Then cancel within each term, and all terms with a constant over a power of x go to zero.

$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^5-x^2}$$ divide everything by x^5

$$\lim_{x\to\infty}\frac{\frac{2x^5}{x^5}+\frac{x^4}{x^5}-\frac{2x^3}{x^5}}{\frac{3x^5}{x^5}-\frac{x^2}{x^5}}$$

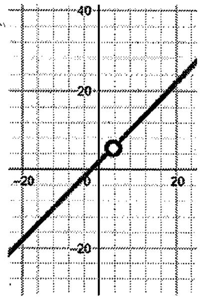
$$\lim_{x\to\infty}\frac{2+\frac{1}{x}-\frac{2}{x^2}}{3-\frac{1}{x^3}}$$

$$\lim_{x\to\infty}\frac{2+0-0}{3-0}=\frac{2}{3}$$

Vertical Asymptotes occur at uncancelled zeros in the denominator

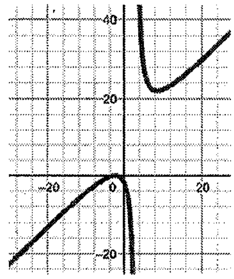
Consider these two limits:

$$\lim_{x\to 3^-}\frac{x^2-x-6}{x-3}$$
$$=\lim_{x\to 3^-}\frac{(x-3)(x+2)}{x-3}$$
$$=\lim_{x\to 3^-}(x+2)=5$$



No Vertical Asymptote because the x-3 could be factored and cancelled

$$\lim_{x\to 3^-}\frac{x^2+5x+6}{x-3}$$
$$=\lim_{x\to 3^-}\frac{(x+3)(x+2)}{x-3}$$
$$=\frac{45}{0}$$



Vertical Asymptote because the x-3 could <u>not</u> be factored and cancelled

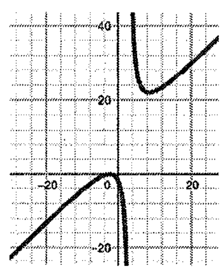
Evaluating Rational Function limits near a vertical asymptote

When direct substitution results in divide by zero and no way factoring removes this, you know the limit evaluates to either positive or negative infinity. To determine which, you consider the **signs** of the numerator and denominator:

$$\lim_{x\to 3^-}\frac{x^2+5x+6}{x-3}$$
$$=\lim_{x\to 3^-}\frac{(x+3)(x+2)}{x-3}$$
$$=\frac{45}{0}\quad=+\infty\ \mathit{or}\ -\infty$$

$$\frac{(+)}{(-)}\longleftarrow$$ negative because as x approach 3 from the left x is less than 3, so x-3 is negative

$$\boxed{=-\infty}$$



Horizontal Asymptotes require evaluating limits as x approaches +/- infinity...

$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^5-x^2}=\frac{\infty}{\infty}$$

$$=\lim_{x\to\infty}\frac{\frac{2x^5}{x^5}+\frac{x^4}{x^5}-\frac{2}{x^5}}{\frac{3x^5}{x^5}-\frac{x^2}{x^5}}$$

$$=\lim_{x\to\infty}\frac{2+\frac{1}{x^4}-\frac{2}{x^5}}{3-\frac{1}{x^3}}$$

$$=\lim_{x\to\infty}\frac{2+0-0}{3-0}=\boxed{\frac{2}{3}}$$

L'Hôpital's Rule gives us another alternative evaluation technique

When you directly evaluate, if you get one of the following indeterminant forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}, \frac{\infty}{-\infty}, \frac{-\infty}{\infty}$$

L'Hôpital's Rule: $\lim\limits_{x\to c}\frac{f(x)}{g(x)}=\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$

$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^5-x^2}$$

Note: separate derivatives, not quotient rule.

$by\ L'H\hat{o}pital's\ Rule:$

$$=\lim_{x\to\infty}\frac{10x^4+4x^3-6x^2}{15x^4-2x}=\frac{\infty}{\infty}$$

$by\ L'H\hat{o}pital's\ Rule:$

$$=\lim_{x\to\infty}\frac{40x^3+12x^2-12x}{60x^3-2}=\frac{\infty}{\infty}$$

$by\ L'H\hat{o}pital's\ Rule:$

$$=\lim_{x\to\infty}\frac{120x^2+24x-12}{180x^2}=\frac{\infty}{\infty}$$

$by\ L'H\hat{o}pital's\ Rule:$

$$=\lim_{x\to\infty}\frac{240x+24}{360x}=\frac{\infty}{\infty}$$

$by\ L'H\hat{o}pital's\ Rule:$

$$=\lim_{x\to\infty}\frac{240}{360}=\boxed{\frac{2}{3}}$$

Conditions to apply L'Hopital's Rule

L'Hopital's rule can be applied repeatedly until the limit either evaluates to a constant or an infinity, but it is important that you check that all conditions for applying L'Hopital's Rule are met first and at each stage:

1) The numerator and denominator functions must both be differentiable on the open interval of interest, and the denonimator must not be zero (except possibly at the target x-value).

2) Must be a valid indeterminant form: $\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}, \frac{\infty}{-\infty}, \frac{-\infty}{\infty}$

$$\lim_{x\to 0}\frac{x^2-4x}{2x-1}$$

$\frac{0}{-1}$

not L'Hopital form

just $\boxed{0}$

$$\lim_{x\to 1}\frac{\cos(\pi x)}{\ln(x)}$$

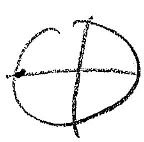
$\frac{\to\cos(\pi)}{\to\ln(1)}$

$\frac{-1}{0} = \pm\infty$

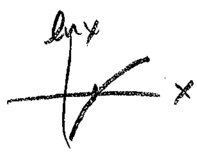
not L'Hopital form

but vertical asymptote

$\lim_{x\to 1^-}\frac{\cos(\pi x)}{\ln(x)} \quad \frac{(-)}{(-)} = +\infty$



$\lim_{x\to 1^+}\frac{\cos(\pi x)}{\ln(x)} \quad \frac{(-)}{(+)} = -\infty$



$\lim_{x\to 1}\frac{\cos(\pi x)}{\ln(x)}$ DNE

$$\lim_{\theta\to 0}\frac{\sin(2\theta)}{\theta} = \frac{0}{0}$$

is L'Hopital form!

by L'Hopital's Rule:

$= \lim_{\theta\to 0}\frac{2\cos(2\theta)}{1}$

$\frac{2(1)}{1}$

$\boxed{2}$

Some other indeterminant forms can be converted to a form where L'Hopital applies

Some limits evaluate to other indeterminant forms where it isn't clear if the value is a constant or an infinity, and three specific other indeterminant forms can be converted into a form where L'Hopital's Rule can be applied. These are:

Product indeterminant forms: $0 \bullet \infty,\ 0 \bullet -\infty$ algebraically manipulate to force into a rational function.

Difference indeterminant forms: $\infty - \infty$ algebraically manipulate to force into a rational function.

Power indeterminant forms: $1^\infty,\ 0^\infty,\ \infty^0$ set expression equal to y, take ln of both sides, then convert right side into a L'Hopital indeterminant form.

Product indeterminant forms:

$0 \bullet \infty,\ 0 \bullet -\infty$

algebraically manipulate to force into a rational function.

$$\lim_{x\to\infty} e^{-x}\sqrt{x}$$

$(0)(\infty)$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{e^x} \quad \frac{\infty}{\infty}$$

by L'Hopital's

$$= \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{e^x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x}$$

$$\frac{\frac{1}{2\infty}}{e^\infty}$$

$$\frac{0}{\infty}$$

$$= \boxed{0}$$

Difference indeterminant forms:

$\infty - \infty$

algebraically manipulate to force into a rational function.

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$$

$$\frac{1}{\ln(1)} - \frac{1}{1-1}$$

$$\frac{1}{0} - \frac{1}{0}$$

$$\infty - \infty$$

$$= \lim_{x\to 1^+}\left(\frac{1(x-1)}{(x-1)\ln x} - \frac{\ln x(1)}{(x-1)\ln x}\right)$$

$$= \lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)\ln x} \quad \frac{0}{0}$$

by L'Hopital's

$$= \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{(x-1)\frac{1}{x}+\ln x} = \frac{1-x^{-1}}{1-x^{-1}+\ln x}$$

$$\frac{0}{0}$$

by L'Hopital's

$$= \lim_{x\to 1^+} \frac{x^{-2}}{x^{-2}+\frac{1}{x}}$$

$$= \lim_{x\to 1^+} \frac{\frac{1}{x^2}}{\frac{1}{x^2}+\frac{1}{x}} = \frac{1}{1+1}$$

$$= \boxed{\frac{1}{2}}$$

Power indeterminant forms:

$1^\infty,\ 0^\infty,\ \infty^0$

set expression equal to y, take ln of both sides, then convert right side into a L'Hopital indeterminant form.

$$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \quad 1^\infty$$

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]$$

$$\ln y = \lim_{x\to\infty}\left[\ln\left(1+\frac{1}{x}\right)^x\right]$$

$$\ln y = \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right]$$

$(\infty)(0)$

$$\ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \quad \frac{0}{0}$$

by L'Hopital's

$$\ln y = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$\ln y = \lim_{x\to\infty} \frac{1}{1+x}$$

$$\frac{1}{1+\infty}$$

$$\frac{1}{\infty} = 0$$

but

$$\ln y = 0$$

$$\log_e y = 0$$

$$y = e^0 = \boxed{1}$$

Slant Asymptotes

Sometimes, if you evaluate a rational function limit at infinity you get an infinite limit, but even though the limit is approaching an infinity, is it also aligning with a straight line (but not a vertical or horizontal asymptote). This occurs with rational functions where numerator and denominator are polynomials and the degree of the numerator is exactly one higher than the degree of the denominator:

$$f(x)=\frac{2x^3+x^2+1}{x^2+1}$$

← deg 3
← deg 2
but...

HA.? $\lim\limits_{x\to\infty} \dfrac{2x^3+x^2+1}{x^2+1}$ $\quad \dfrac{\infty}{\infty}$

L'Hop

$= \lim\limits_{x\to\infty} \dfrac{6x^2+2x}{2x}$ $\quad \dfrac{\infty}{\infty}$

L'Hop

$= \lim\limits_{x\to\infty} \dfrac{12x+2}{2}$ $\quad \dfrac{\infty}{2}$

$= \infty$

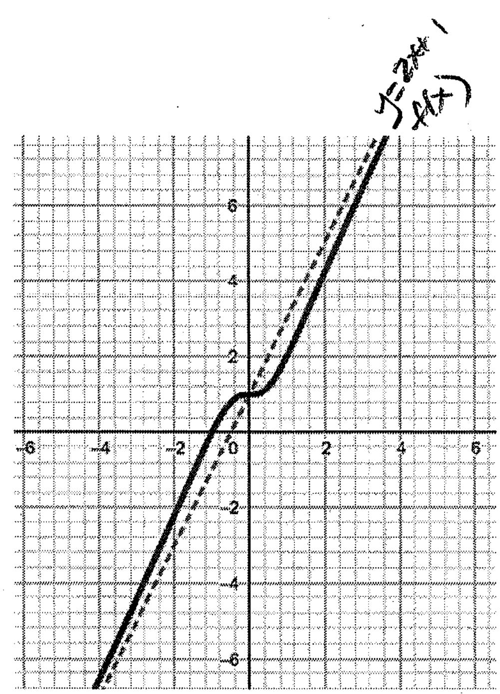
(no H.A.)

$$\begin{array}{r} 2x+1 \\ x^2+0x+1 \,\overline{)\,2x^3+x^2+0x+1} \\ -(2x^3+0x^2+2x) \\ \hline x^2-2x+1 \\ -(x^2+0x+1) \\ \hline -2x \end{array}$$

so $\dfrac{2x^3+x^2+1}{x^2+1} = \boxed{2x+1} + \dfrac{-2x}{x^2+1}$

as $x\to\infty$ ↑ goes to zero

and curve approaches slant asymptote $\boxed{y=2x+1}$



Summary Curve Sketching Procedure

With this, admittedly long, procedure, you can find everything you need to answer any question about a given function and sketch the function without a calculator.

Precalc

1) Find the domain.

2) Find the x- and y-intercepts (skip x-intercepts if too hard to factor)

3) Check for vertical asymptotes (any uncancelled zeros in denominator)?

4) Check for horizontal asymptotes by
$\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$

5) Check for slant asymptote if rational function w/degree different by one.

1st derivative

6) Compute $f'(x)$ and find critical values where $f'(x)=0$ *or* $f'(x)$ *is undefined*

7) Divide the domain into intervals split at the critical values and test the sign of $f'(x)$ in each interval to make a 'slope' map.

8) Plug critical value x's into orig. function and list local max and min points.

2nd derivative

9) Compute $f''(x)$ and find inflection point x-values where $f''(x)=0$ *or* $f''(x)$ *is undefined*

10) Divide the domain into intervals split at the inflection points and test the sign of $f''(x)$ in each interval to make a concavity map.

11) Plug inflection x's into original function to get inflection points, then sketch.

Fully analyze and sketch without a calculator: $f(x)=\dfrac{-x^2-4x-7}{x+3}$

precalc

$\boxed{\text{domain: } (-\infty,-3)\cup(-3,\infty)}$

y int: $f(0)=-\frac{7}{3}$ $\boxed{(0,-\frac{7}{3})}$

x ints: $-x^2-4x-7=0$

$x^2-4x+7=0$

$x=\frac{4\pm\sqrt{16-4(7)}}{2}$

$=\frac{4\pm\sqrt{-}}{2}$ $\boxed{\text{no x ints}}$

$\boxed{\text{Vertical asymptote at } x=-3}$

Horizontal asymptote?

$\lim_{x\to\infty}\frac{-x^2-4x-7}{x+3}\ \frac{-\infty}{\infty}$

L'Hop

$=\lim_{x\to\infty}\frac{-2x-4}{1}=-\infty$

$\lim_{x\to-\infty}-\frac{x^2-4x-7}{x+3}\ \frac{-\infty}{-\infty}$

L'Hop

$=\lim_{x\to-\infty}\frac{-2x-4}{1}\ \infty$

$\boxed{\text{no H.A.}}$

Slant asymptote:

$$\begin{array}{r} -x-1 \\ x+3\,\overline{)\,-x^2-4x-7} \\ -(-x^2-3x) \\ -x-7 \\ -(-x-3) \\ -4 \end{array}$$

slant asymptote $\boxed{y=-x-1}$

f'(x)

$f'(x)=\frac{(x+3)(-2x-4)-(-x^2-4x-7)(1)}{(x+3)^2}$

$=\frac{-2x^2-4x-6x-12+x^2+4x+7}{(x+3)^2}$

$=\frac{-x^2-6x-5}{(x+3)^2}=\frac{-(x^2+6x+5)}{(x+3)^2}$

$=\frac{-(x+1)(x+5)}{(x+3)^2}$

critical: $x=-1,-3,-5$

-5 -3 -1

$f'(-6)$: $\frac{(-)(-)(-)}{(+)}$ $-$

$f'(-4)$: $\frac{(-)(-)(+)}{(+)}$ $+$

$f'(-2)$: $\frac{(-)(-)(+)}{(+)}$ $+$

$f'(0)$: $\frac{(-)(+)(+)}{(+)}$ $-$

$\boxed{\begin{array}{l}\text{increasing } (-5,-3)\cup(-3,-1)\\ \text{decreasing } (-\infty,-5)\cup(-1,\infty)\end{array}}$

$f(-5)=\frac{-25+20-7}{-2}=6$

$f(-1)=\frac{-1+4-7}{2}=-2$

$\boxed{\begin{array}{l}(-5,6)\text{ is rel. minimum}\\ (-1,-2)\text{ is rel. maximum}\end{array}}$

f''(x)

$f''(x)=\frac{(x+3)^2(-2x-6)-(-x^2-6x-5)(2(x+3))}{(x+3)^4}$

$=\frac{-2(x+3)^3-2(x+3)(-x^2-6x-5)}{(x+3)^4}$

$=\frac{-2(x+3)[(x+3)^2-x^2-6x-5]}{(x+3)^4}$

$=\frac{-2[x^2+6x+9-x^2-6x-5]}{(x+3)^3}$

$=\frac{-8}{(x+3)^3}$ inflection at $x=-3$

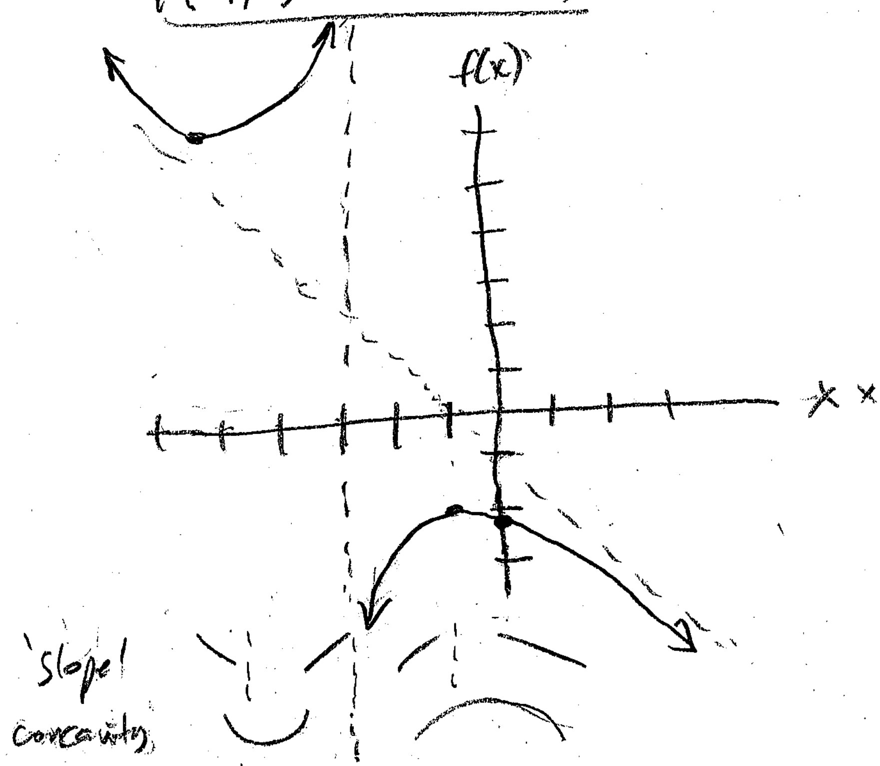
-3

$f''(-4)$: $\frac{(-)}{(-)}$ $+$

$f''(0)$: $\frac{(-)}{(+)}$ $-$

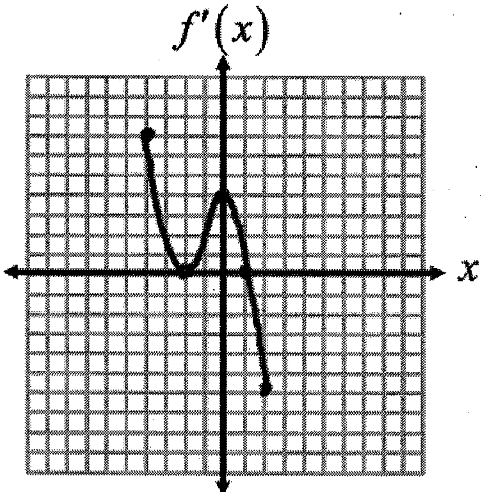
$\boxed{\begin{array}{l}\text{concave up } (-\infty,-3)\\ \text{concave down } (-3,\infty)\end{array}}$

no inflection point at $x=-3$ b/c vert asym. (no point)



Deducing things about f(x) from a graph of f'(x) or f"(x)

Making a table of information in all intervals of the given graph can provide clues...



$(-4,-2)$ $(-2,0)$ $(0,1)$ $(1,2)$

$f'(x)$ + + + −

decreasing increasing decreasing decreasing

a) On what intervals is f increasing? when f' +

$\boxed{(-4,-2) \cup (-2,1)}$

b) On what intervals is the graph of f concave up? when f' is increasing

$\boxed{(-2,0)}$

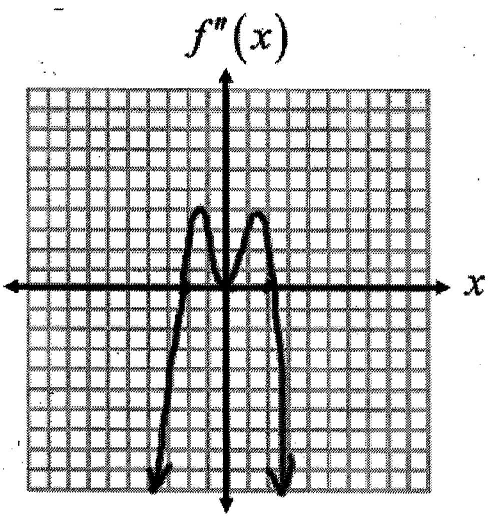
c) At what x-values does f have relative extrema?

$x=-2$ f' + → + no change when $f'(x)=0$ or DNE

$x=1$ f' + → − relative max at $x=1$

d) At what x-values does the graph of f have a point of inflection? when $f''(x)=0$ when $f'(x)$ has horiz tangent

at $\boxed{x=-2 \text{ \& } x=0}$



$(-\infty,-2)$ $(-2,?)$ $(?,0)$ $(0,?)$ $(?,2)$ $(2,\infty)$

$f''(x)$ − + + + + −

increasing increasing decreasing increase decrease decrease

a) On what intervals is f concave up and concave down? sign of $f''(x)$

concave up $(-2,2)$

concave down $(-\infty,-2) \cup (2,\infty)$

b) On what intervals is the graph of f' increasing or decreasing? same as concavity

f' increasing when concave up $(-2,2)$

f' decreasing when concave down $(-\infty,-2) \cup (2,\infty)$

c) At what x-values does f have ~~relative extrema~~? points of inflection?

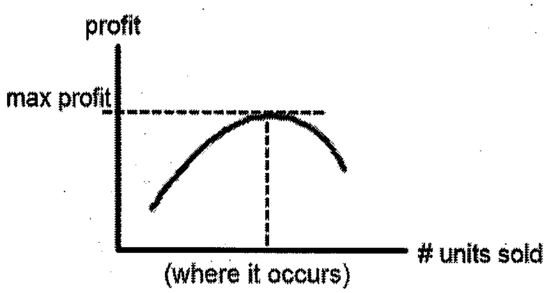
when $f''(x)=0$ and sign of $f''(x)$ changes

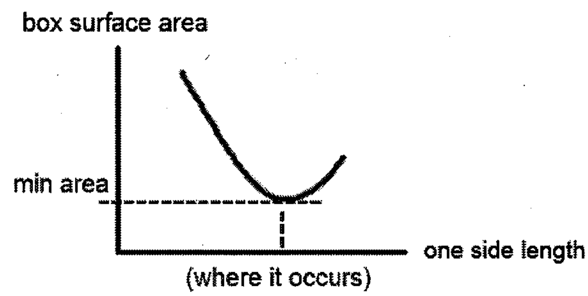
at $\boxed{x=-2 \text{ \& } x=2}$

Unit 4-7: Optimization
Larsen: 3.6 (Stewart: 4.7)

Optimization

Optimization refers to finding the optimum value for a real-world problem - finding the values which make a value either a maximum or a minimum.





Step 1: Identify the quantity to be maximized or minimized.
Step 2: Assign symbols to represent other variables in the problem.
Use an illustration to assist you.
Step 3: Determine the relationships among these variables.
Step 4: Express the quantity to be optimized as a function of **one** of these variables. (Be sure to state the domain).
Step 5: Use first-derivative test to locate relative extrema (where optimum value occurs).
Step 6: Use second-derivative test to verify you have min or max as appropriate.
Step 7: Re-read the problem to make sure you are answering the question (may want the optimum value, where it occurs, or something else).

Terminology to know regarding business problems

Cost: C(x) represents the cost to produce x units.

Demand: p(x) represents the price that is associated with a specific number of units sold.

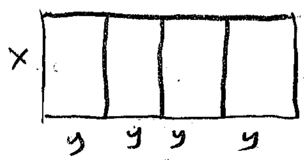
Revenue: R(x) represents the sales (money brought in) from selling x units. If you have a demand (price) function, R(x) = (#units sold)(price) = x * p(x).

Profit: Is the net money made selling x units, revenue - cost: P(x) = R(x) - C(x).

'Marginal': Is the amount to add one additional unit, so it represents the 'derivative of'...marginal cost = C'(x), marginal revenue = R'(x).

Maximum Profit: Since P(x) = R(x) - C(x), P'(x) = R'(x) - C'(x) and max profit occurs when P'(x) = 0, so max profit occurs when R'(x) = C'(x).

Ex) Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?



max $A = 4xy$ $\quad$ $F = 5x + 8y = 750$

$8y = 750 - 5x$

$y = \dfrac{750 - 5x}{8}$

$A = 4x\left(\dfrac{750-5x}{8}\right)$

$A = \frac{1}{2}x(750 - 5x)$

$A = 375x - \frac{5}{2}x^2$

$A' = 375 - 5x = 0$

$5x = 375$

$x = \frac{375}{5} = 75$

$y = \dfrac{750 - 5(75)}{8} = \dfrac{375}{8}$

verify it is a maximum:

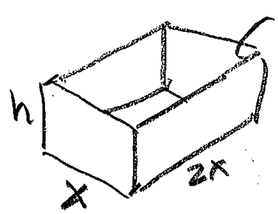
$A'' = -5$ concave down rel max ✓

$A_{max} = 4xy = 4(75)\left(\frac{375}{8}\right)$

$= \dfrac{112500}{8}$

$= \dfrac{28125}{2} \boxed{= 14062.5 \text{ ft}^2}$

Ex) A rectangular storage container with an open top is to have a volume of $10 m^3$. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest container.



$V = 10 \text{ m}^3$

min cost = \$10/m (base area) + \$6/m (sides area)

$V = 2x^2h = 10$

$h = \frac{10}{2x^2} = \frac{5}{x^2}$

$C = 10(2x^2) + 6(2(2xh) + 2xh)$

$C = 20x^2 + 6\left(4x\left(\frac{5}{x^2}\right) + 2x\left(\frac{5}{x^2}\right)\right)$

$C = 20x^2 + 6\left(6x\frac{5}{x^2}\right)$

$C = 20x^2 + \frac{180}{x} = 20x^2 + 180x^{-1}$

$C' = 40x - 180x^{-2} = 40x - \frac{180}{x^2} = 0$

$40x = \frac{180}{x^2}$ $\quad$ $40x^3 = 180$

$x^3 = \frac{18}{4} = \frac{9}{2}$

$x = \pm\sqrt[3]{\frac{9}{2}}$ (not neg.)

$x = \sqrt[3]{\frac{9}{2}}$

verify min

$C'' = 40 + 360x^{-3}$

pos x $\Rightarrow C''+$

concave down, rel. min cost

dimensions:

$x = \sqrt[3]{\frac{9}{2}} \approx 1.65096 m$

$h = \dfrac{5}{\left(\frac{9}{2}\right)^{2/3}} \approx 1.8344 m$

$C_{min} = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \dfrac{180}{\sqrt[3]{\frac{9}{2}}} \approx \boxed{\$163.54}$

Ex) The cost function is $C(x)=680+4x+0.01x^2$.

If a company charges a price p for each unit x, it will be able to sell $x(p)=500(12-p)$ units.

Find the production level that will maximize profit.

$x=500(12-p)$

$12-p=\frac{x}{500}$

price $p(x)=12-\frac{x}{500}$ (demand function)

$R(x)=p\cdot x=12x-\frac{1}{500}x^2$

profit $P(x)=R(x)-C(x)$

$P(x)=12x-\frac{1}{500}x^2-680-4x-0.01x^2$

max profit:

$P'=12-\frac{1}{250}x-4-0.02x=0$

$P'=8-0.024x=0$

$.024x=8$

$x=\frac{8}{.024}=\frac{1000}{3}\approx \boxed{333.333 \text{ units}}$

verify max

$P''=-0.024$

concave down

so this is a

rel. max profit

A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted. (max area)

$A=2rh+\frac{1}{2}\pi r^2$

$P=2r+2h+\frac{1}{2}(2\pi r)=30$

$2r+2h+\pi r=30$

$2h=30-2r-\pi r$

$h=15-r-\frac{\pi}{2}r$

$A=2r\left(15-r-\frac{\pi}{2}r\right)+\frac{1}{2}\pi r^2$

$A=30r-2r^2-\pi r^2+\frac{1}{2}\pi r^2$

$A'=30-4r-2\pi r+\pi r=0$

$r(\pi-2\pi-4)=-30$

$r=\frac{-30}{-\pi-4}=\frac{30}{\pi+4}$

$\boxed{r=\frac{30}{\pi+4} \text{ ft}}\approx 4.201 \text{ ft}$

verify max!

$A''=-4-2\pi+\pi$

$-4-\pi$ $(-)$

concave down

so rel. max area

$h=15-r-\frac{\pi}{2}r$

$=15-\left(1+\frac{\pi}{2}\right)r$

$=15-\left(1+\frac{\pi}{2}\right)\frac{30}{\pi+4}$

$=\frac{15}{1}-\frac{30\left(1+\frac{\pi}{2}\right)}{\pi+4}$

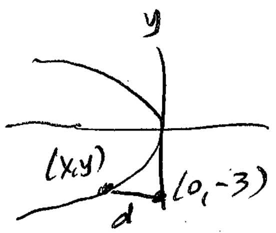
$=\frac{15\pi+60-30-15\pi}{\pi+4}$

$\boxed{h=\frac{30}{\pi+4}}$

so max area when $r=h$ (both $\frac{30}{\pi+4}$ ft)

Ex) Find the point on the parabola $x + y^2 = 0$ that is closest to the point $(0, -3)$.

$x = -y^2$



$\min d = \sqrt{(x-0)^2 + (y-(-3))^2}$

$d = \sqrt{x^2 + (y+3)^2}$

on parabola, so $x + y^2 = 0$

$\leftarrow x = -y^2$

$d = \sqrt{(-y^2)^2 + (y+3)^2} = \sqrt{y^4 + y^2 + 6y + 9} = (y^4 + y^2 + 6y + 9)^{1/2}$

$\left(\frac{dd}{dy}\right)$ $d' = \frac{1}{2}(y^4 + y^2 + 6y + 9)^{-1/2}(4y^3 + 2y + 6) = \frac{4y^3 + 2y + 6}{2\sqrt{y^4 + y^2 + 6y + 9}} = 0$

when $4y^3 + 2y + 6 = 0$

try factoring by synth div: maybe $y = -1$?

$$\begin{array}{r|rrrr} -1 & 4 & 0 & 2 & 6 \\ & & -4 & 4 & -6 \\ \hline & 4 & -4 & 6 & \boxed{0} \end{array}$$

$4y^2 - 4y + 6$

$4y^3 + 2y + 6 = 0$

$(y+1)(4y^2 - 4y + 6) = 0$

$2(y+1)(2y^2 - 2y + 3) = 0$

$y = -1$ or $y = \frac{2 \pm \sqrt{4 - 4(2)(3)}}{2(2)}$ (no real ans)

verify min: d'' too hard so ...

is a local min for distance

-1

$d'(-2) = -3.63$ $\quad$ $d'(0)$ 1

so $y = -1$

$x = -y^2 = -(-1)^2 = -1$

$\boxed{\text{point is } (-1, -1)}$

Ex) During the summer months Terry makes and sells necklaces on the beach. Last summer he sold the necklaces for \$10 each and his sales averaged 20 per day. When he increased the price by \$1, he found that he lost two sales per day.

a) Find the demand function, assuming that it is linear.

b) If the material for each necklace costs Terry \$6, what should the selling price be to maximize his profit?

(a)

x (units)	\$
20	10
18	11

$m = \frac{10-11}{20-18} = \frac{-1}{2}$

$p - 10 = -\frac{1}{2}(x - 20)$

(price) $\boxed{p(x) = -\frac{1}{2}x + 20}$

(b) $R(x) = p \cdot x = -\frac{1}{2}x^2 + 20x$

$C(x) = 6x$

profit $P(x) = R(x) - C(x)$

$P = -\frac{1}{2}x^2 + 20x - 6x = -\frac{1}{2}x^2 + 14x$

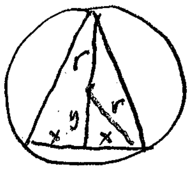
$P' = -x + 14 = 0$

$x = 14$ units

so price should be $p(14) = -\frac{1}{2}(14) + 20 = \boxed{\$13 \text{ per necklace}}$

Ex) Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

(r is a constant throughout problem)



max $A = \frac{1}{2}\,\text{base}\cdot\text{height}$ $x^2+y^2=r^2$

$A = \frac{1}{2}(2x)(y+r)$ $x^2=r^2-y^2$

$A = x(y+r)$ $\leftarrow x=\sqrt{r^2-y^2}$

$A=\sqrt{r^2-y^2}\,(y+r)$ (remember, r is a constant :))

$\left(\frac{dA}{dy}\right)$ $A'=\sqrt{r^2-y^2}\,(1)+(y+r)\frac{1}{2}(r^2-y^2)^{-1/2}(-2y)$

$A'=\sqrt{r^2-y^2}-\frac{y(y+r)}{\sqrt{r^2-y^2}}$

$A'=\frac{r^2-y^2-y^2-yr}{\sqrt{r^2-y^2}}=\frac{r^2-2y^2-yr}{\sqrt{r^2-y^2}}=0$

when $r^2-2y^2-yr=0$

$-2y^2-ry+r^2=0$

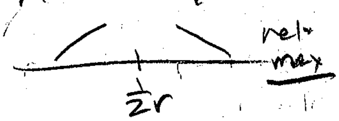
$(-2y+r)(y+r)=0$

$2y=r$ $y=-r$ not possible

$\boxed{y=\frac{1}{2}r}$

verify max

A'' too hard, so ...



$A'(y=0)$ $\frac{r^2}{r}$ $+$ $A''(\frac{3}{4}r)$ $\frac{r^2-(1.5r)^2-.75r^2}{(+)}$ $-$

dimensions:

$y=\frac{1}{2}r$

$x=\sqrt{r^2-(\frac{1}{2}r)^2}$

$x=\sqrt{r^2-\frac{1}{4}r^2}=\sqrt{\frac{3}{4}r^2}$

$\boxed{\text{base}} \neq 2x=\boxed{2\sqrt{\frac{3}{4}r^2}=\sqrt{3}r}$

$\boxed{\text{height}} \neq y+r=\frac{1}{2}r+r=\boxed{\frac{3}{2}r}$

ex. Suppose you run a small independent furniture business. Your assistant signs a deal with a customer to deliver up to 400 chairs, the exact number to be determined by the customer later. The price will be \$90 per chair up to 300 chairs, and above 300, the price will be reduced by \$0.25 per chair (on the whole order) for every additional chair over 300 order. What are the largest and smallest revenues your company can make under this deal?

chairs x	price/chair
300	90
301	89.75

$m=\frac{90-89.75}{300-301}=-.25$

$p-90=-.25(x-300)$

$p(x)=-\frac{1}{4}x+165$

really, $p(x)=\begin{cases}90, & x\le 300\\ -\frac{1}{4}x+165, & x>300\end{cases}$

$R(x)=\text{price}\times\text{units}=(-\frac{1}{4}x+165)x$

if $x\le 300$

$R=90x$

$R'=90$

no extrema

if $x>300$

$R=-\frac{1}{4}x^2+165x$

$R'=-\frac{1}{2}x+165=0$

$\frac{1}{2}x=165$

$x=330$

$R''=-\frac{1}{2}$ max revenue at 330 chairs

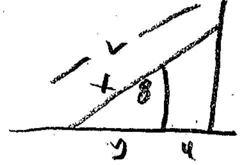
$R_{max}=-\frac{1}{4}(330)^2+165(330)=\boxed{\$27225}$

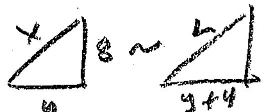
sell up to 400 chairs...

$R(400)=-\frac{1}{4}(400)^2+165(400)$

$\boxed{R_{min}=\$26000 \text{ for } 400 \text{ chairs}}$

($\leftarrow$ assumes you do sell > 300, if you sell < 300 chairs then min $R=0$ for 0 chairs) ex. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?





$y^2+64=x^2$

$y=\sqrt{x^2-64}$

$\frac{L}{x}=\frac{y+4}{y}$

$\frac{L}{x}=\frac{\sqrt{x^2-64}+4}{\sqrt{x^2-64}}$

$L(x)=x+\frac{4x}{\sqrt{x^2-64}}$

$L'=1+\frac{\sqrt{x^2-64}\,(4)-4x\left(\frac{1}{2}(x^2-64)^{-1/2}(2x)\right)}{x^2-64}$

$=1+\frac{4\sqrt{x^2-64}-\frac{4x^2}{\sqrt{x^2-64}}}{x^2-64}$

$=\frac{(x^2-64)+4(x^2-64)^{1/2}-4x^2(x^2-64)^{-1/2}}{x^2-64}$

$=\frac{(x^2-64)^{-1/2}\left[(x^2-64)^{3/2}+4(x^2-64)-4x^2\right]}{x^2-64}$

$=\frac{(x^2-64)^{3/2}-256}{(x^2-64)^{3/2}}=0$ when $(x^2-64)^{3/2}-256=0$

$(x^2-64)^{3/2}=256$

(use calculator intersect)

at $x=10.213593$ ft

(must be positive)

verify mini

$L'(10)$	$10.213...$	$L'(11)$
$-.1852$		$+.405$

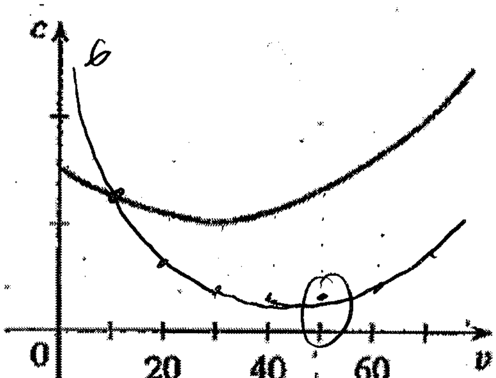
is a rel. min for x

so $L_{min}=x+\frac{4x}{\sqrt{x^2-64}}$ plug in $x=10.213...$

$\boxed{L_{min}=16.648 \text{ ft}}$

46. The graph shows the fuel consumption c of a car (measured in gallons per hour) as a function of the speed v of the car. At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that $c(v)$ is minimized for this car when $v \approx 30$ mi/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in gallons *per mile*. Let's call this consumption G. Using the graph, estimate the speed at which G has its minimum value.

$G=\frac{c}{v}=\frac{g/h}{mi/h}=g/mi$



(divide c by v to get value for G)

around 50-52 mph?