AP Calc BC – Lesson Notes – Unit 4: Integral - Evaluation

Unit 4-1: Antiderivative evaluation using "shortcuts"

Antiderivatives

If we are given a function, we can find the derivative:

$$f(x)=3x^4 \longrightarrow f'(x)=12x^3$$

If we are given a derivative function, could we find the function from which it came:

$$f'(x)=12x^3 \longrightarrow$$

'Reversing' the process of finding the derivative is called finding the **antiderivative**.

Symbols for antiderivatives

If $f(x)=2x$

$F(x)=x^2$ is an antiderivative of $f(x)$

But the following are all also antiderivatives of $f(x)$:

$$F(x)=x^2+1$$
$$F(x)=x^2-22$$
$$F(x)=x^2+15{,}432{,}167$$
$$F(x)=x^2+\frac{2}{7}$$

If $f(x)=2x$

All the antiderivatives of $f(x)$ are of the form:

$$F(x)=x^2+C$$

The process of taking an antiderivative of $f(x)$ is represented with the **integral sign** like this:

$$\int f(x)\,dx=F(x)+C$$

'integration constant'

Compare notation to taking a derivative...

$$\frac{d}{dx}\left[f(x)\right]=f'(x) \qquad \int f(x)\,dx=F(x)+C$$

$\frac{d}{dx}\left[\quad\right]$ = 'find derivative of' $\qquad \int(\quad)\,dx$ = 'find antiderivative of'

Antiderivative Shortcuts

For every basic derivative shortcut, we have an antiderivative shortcut...

$\frac{d}{dx}[C]=0$	$\int 0\,dx=C$
$\frac{d}{dx}[kx]=k$	$\int k\,dx=kx+C$
$\frac{d}{dx}[x^n]=nx^{n-1}$	$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C\quad(n\neq -1)$
$\frac{d}{dx}[e^x]=e^x$	$\int e^x\,dx=e^x+C$
$\frac{d}{dx}[a^x]=(\ln a)\,a^x$	$\int a^x\,dx=\left(\frac{1}{\ln a}\right)a^x+C$
$\frac{d}{dx}[\ln x]=\frac{1}{x}\;(x>0)$	$\int \frac{1}{x}\,dx=\ln\lvert x\rvert+C$
$\frac{d}{dx}[\sin x]=\cos x$	$\int \cos x\,dx=\sin x+C$
$\frac{d}{dx}[\cos x]=-\sin x$	$\int \sin x\,dx=-\cos x+C$
$\frac{d}{dx}[\tan x]=\sec^2 x$	$\int \sec^2 x\,dx=\tan x+C$
$\frac{d}{dx}[\sec x]=\sec x\tan x$	$\int \sec x\tan x\,dx=\sec x+C$
$\frac{d}{dx}[\csc x]=-\csc x\tan x$	$\int \csc x\cot x\,dx=-\csc x+C$
$\frac{d}{dx}[\cot x]=-\csc^2 x$	$\int \csc^2 x\,dx=-\cot x+C$
$\frac{d}{dx}[\arcsin x]=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx=\arcsin x+C$
$\frac{d}{dx}[\arccos x]=\frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{1-x^2}}\,dx=\arccos x+C$
$\frac{d}{dx}[\arctan x]=\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx=\arctan x+C$
$\frac{d}{dx}[\mathrm{arc}\sec x]=\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx=\mathrm{arc}\sec x+C$
$\frac{d}{dx}[\mathrm{arc}\csc x]=\frac{-1}{x\sqrt{x^2-1}}$	$\int \frac{-1}{x\sqrt{x^2-1}}\,dx=\mathrm{arc}\csc x+C$
$\frac{d}{dx}[\mathrm{arc}\cot x]=\frac{-1}{1+x^2}$	$\int \frac{-1}{1+x^2}\,dx=\mathrm{arc}\cot x+C$

Antiderivative Properties

Some of the properties for derivatives are similar for antiderivatives...

$$\frac{d}{dx}[k\,f(x)] = k\,f'(x) \qquad \int k\,f(x)\,dx = k\int f(x)\,dx$$

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) \qquad \int (f(x) \pm g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

...others are different (more about these later)...

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot g'(x) + g(x)\cdot f'(x) \longrightarrow \text{(Integration by Parts)}$$

(Product Rule)

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x) \longrightarrow \text{(Integration by Substitution)}$$

(Chain Rule)

Examples

#1. Find the antiderivative:

$$\int \left(3e^x + 7\sec^2(x)\right) dx$$

$$\boxed{3e^x + 7\tan(x) + C}$$

#2. Find $f(x)$ given $f''(x)$:

$f''(x) = 1 + x^{4/5}$

$f'(x) = \int (1 + x^{4/5})\,dx$

$f'(x) = x + \frac{5}{9}x^{9/5} + C$

$f(x) = \int (x + \frac{5}{9}x^{9/5} + C)\,dx$

$$\boxed{f(x) = \frac{1}{2}x^2 + (\frac{5}{9})(\frac{5}{14})x^{14/5} + Cx + D}$$

If you are given an 'x-y pair', you can solve for the integration constant

#3. Find $f(x)$ given $f'(x)$:

$f'(x) = 3\sqrt{x} - \frac{1}{\sqrt{x}}, \qquad f(1) = 2$

$f'(x) = 3x^{1/2} - x^{-1/2}$

$f(x) = \int (3x^{1/2} - x^{-1/2})\,dx$

$f(x) = 3(\frac{2}{3})x^{3/2} - 2(x^{1/2}) + C$

$f(x) = 2x^{3/2} - 2x^{1/2} + C$

$2 = 2(1)^{3/2} - 2(1)^{1/2} + C$

$2 = 2 - 2 + C \rightarrow C = 2$

$$\boxed{f(x) = 2x^{3/2} - 2x^{1/2} + 2}$$

If you must integrate twice, you'll need two 'x-y pairs' to find both constants

#4. Find $f(x)$ given $f''(x)$:

$$f''(x)=12x^2-6x+2, \quad f(0)=1, \quad f(2)=11$$

$f'(x)=\int(12x^2-6x+2)dx$

$f'(x)=4x^3-3x^2+2x+C$

$f(x)=\int(4x^3-3x^2+2x+C)dx$

$f(x)=x^4-x^3+x^2+Cx+D$

System:

$$\begin{cases}(0)^4-(0)^3+(0)^2+C(0)+D=1\\(2)^4-(2)^3+(2)^2+C(2)+D=11\end{cases}$$

$$\begin{cases}D=1\\12+2C+D=11\end{cases}$$

$12+2C+(1)=11$

$2C=-2, \quad C=-1$

$C=-1, \ D=1$

$$\boxed{f(x)=x^4-x^3+x^2-x+1}$$

Application: Physics

Distance (displacement)	$s(t)$		
	$\downarrow \frac{d}{dx}$	$\uparrow \int dx$	
Velocity (magnitude = speed)	$v(t)=s'(t)$		
	$\downarrow \frac{d}{dx}$	$\uparrow \int dx$	
Acceleration	$a(t)=v'(t)=s''(t)$		

In Earth's gravity, acceleration is a constant...

$$a(t)=-32\ ^{ft}/_{s^2} \qquad a(t)=-9.81\ ^{m}/_{s^2}$$

If an object is only being acted upon by gravity (called 'ballistic motion'), then we start with acceleration and use antiderivatives to find functions for velocity, then displacement.

#5. A particle moves along the x-axis such that the velocity in cm^2/sec at time t in seconds is given by $v(t)=3t^2-2t+4$
If the particle is at $x = 3$ when $t = 2$ seconds, what is the function for the position of the particle, $x(t)$?

$x(t)=\int v(t)dt=\int(3t^2-2t+4)dt$

$x(t)=t^3-t^2+4t+C$

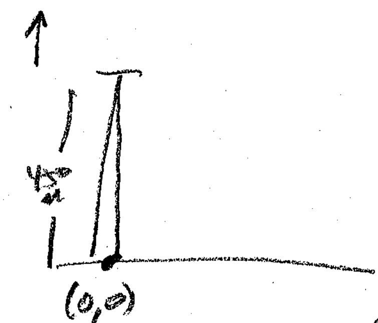
$3=(2)^3-(2)^2+4(2)+C$

$3=12+C$

$C=-9$

$$\boxed{x(t)=t^3-t^2+4t-9}$$

up = positive; s(t)



$g = -9.8 \text{ m/s}^2$ (downward)

#6. A stone is thrown downward with a speed of 5 m/s from the top of a 450 m tall tower.

a) Find the distance of the stone above the ground level at time *t*.

b) How long does it take the stone to reach the ground?

c) With what velocity does the stone strike the ground?

a) $a(t) = -9.8$

$v(t) = \int a(t)\,dt = \int -9.8\,dt = -9.8t + C$

$v(0) = -5$ m/s

$-5 = -9.8(0) + C \Rightarrow C = -5$

$v(t) = -9.8t - 5$

$s(t) = \int v(t)\,dt = \int (-9.8t - 5)\,dt = -4.9t^2 - 5t + D$

$s(0) = 450$

$450 = -4.9(0)^2 - 5(0) + D \Rightarrow D = 450$

$\boxed{s(t) = -4.9t^2 - 5t + 450}$

b) on ground when $s(t) = 0$, at $\boxed{t = 9.087 \text{ sec}}$

c) $v(9.0865163) = -9.8(9.0865163) - 5$

$= \boxed{-94.048 \text{ m/sec}}$

Unit 4-2: Riemann Sums and the Definite Integral

Sigma Notation

In various parts of the course, we will need to 'sum' things (add them, accumulate) and one way to compactly represent summation is with **Sigma Notation**:

$$\sum_{i=c}^{d} a_i = a_c + a_{c+1} + a_{c+2} + \dots + a_d$$

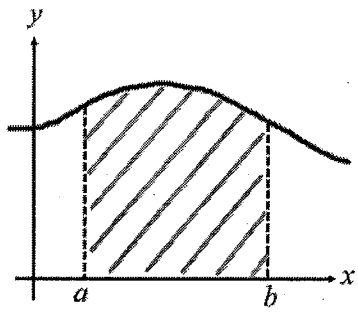
A couple of examples...

$$\sum_{k=0}^{3}(2k+3) = (2(0)+3)+(2(1)+3)+(2(2)+3)+(2(3)+3)$$
$$=(3)+(5)+(7)+(9)=24$$

$$\sum_{i=2}^{5}(i^2) = (2^2)+(3^2)+(4^2)+(5^2)$$
$$=(4)+(9)+(16)+(25)=54$$

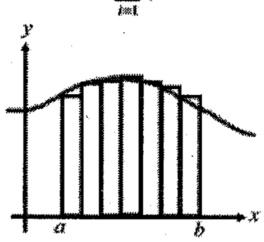
Finding Area under a function curve using a Summation

Imagine we wanted to find the approximate area under a function curve between two x-values....

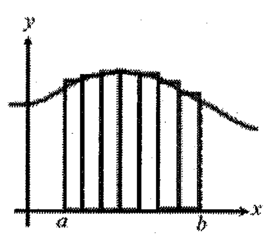


We could use a summation to add the areas of rectangles which approximately fill the area. We can use equal-width rectangles, and let the function curve's value establish the height for each rectangle...

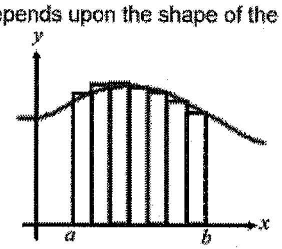
$$Area = \sum_{i=1}^{n}(\textit{individual rectangle areas})$$ (whether these estimates underestimate or overestimate the area depends upon the shape of the curve)



...using the lower edge of the rectangle's x to establish height (called a left sum)



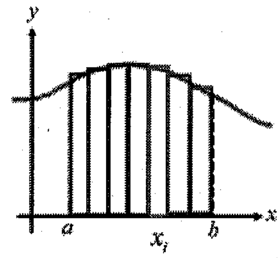
...using the middle of the rectangle's x to establish height (called 'midpoint rule')



...using the upper edge of the rectangle's x to establish height (called an right sum)

Finding Area under a function curve using a Summation

Whichever rule we use to choose the x-value, for each rectangle, there is chosen x-value which we can plug into the function to establish that rectangle's height.



If we divide the x-distance between a and b into n rectangles, the width of each rectangle will be: $width = \frac{b-a}{n}$.

To find the x value for the i^{th} rectangle, we start at a and add i widths: $x_i = a + i\left(\frac{b-a}{n}\right)$

For the left side x, i starts at 0. For the right side x, i starts at 1.
(We would need a more complex expression to find the middle x.)

The height of this rectangle is then... $height = f(x_i)$

...and this rectangle's area is: $area = height \cdot width = f(x_i)\cdot\left(\frac{b-a}{n}\right)$

We then sum the individual rectangle areas: $Area = \sum_{i=1}^{n} f(x_i)\cdot\left(\frac{b-a}{n}\right)$ complicated notation for a simple idea!

#1. Use the right side of the rectangles with n = 4 to approximate the area of the region bounded by the x-axis and the graph of the function over the given interval.

$$f(x) = x^2 + 4x, \quad [0,4]$$

interval	x_i	$f(x_i)$	$\cdot\ \Delta x$	= area
[0,1]	1	5	· 1	= 5
[1,2]	2	12	· 1	= 12
[2,3]	3	21	· 1	= 21
[3,4]	4	32	· 1	= 32

area $\approx [5\cdot 1 + 12\cdot 1 + 21\cdot 1 + 32\cdot 1] = 70$

Finding Area under a function curve using the Limit Definition

Of course, this approximation will be more accurate if we include more rectangles!

So we make the number of rectangles infinite by taking the limit as the number of rectangles, n, approaches infinity:

$$Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\left(\frac{b-a}{n}\right)$$

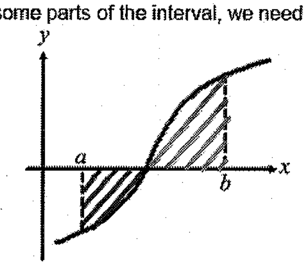
The Fundamental Theorem of Calculus (which we'll learn more about later) shows that this limit is actually the same as a definite integral:

$$Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\left(\frac{b-a}{n}\right)$$

$$Area = \int_a^b f(x)\,dx$$

What if f(x) is negative?

If a function is negative for some parts of the interval, we need to decide how to handle that situation.



Usually, we want the negative part to 'cancel' out part of the positive part of the sum, and this is how we interpret negative f(x) when the calculation is representing a definite integral.

If, for some reason, we instead want the physical area, we need to break the function up so we can calculate the positive and negative areas separately and 'negate' the negative areas so they become positive areas to add to the total area.

Riemann Sums

When we use shapes to compute an area for a curve we call the approximate area a **Riemann Sum.**
Here is the formal definition:

> *Let f be defined on the closed interval $[a,b]$,*
> *and let Δ be a partition of $[a,b]$ given by*
> $a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b$
> *where Δx_i is the width of the ith subinterval: $[x_{i-1}, x_i]$*
> *If c_i is any point in the ith subinterval, then the sum*
> $$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$
> *is called a* Riemann Sum *of f for the partition Δ.*

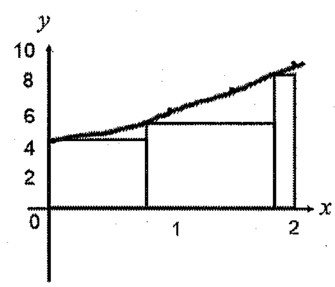
Some things to note:

- The function curve can be either positive or negative and negative cancels positive.
- The shapes used don't have to have equal widths.
- In fact, the shapes don't have to be rectangles.

#2. The values of a function are shown in a table for specific x-values. Evaluate a Riemann Sum using rectangular partitions, left endpoints, and the specific subintervals given:

x	0.00	0.50	0.75	1.00	1.50	1.75	2.00
y	4.32	4.58	5.79	6.14	7.64	8.08	8.14

subintervals: [0.00, 0.75] [0.75, 1.75] [1.75, 2.00]



Interval	x_i	$f(x_i) \cdot \Delta x$	= area
[0, 0.75]	0	4.32 · 0.75	= 3.24
[0.75, 1.75]	0.75	5.79 · 1	= 5.79
[1.75, 2]	1.75	8.08 · 0.25	= 2.02

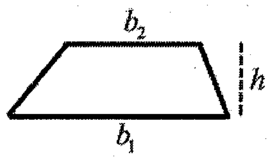
area ≈ $[(4.32)(0.75) + (5.79)(1) + (8.08)(0.25)] = 11.05$

Riemann Sums using the Trapezoidal Rule

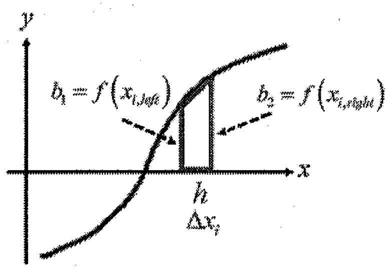
Riemann Sums allow us to use any reasonable shape to estimate the area of a subinterval, and one that is common and may appear on the AP Calculus Exam is called the **Trapezoidal Rule**.

The area of a trapezoid is given by the formula

$$area_{trapezoid} = \frac{1}{2}(b_1 + b_2)h$$

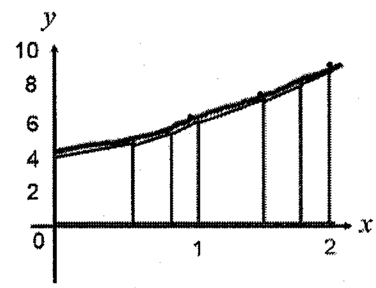


To use a trapezoid as an area element, we turn it on it side, so that the 'height' of the trapezoid is the width of the subinterval, and the 'bases' are the left and right endpoint function values:



#3. The values of a function are shown in a table for specific x-values. Evaluate a Riemann Sum using the Trapezoidal Rule for all the partitions included in the table.

x	0.00	0.50	0.75	1.00	1.50	1.75	2.00
y	4.32	4.58	5.79	6.14	7.64	8.08	8.14



interval	f_L	f_R	Δx	$area = \frac{1}{2}(f_L + f_R)\Delta x$
[0, 0.5]	4.32	4.58	0.5	2.225
[0.5, 0.75]	4.58	5.79	0.25	1.29625
[0.75, 1]	5.79	6.14	0.25	1.49125
[1, 1.5]	6.14	7.64	0.5	3.445
[1.5, 1.75]	7.64	8.08	0.25	1.965
[1.75, 2]	8.08	8.14	0.25	2.0275

area $\approx$ sum $=$ 12.45

The Definite Integral

Riemann Sums allow any region with negative $f(x)$ to add negative contributions to the accumulation. This turns out to be extremely useful, as we will explore in later sections.

For now, we will also use a notation called the **Definite Integral** to represent a Riemann Sum with infinitely many subintervals:

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i}$$

Definite Integral

$$\int_a^b f(x)\,dx$$

...means to compute a value for the summation of the area between the function curve with negative contributions for negative *f(x)* regions.

$\boxed{\text{Result = a number}}$

Indefinite Integral

$$\int f(x)\,dx$$

...means to find the antiderivative of the function *f(x)*, including the integration constant.

$\boxed{\text{Result = a family of functions}}$

#4. Approximate the value of $\int_1^7 x^2\,dx$ using a right Riemann Sum with 3 equal-size intervals.

width $= \frac{7-1}{3} = \frac{6}{3} = 2$

interval	x_i	$f(x_i)$	$\cdot\ \Delta x$	$=$ area
[1,3]	3	$(3)^2=9$	$\cdot 2$	18
[3,5]	5	$(5)^2=25$	$\cdot 2$	50
[5,7]	7	$(7)^2=49$	$\cdot 2$	98

$$\int_1^7 x^2\,dx \approx [18+50+98] = 166$$

Properties of the Definite Integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
where c is between a and b

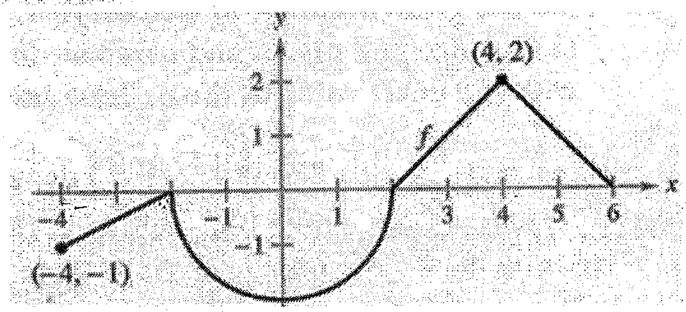
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$\int_a^b [f(x)dx \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\textit{If } f(x) \le g(x) \textit{ on } [a,b], \textit{ then } \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Using properties and geometry to evaluate definite integrals

Evaluate the definite integrals using the graph of $f(x)$



#1. $\int_{-4}^{2} f(x)\,dx$

Interval	area
$[-4,-2]$	$\ominus \frac{1}{2}(2)(1) = -1$
$[-2,2]$	$\ominus \frac{1}{2}\pi(2)^2 = -2\pi$

$\int_{-4}^{2} f(x)\,dx = -1 - 2\pi$

#2. $\int_{0}^{6} f(x)\,dx$

interval	area
$[0,2]$	$\ominus \frac{1}{4}\pi(2)^2 = -\pi$
$[2,6]$	$\frac{1}{2}(4)(2) = 4$

$\int_{0}^{6} f(x)\,dx = 4 - \pi$

#3. $\int_{6}^{0} f(x)\,dx = -\int_{0}^{6} f(x)\,dx = -(4-\pi) = \pi - 4$

Important! In order to use geometry to evaluate a definite integral the lower limit of integration must be lower than the upper limit of integration. If they are reversed, use properties to reverse them.

(Technically, this isn't called a Riemann Sum because the areas aren't all being found using a height times a width and the result is an exact value, not an approximation).

Finding the Definite Interval a Riemann Sum is approximating

We solved this problem previously...

Approximate the value of $\int_1^7 x^2\,dx$ using a right Riemann Sum with 3 equal-size intervals.

interval	x_i	$f(x_i)$	*	Δx	=	area
[1, 3]	3	$(3)^2$	*	2	=	18
[3, 5]	5	$(5)^2$	*	2	=	50
[5, 7]	7	$(7)^2$	*	2	=	98
						166

$$\int_1^7 x^2\,dx \approx 166$$

$$\approx\left[(3)^2\cdot 2+(5)^2\cdot 2+(7)^2\cdot 2\right]$$

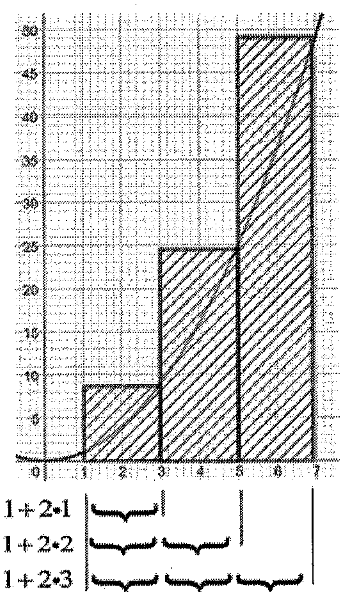
$$\approx 2\left[(3)^2+(5)^2+(7)^2\right] \qquad \textit{structure = integrand}$$

$$\approx 2\Big[\underbrace{(1+2\cdot 1)^2}_{n=1}+\underbrace{(1+2\cdot 2)^2}_{n=2}+\underbrace{(1+2\cdot 3)^2}_{n=3}\Big]$$

$$\textit{width}=\frac{b-a}{n}=\frac{7-1}{3}=2 \qquad a \qquad \textit{width}$$

$$\approx\sum_{n=1}^{3}2\Big[\underbrace{(1+2n)^2}_{\textit{structure = integrand}}\Big]$$

width $\quad a$



#4. $\sum_{n=1}^{3}2\left[(1+2n)^2\right]$ **approximates what definite integral?**

1) $a=1$

2) width $=\dfrac{b-a}{n}=2$, $a=1$, $n=3$, so $\dfrac{b-1}{3}=2$

$b-1=6$

$b=7$

3) $(1+2n)^2 \rightarrow x^2$

4) $\sum_{n=1}^{\infty} 2[(1+2n)^2] \approx \int_1^7 x^2\,dx$

Riemann Sum to Definite Integral

What if there were an infinite number of rectangles?

#5. $\lim_{n\to\infty}\frac{2}{n}\left[\left(1+\frac{2}{n}\right)^3+\left(1+\frac{4}{n}\right)^3+\left(1+\frac{6}{n}\right)^3+....+\left(1+\frac{2n}{n}\right)^3\right]$ is equivalent to what definite integral?

$$\lim_{n\to\infty}\frac{2}{n}\left[\left(1+\frac{2}{n}\right)^3+\left(1+\frac{4}{n}\right)^3+\left(1+\frac{6}{n}\right)^3+....+\overbrace{\left(1+\frac{2n}{n}\right)}^{\textit{structure = integrand}}{}^3\right]$$

$width = \frac{b-a}{n}$

a

this must be width times n

#5. $\lim_{n\to\infty}\frac{2}{n}\left[\left(1+\frac{2}{n}\right)^3+\left(1+\frac{4}{n}\right)^3+\left(1+\frac{6}{n}\right)^3+....+\left(1+\frac{2n}{n}\right)^3\right]$ is equivalent to what definite integral?

1) $a=1$

2) width $= \frac{b-a}{n} = \frac{2}{n}$, $b-a=2$
$b-1=2 \rightarrow b=3$

3) $\left(1+\frac{2}{n}\right)^3 \rightarrow x^3$

4) this is $\approx \int_1^3 x^3\,dx$

Find the definite integral which the following Riemann Sum approximates:

#6. $\lim_{n\to\infty}\frac{1}{n}\left[\left(3+\frac{1}{n}\right)^2+\left(3+\frac{2}{n}\right)^2+\left(3+\frac{3}{n}\right)^2+....+\left(3+\frac{n}{n}\right)^2\right]$

1) $a=3$

2) width $= \frac{b-a}{n} = \frac{1}{n}$, $b-a=1$
$b-3=1$, $b=4$

3) $\left(3+\frac{1}{n}\right)^2 \rightarrow x^2$

4) this is $\approx \int_3^4 x^2\,dx$

Unit 4-3: The Fundamental Theorem of Calculus - Evaluating Definite Integrals

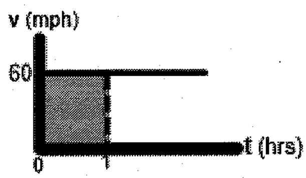
The Fundamental Theorem of Calculus

Start with two seemingly different ideas (that use the same notation)...

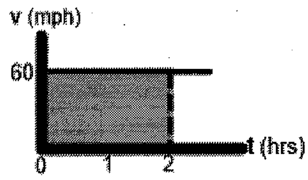
$$\int f(x)\,dx \qquad\qquad \int_a^b f(x)\,dx$$

Antiderivative of $f(x)$, $F(x)$ — Area under $f(x)$ curve from x=a to x=b

Consider a car traveling at a constant 60 mph:



$$\begin{aligned} \textit{area of rect} &= v(t)\cdot\Delta t \\ &= 60\,\frac{miles}{hr}\cdot 1\,hr \\ &= 60\ miles \\ &= \textit{distance traveled from } 0-1\ hrs \end{aligned}$$



$$\begin{aligned} \textit{area of rect} &= v(t)\cdot\Delta t \\ &= 60\,\frac{miles}{hr}\cdot 2\,hr \\ &= 120\ miles \\ &= \textit{distance traveled from } 0-2\ hrs \end{aligned}$$

Area under the velocity curve = the total (accumulated) distance traveled

But we also know that the velocity function is the derivative of the distance (displacement) function... $v(t)=s'(t)$

...and therefore the distance function is the antiderivative of the velocity function.

$$s(t)=\int v(t)\,dt$$

$$s(t)=V(t)$$ antiderivative of the velocity function

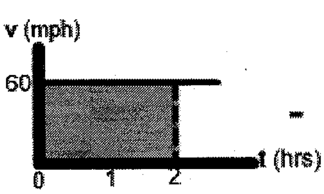
the total distance traveled = antiderivative of velocity

Since the total distance traveled = area under the velocity curve
and the total distance traveled = antiderivative of velocity

Area under the velocity curve = Antiderivative of velocity

$$\int_0^t v(t)\,dt = V(t)$$

And if we wanted to find the distance traveled between time t=1 and t=2
we could subtract one area from the other:



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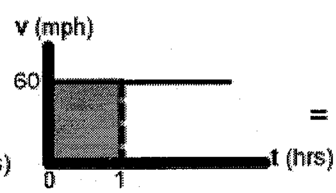
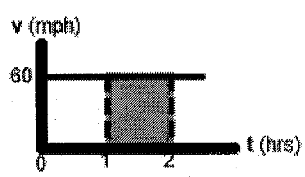


$=$

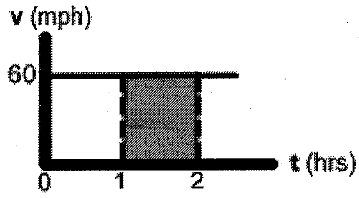


$$\int_0^2 v(t)\,dt = V(2) \qquad \int_0^1 v(t)\,dt = V(1) \qquad \int_1^2 v(t)\,dt = V(2)-V(1)$$

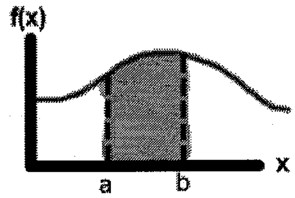
The accumulated distance depends upon finding the values of the antiderivative of velocity (distance) but only at the ends of the interval.

The Fundamental Theorem of Calculus

This idea occurred to two people: Issac Newton and Gottfried Leibniz who further showed that this idea, that the area under a function curve over an x-interval is equal to the antiderivative evaluated at the endpoints, is generalizable to all functions, not just constant functions...



$$\int_1^2 v(t)\,dt = V(2) - V(1)$$



$$\int_a^b f(x)\,dx = F(b) - F(a)$$

...and is called the ***Fundamental Theorem of Calculus (part 2).***

Examples

#1. Evaluate $\int_2^4 \left(x^2 + 2x\right) dx$

$$\frac{x^3}{3} + \frac{2x^2}{2} + c$$

$$\left[\frac{x^3}{3} + x^2 + c\right]_2^4$$

$$\left[\frac{(4)^3}{3} + (4)^2 + c\right] - \left[\frac{(2)^3}{3} + (2)^2 + c\right]$$

$$\frac{(4)^3}{3} + (4)^2 + \cancel{c} - \frac{(2)^3}{3} - (2)^2 - \cancel{c}$$

$$\boxed{\frac{(4)^3}{3} + (4)^2 - \frac{(2)^3}{3} - (2)^2}$$

$$\left(\frac{92}{3}\right)$$

#2. Evaluate $\int_{-1}^3 x^5 dx$

$$\left[\frac{x^6}{6}\right]_{-1}^3$$

$$\boxed{\left[\frac{(3)^6}{6}\right] - \left[\frac{(1)^6}{6}\right]}$$

Proof of The Fundamental Theorem of Calculus (Part 2)

> If a function f is continuous on the closed interval [a,b] and F is an antiderivative of f on the interval [a,b], then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof:

Let Δ be any partition of [a,b]: $a = x_0 < x_1 < x_2 \ldots . < x_{n-1} < x_n = b$

The right hand side can be replaced with the endpoint values, along with the values between these endpoints in the interval, each one added and subtracted back off...

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + F(x_{n-2}) \ldots - F(x_1) + F(x_1) - F(x_0)$$

Regrouping into pairs in order...

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \ldots\ldots + F(x_1) - F(x_0)$$

This can now be expressed as a summation...

$$F(b) - F(a) = \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1}) \right]$$

By the Mean Value Theorem, we know that there exists a number c_i in the i^{th} subinterval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

therefore $\quad F(x_i) - F(x_{i-1}) = F'(c_i) \bullet (x_i - x_{i-1})$

Because F is the antiderivative of f, $F'(c_i) = f(c_i)$ and you define $\Delta x_i = x_i - x_{i-1}$, then...

$$F(x_i) - F(x_{i-1}) = f(c_i)\,\Delta x_i$$

From earlier: $\quad F(b) - F(a) = \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1}) \right]$

$$F(b) - F(a) = \sum_{i=1}^{n} \left[f(c_i)\,\Delta x_i \right]$$

This result tells us that by repeatedly applying the Mean Value Theorem, you can always find a collection of c_i values such that the *constant F(b) - F(a)* is a Riemann Sum of f on [a,b] for any partition.

Now taking a limit to apply infinitely many subintervals:

$$F(b) - F(a) = \lim_{n \to \infty} \sum_{i=1}^{n} \left[f(c_i)\,\Delta x_i \right]$$

$$\boxed{F(b) - F(a) = \int_a^b f(x)\,dx}$$

Evaluate...

#3. $g(x)=\int_2^x t^2\,dt$

$\left[\frac{1}{3}t^3\right]_2^x$

$\boxed{\left[\frac{1}{3}(x)^3\right]-\left[\frac{1}{3}(2)^3\right]}$

#4. $g(x)=\int_3^{x^2} 3t^3\,dt$

$\left[\frac{3}{4}t^4\right]_3^{x^2}$

$\boxed{\left[\frac{3}{4}(x^2)^4\right]-\left[\frac{3}{4}(3)^4\right]}$

What if we took the derivative of the result of the last two examples?

#5. $g(x)=\int_2^x t^2\,dt = \frac{d}{dx}\left[\int_2^x t^2 dt\right]=(x)^2$

Find $g'(x)$

$\frac{d}{dx}\left[\frac{1}{3}(x)^3-\frac{1}{3}(2)^3\right]$

$(x)^2 - 0$

x^2

#6. $g(x)=\int_3^{x^2} 3t^3\,dt = \frac{d}{dx}\left[\int_3^{x^2} 3t^3 dt\right]=3(x^2)^3 2x$

Find $g'(x)$

$\frac{d}{dx}\left[\frac{3}{4}(x^2)^4-\frac{3}{4}(3)^4\right]$

$3(x^2)^3\cdot 2x - 0$

$3(x^2)^3\cdot(2x)$

This can be extended by the chain rule to a more general (and useful) form:

> If a function f is continuous on an open interval I containing a, then, for every x in the interval
>
> $$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right]=f(g(x))\cdot g'(x)$$

This result is known as the 'other part' of the Fundamental Theorem of Calculus (part 1 in our textbook):

> If a function f is continuous on an open interval I containing a, then, for every x in the interval
>
> $$\frac{d}{dx}\left[\int_a^{x} f(t)\,dt\right]=f(x)$$

...or even more generally:

> If a function f is continuous on an open interval I containing a, then, for every x in the interval
>
> $$\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)\,dt\right]=f(g(x))\cdot g'(x)-f(h(x))\cdot h'(x)$$

Evaluate

#7. $\frac{d}{dx}\left[\int_{x^2}^{3x^4} f(t)\,dt\right]$

$\boxed{f(3x^4)\cdot 12x^3 - f(x^2)\,2x}$

#8. $\frac{d}{dx}\left[\int_{x^3}^{2} f(t)\,dt\right]$

$\cancel{f(2)\cdot 0} - f(x^3)\,3x^2$

— or —

$-\frac{d}{dx}\left[\int_{2}^{x^3} f(t)\,dt\right]$

$\boxed{-\ f(x^3)\,3x^2}$

#9. Find the interval on which the curve is concave upward.

$$y=\int_0^x \frac{1}{1+t+t^2}\,dt$$

$y' = \frac{d}{dx}\left[\int_0^x \frac{1}{1+t+t^2}\,dt\right] = \frac{1}{1+x+x^2} = (1+x+x^2)^{-1}$

$y'' = -(1+x+x^2)^{-2}(1+2x) = \frac{-(1+2x)}{(1+x+x^2)^2}$ $\quad \frac{(-)(?)}{(+)}$

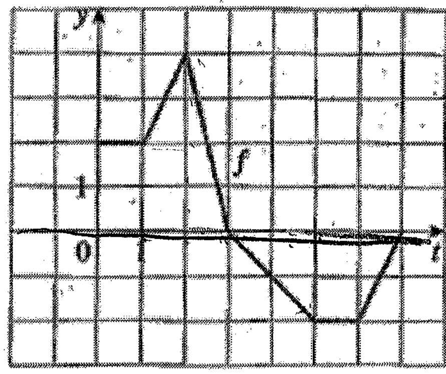
y is concave up when $y''>0$ when $1+2x<0$

$2x < -1$

$\boxed{x < -\frac{1}{2}}$

Examples

#10. Let $g(x)=\int_0^x f(t)dt$ where the graph of f is shown:



a) Evaluate $g(5)$

b) On what interval is $g(x)$ increasing?

c) Where does $g(x)$ have a maximum value?

a) $g(5)=\int_0^5 f(t)dt = 2+2+\frac{1}{2}(1)(2)+\frac{1}{2}(1)(4)-\frac{1}{2}(2)(2)$
$= \boxed{5}$

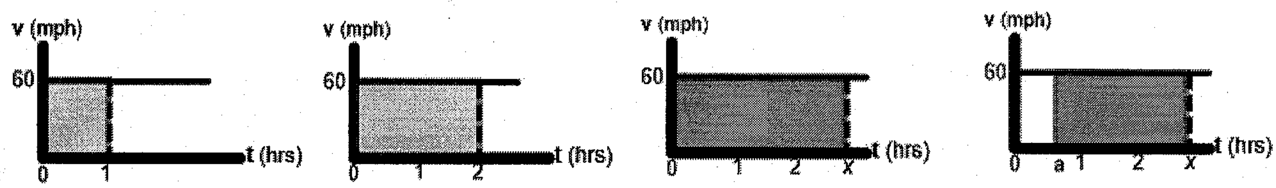
b) $g(x)$ is increasing as the area under the $f(t)$ curve is increasing. $\boxed{(0,3)}$

c) $g(x)$ max at $\boxed{t=3}$

Unit 4-4: Start Plus Accumulation,
Displacement vs. Total Distance Travelled
Average Value of a Function vs Average Rate of Change

Start Plus Accumulation Method (SPAM)

We first considered the Fundamental Theorem of Calculus using the example of a car driving at a constant velocity, and determined that the area under the function curve represents the accumulated distance traveled from time t = 0:



If we leave the ending value as a variable, *x*, then we are producing a <u>function</u> of time (with variable *x* allowing us to set the stop time later) and this function gives us the accumulated distance from zero to that time, *x*.

$\int_0^1 v(x)\,dx$ $=V(1)-V(0)$ $=s(1)-s(0)$ *a number representing distance travelled*	$\int_0^2 v(x)\,dx$ $=V(2)-V(0)$ $=s(2)-s(0)$ *a number representing distance travelled*	$\int_0^x v(t)\,dt$ $=V(x)-V(0)$ $=s(x)-s(0)$ *a <u>function</u> which gives us distance travelled for any x*	$\int_a^x v(t)\,dt$ $=V(x)-V(a)$ $=s(x)-s(a)$ *a <u>function</u> which gives us distance travelled from time t = a to any t = x*

The resulting function is sometimes called an <u>**accumulation function**</u>.

Although we've shown this for a constant velocity function, this can be shown to be true for any function. Conceptually, this means that if you have a function which represents the <u>derivative</u> of a quantity (how it is changing, for example, over time), then integrating that function from a constant to *x* will give a function for how that quantity <u>accumulates</u>.

We can use this fact to solve problems where we are given the a quantity at a known time and the rate at which that quantity is changing (its derivative) and then asked to find the quantity at a 2nd, unknown time.

#1. A liquid flows into a storage tank at a rate of $(180+3t)$ liters per minute. If there is 40 liters of liquid in the tank at time t = 2 minutes, how much liquid is in the tank at t = 10 minutes?

$$\int_2^{10} w'(t)\,dt = w(10)-w(2)$$

$$\int_2^{10} (180+3t)\,dt = w(10)-40$$

$$\left[180t+\tfrac{3}{2}t^2\right]_2^{10}$$

$$\left[180(10)+\tfrac{3}{2}(10)^2\right]-\left[180(2)+\tfrac{3}{2}(2)^2\right] = w(10)-40$$

$$1584 = w(10)-40$$

$$\boxed{w(10) = 1624 \text{ liters}}$$

Start Plus Accumulation Method (SPAM)

We can summarize this idea with what we are calling the 'Start Plus Accumulation Method (SPAM)' (not an official term). Various textbooks use other names for this (our textbook calls this the 'Net Change Theorem'):

$$\boxed{f(t_{end}) = f(t_{start}) + \int_{t_{start}}^{t_{end}} f'(x)\,dx}$$

#2. A hot air balloon's height above the ground is changing at a rate given by $h'(t) = -110t + 550$ where h is in feet and t is in hours.
If the hot air balloon is on the ground at time $t = 0$, what is the height of the balloon at $t = 2$ hours?

$h(2) = h(0) + \int_0^2 (-110t + 550)\,dt$

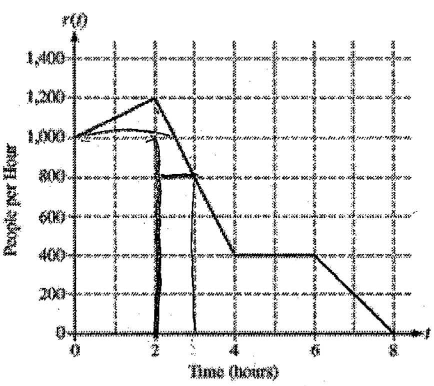
$h(2) = 0 + \left[-55t^2 + 550t\right]_0^2$

$h(2) = 0 + \left[-55(2)^2 + 550(2)\right] - (0)$

$\boxed{h(2) = 880 \text{ ft}}$

#3. There are 700 people in line for a popular amusement-park ride when the ride begins operation in the morning. Once it begins operation, the ride accepts passengers until the park closes 8 hours later. While there is a line, people move onto the ride at a rate of 800 people per hour. The graph above shows the rate, $r(t)$, at which people arrive at the ride throughout the day. Time t is measured in hours from the time the ride begins operation.

How many people are in line for the ride at $t = 3$ hours?



$P(t) = r(t) - 800$

$P(3) = P(0) + \int_0^3 [r(t) - 800]\,dt$

$P(3) = P(0) + \int_0^3 r(t)\,dt - \int_0^3 800\,dt$

$P(3) = 700 + \left[(2)(1000) + (2)(200) + (1)(400) + (1)(800)\right] - \left[800t\right]_0^3$

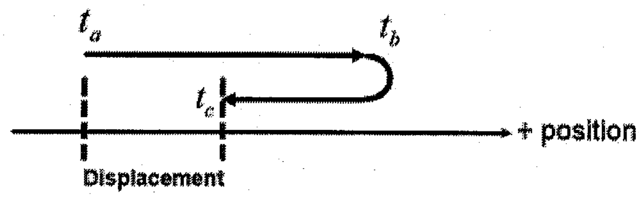
$P(3) = 700 + 3600 - \left[800(3) - 800(0)\right]$

$\boxed{P(3) = 1900 \text{ people}}$

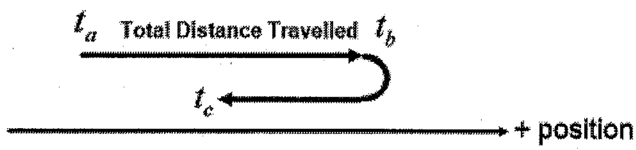
Working with velocity: Displacement vs. Total Distance Travelled

If the rate of change we are working with is velocity (so we are accumulating position or distance) we need to be cognizant of an important distinction:

Displacement
is the different between the start and end positions



Total Distance Travelled
is the length of the complete path taken



#4. The velocity function, in feet per second, is $v(t) = t^2 - t - 12 \quad for \ 1 \le t \le 5$ for a particle moving along a straight line.

a) Find the displacement over the interval.

b) Find the total distance that the particle travels over the given interval.

a) displacement $= s(5) - s(1) = \int_1^5 (t^2 - t - 12)\,dt$

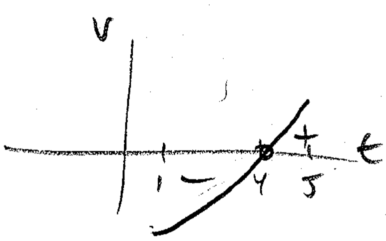
$= \left[\frac{1}{3}t^3 - \frac{1}{2}t^2 - 12t\right]_1^5$

$= \left[\frac{1}{3}(5)^3 - \frac{1}{2}(5)^2 - 12(5)\right] - \left[\frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 - 12(1)\right]$

* show math *

$= \boxed{-18.667}$

b) total distance traveled $= -\int_1^4 (t^2 - t - 12)\,dt + \int_4^5 (t^2 - t - 12)\,dt$



$-(-22.5) + (3.833\overline{3})$

$= \boxed{26.333}$

Unit 4-5: Integration by Substitution

Integration by Substitution

Some integrals cannot be evaluated by using the basic integration formulas, so we need other integration techniques. One of these is <u>integration by substitution</u> which is based on the Chain Rule.

<u>Derivative using Chain Rule</u>

$y=\left(x^2+5\right)^4$

useful when one function is 'inside' of another

$u=x^2+5 \qquad y=u^4$

$\frac{du}{dx}=2x \qquad \frac{dy}{du}=4u^3$

$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx}=4u^3(2x)$

$\frac{dy}{dx}=4\left(x^2+5\right)^3(2x)$

<u>Integration by Substitution</u>

$\int 4\left(x^2+5\right)^3 2x\,dx$

$u=x^2+5$

$\frac{du}{dx}=2x$

$du=2x\,dx$

$4\int u^3du$

$4\left[\frac{1}{4}u^4\right]+C$

u^4+C

$\left(x^2+5\right)^4+C$

1) define the 'inside' function to be u.

2) Find du/dx and solve for du to get a 'toolkit' with du and u.

3) Substitution all expressions with x and dx in the original integral to obtain a new integral using u as the variable.

4) Integrate, then resubstitute u to use the original x variable.

#1. $\int x^2\sqrt{2x^3-3}\,dx$

$u=2x^3-3$

$\frac{du}{dx}=6x^2$

$du=6x^2dx$

$x^2dx=\frac{1}{6}du$

$\int\sqrt{u}\,\frac{1}{6}du$

$\frac{1}{6}\int u^{1/2}du$

$\frac{1}{6}\frac{2}{3}u^{3/2}+C$

$\boxed{\frac{1}{9}\left(2x^3-3\right)^{3/2}+C}$

#2. $\int x^3\left(1-x^4\right)^5 dx$

$u=1-x^4$

$\frac{du}{dx}=-4x^3$

$du=-4x^3dx$

$x^3dx=-\frac{1}{4}du$

$\int u^5\left(-\frac{1}{4}du\right)$

$-\frac{1}{4}\int u^5du$

$-\frac{1}{4}\frac{u^6}{6}+C$

$\boxed{-\frac{1}{24}\left(1-x^4\right)^6+C}$

#3. $\int\frac{x^2}{\sqrt{1-x}}dx$

$u=1-x \quad , \quad x=1-u$

$\frac{du}{dx}=-1$

$du=-dx$

$dx=-du$

$\int\frac{(1-u)^2}{\sqrt{u}}(-du)$

$-\int u^{-1/2}(1-u)^2du$

$-\int u^{-1/2}\left(1-2u+u^2\right)du$

$-\int\left(u^{-1/2}-2uu^{-1/2}+u^2u^{-1/2}\right)du$

$-\int\left(u^{-1/2}-2u^{1/2}+u^{3/2}\right)du$

$-\left[2u^{1/2}-2\left(\frac{2}{3}\right)u^{3/2}+\frac{2}{5}u^{5/2}\right]$

$\boxed{-2(1-x)^{1/2}+\frac{4}{3}(1-x)^{3/2}-\frac{2}{5}(1-x)^{5/2}+C}$

#4. $\int\frac{1}{x^2}\sqrt{1+\frac{1}{x}}\,dx$

$u=1+\frac{1}{x}=1+x^{-1}$

$\frac{du}{dx}=-x^{-2}=-\frac{1}{x^2}$

$du=-\frac{1}{x^2}dx$

$\frac{1}{x^2}dx=-du$

$\int\sqrt{u}\,(-du)$

$-\int u^{1/2}du$

$-\frac{2}{3}u^{3/2}+C$

$\boxed{-\frac{2}{3}\left(1+\frac{1}{x}\right)^{3/2}+C}$

#5. $\int \frac{\sin x}{1+\cos^2 x}\,dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x\,dx$

$\sin x\,dx = -du$

$\int \frac{1}{1+u^2}(-du)$

$-\int \frac{1}{1+u^2}\,du$

$-\arctan(u) + C$

$\boxed{-\arctan(\cos x) + C}$

#6. $\int \sec x \tan x \sqrt{1+\sec x}\,dx$

$u = 1 + \sec x$

$\frac{du}{dx} = \sec x \tan x$

$du = \sec x \tan x\,dx$

$\int \sqrt{u}\,du$

$\int u^{1/2}\,du$

$\frac{2}{3}u^{3/2} + C$

$\boxed{\frac{2}{3}(1+\sec x)^{3/2} + C}$

u-Substitution will work when the integrand contains an 'inside' function inside of an 'outside' function and **the integrand contains the derivative of the inside function**.

"Reverse chain rule"

We can take advantage of this to show the work differently where we don't have to actually write the 'u'...and just recognizing that the derivative of the inside function gets 'absorbed' when we do the integration.

Showing the u-substitution

#7. $\int 3(3x-2)^5\,dx$

$u = 3x - 2$

$\frac{du}{dx} = 3$

$du = 3dx$

$\int u^5\,du$

$\frac{u^6}{6} + C$

$\boxed{\frac{(3x-2)^6}{6} + C}$

#9. $\int \frac{x^2}{\sqrt{4-x^3}}\,dx$

$u = 4 - x^3$

$\frac{du}{dx} = -3x^2$

$du = -3x^2\,dx$

$x^2\,dx = -\frac{1}{3}du$

$\int \frac{1}{\sqrt{u}}\left(-\frac{1}{3}du\right)$

$-\frac{1}{3}\int u^{-1/2}\,du$

$-\frac{1}{3}\, 2u^{1/2} + C$

$\boxed{-\frac{2}{3}(4-x^3)^{1/2} + C}$

Recognizing 'absorption' of the inside function derivative

#8. $\int 3(3x-2)^5\,dx$

inside: $3x-2$

$(3\,dx)$

$\boxed{\frac{(3x-2)^6}{6} + C}$

#10. $\int \frac{x^2}{\sqrt{4-x^3}}\,dx$

inside: $4-x^3$

$(-3x^2\,dx)$

$\int (4-x^3)^{-1/2}x^2\,dx$

$-\frac{1}{3}\int (4-x^3)^{-1/2}(-3)x^2\,dx$

$\boxed{-\frac{1}{3}2(4-x^3)^{1/2} + C}$

Showing the u-substitution

Recognizing 'absorption' of the inside function derivative

#11. $\int (2x^2-3x)^4(4x-3)\,dx$

$u=2x^2-3x$

$\frac{du}{dx}=4x-3$

$du=(4x-3)dx$

$\int u^4\,du$

$\frac{u^5}{5}+C$

$\boxed{\frac{(2x^2-3x)^5}{5}+C}$

#12. $\int (2x^2-3x)^4(4x-3)\,dx$

inside: $2x^2-3x$

$(4x-3)dx$

$\boxed{\frac{(2x^2-3x)^5}{5}+C}$

#13. $\int \cos(x^2+3x)(6x+9)\,dx$

$u=x^2+3x$

$\frac{du}{dx}=2x+3$

$du=(2x+3)dx$

$3\int \cos(x^2+3x)(2x+3)dx$

$3\int \cos(u)\,du$

$3\sin(u)+C$

$\boxed{3\sin(x^2+3x)+C}$

#14. $\int \cos(x^2+3x)(6x+9)\,dx$

inside: x^2+3x

$(2x+3)dx$

$3\int \cos(x^2+3x)(2x+3)dx$

$\boxed{3\sin(x^2+3x)+C}$

#15. $\int \cos(5x)\,dx$

$u=5x$

$\frac{du}{dx}=5$

$du=5dx$

$dx=\frac{1}{5}du$

$\int \cos(u)\frac{1}{5}du$

$\frac{1}{5}\int \cos(u)du$

$\frac{1}{5}\sin(u)+C$

$\boxed{\frac{1}{5}\sin(5x)+C}$

#16. $\int \cos(5x)\,dx$

inside: $5x$

$5dx$

$\frac{1}{5}\int \cos(5x)5dx$

$\boxed{\frac{1}{5}\sin(5x)+C}$

You may choose whichever method seems best to you for that problem

Using Integration by Substitution to evaluate Definite Integrals

You can also use Integration by Substitution when evaluating definite integrals.
There are three variations on how to show work...

Showing the u-substitution and staying in u **(this is the most formal and the way AP shows work in their rubrics)**

$$\cancel{\int_0^{\pi/3} u^{-2}\sin\theta\, d\theta}$$

Never write anything like this which has a mixture of variables. In any integral, everything in the integral must use the same variable.

$$\int_0^{\pi/3} \frac{\sin\theta}{(\cos\theta)^2}\, d\theta$$

$$-\int_1^{1/2} u^{-2}\, du$$

$$-\left[(-1)u^{-1}\right]_1^{1/2}$$

$$\left[\frac{1}{u}\right]_1^{1/2}$$

$$\left[\frac{1}{\left(\frac{1}{2}\right)}\right]-\left[\frac{1}{1}\right]$$

$u=\cos\theta$ ← Use the toolkit to also convert the limits of integration to u-values:

$\theta=0 \rightarrow u=\cos(0)=1$

$\theta=\pi/3 \rightarrow u=\cos(\pi/3)=1/2$

$$\frac{du}{d\theta}=-\sin\theta$$

$$du=-\sin\theta d\theta$$

$$\sin\theta d\theta=-du$$

You don't have to show this side work, but when you write the integral again using u, make sure the limits are also u-values

Showing the u-substitution and switching back to the original variable

You can avoid converting the limits of integration to the new variable, but there is a problem: You aren't allowed to show mixed variables in the integral but must show the limits somehow.

This notation is allowed and considered correct (although not preferred by AP graders)

$$\int_0^{\pi/3} \frac{\sin\theta}{(\cos\theta)^2}\, d\theta$$

$$-\int_{\theta=0}^{\theta=\pi/3} u^{-2}\, du$$

$$-\left[(-1)u^{-1}\right]_{\theta=0}^{\theta=\pi/3}$$

$$\left[\frac{1}{u}\right]_{\theta=0}^{\theta=\pi/3}$$

$$\left[\frac{1}{\cos\theta}\right]_0^{\pi/3}$$

You can drop the special notation once you've substituted back to the original variable.

$$\left[\frac{1}{\cos\left(\frac{\pi}{3}\right)}\right]-\left[\frac{1}{\cos(0)}\right]$$

$u=\cos\theta$

$$\frac{du}{d\theta}=-\sin\theta$$

$$du=-\sin\theta d\theta$$

$$\sin\theta d\theta=-du$$

Usually, this method is more work, because the u-substitution generally makes things simpler, so going back to the original variable usually involves more complicated expressions.

Using the 'recognizing absorption of the inside derivative' and never converting to u

You can avoid converting the limits of integration to the new variable, but there is a problem: You aren't allowed to show mixed variables in the integral but must show the limits somehow.

$$\int_0^{\pi/3} \frac{\sin\theta}{(\cos\theta)^2}\, d\theta$$

$$-\int_0^{\pi/3} (\cos\theta)^{-2}(-\sin\theta)\, d\theta$$

$$-\left[\frac{(\cos\theta)^{-1}}{-1}\right]_0^{\pi/3}$$

$$\left[\frac{1}{\cos\theta}\right]_0^{\pi/3}$$

$$\left[\frac{1}{\cos\left(\frac{\pi}{3}\right)}\right]-\left[\frac{1}{\cos(0)}\right]$$

#1. $\int_1^3 x\sqrt{3x^2-2}\,dx$

$u=3x^2-2$

$\frac{du}{dx}=6x$

$du=6x\,dx$

$x\,dx=\frac{1}{6}du$

$$\int_1^{25}\sqrt{u}\,\frac{1}{6}\,du$$

$$\frac{1}{6}\int_1^{25}u^{1/2}\,du$$

$$\left[\frac{1}{6}\left(\frac{2}{3}\right)u^{3/2}\right]_1^{25}$$

$$\boxed{\frac{1}{9}\left[25^{3/2}-1^{3/2}\right]}$$

$$\int_1^3 x\sqrt{3x^2-2}\,dx$$

inside: $3x^2-2$

$(6x)\,dx$

$$\frac{1}{6}\int_1^3(3x^2-2)^{1/2}6x\,dx$$

$$\left[\frac{1}{6}\left(\frac{2}{3}\right)(3x^2-2)^{3/2}\right]_1^3$$

$$\boxed{\frac{1}{9}\left[(3(3)^2-2)^{3/2}-(3(1)^2-2)^{3/2}\right]}$$

#2. $\int_0^{1/2}\frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

$u=\sin^{-1}x$

$\frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}$

$du=\frac{1}{\sqrt{1-x^2}}\,dx$

$$\int_{x=0}^{x=1/2}u\,du$$

$$\left[\frac{1}{2}u^2\right]_{x=0}^{x=1/2}$$

$$\left[\frac{1}{2}(\sin^{-1}(x))^2\right]_0^{1/2}$$

$$\boxed{\frac{1}{2}\left[(\sin^{-1}(1/2))^2-(\sin^{-1}(0))^2\right]}$$

$$\int_0^{1/2}\frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

"inside" $=\sin^{-1}x$

$\left(\frac{1}{\sqrt{1-x^2}}\right)dx$

harder to see, but

$$\left[\frac{(\sin^{-1}x)^2}{2}\right]_0^{1/2}$$

$$\boxed{\frac{1}{2}\left[(\sin^{-1}(\tfrac{1}{2}))^2-(\sin^{-1}(0))^2\right]}$$

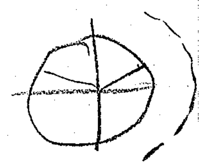
BTW ... If this no calc. MCQ how would you evaluate $\sin^{-1}\left(\frac{1}{2}\right)$?

$\sin^{-1}\left(\frac{1}{2}\right)=\theta$

$\sin\theta=\frac{1}{2}$

$\theta=\pi/6$

so $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$



$\sin^{-1}$ works here only

Unit 4-6: Integration using Algebra Techniques

Other Integration Techniques

Some integrals don't resolve using basic antiderivative shortcuts or integration by substitution immediately, but you can do some additional work to put them into a form where these techniques will work. Here, we examine a few such techniques.

Inverse Trig forms with constants

We've memorized this form: $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

But what if we need to integrate this: $\int \frac{1}{4+x^2}\,dx$

What if we manipulated constants to get a 1 in lower left?

$$\int \frac{1}{4(1+\frac{1}{4}x^2)}\,dx$$

$$\frac{1}{4}\int \frac{1}{1+(\frac{1}{2}x)^2}\,dx$$

$$\frac{1}{4}\int \frac{1}{1+u^2}\,2du$$

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du$$

$$\frac{1}{2}\arctan(u)+C$$

$$\boxed{\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C}$$

$u=\frac{1}{2}x$

$\frac{du}{dx}=\frac{1}{2}$

$du=\frac{1}{2}dx$

$dx=2du$

It can be shown that the following are true...
(we will learn the technique used to derive these later - trigonometric substitution)

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C$$

(We don't usually use the cosine, cotangent, or cosecant forms because they are just the negatives of these).

We can update our table of antiderivative shortcuts to memorize...

Updated antiderivative list

$$\frac{d}{dx}[C]=0 \qquad \int 0\,dx=C$$

$$\frac{d}{dx}[kx]=k \qquad \int k\,dx=kx+C$$

$$\frac{d}{dx}\left[x^n\right]=nx^{n-1} \qquad \int x^n\,dx=\frac{x^{n+1}}{n+1}+C \quad (n\neq -1)$$

$$\frac{d}{dx}\left[e^x\right]=e^x \qquad \int e^x\,dx=e^x+C$$

$$\frac{d}{dx}\left[a^x\right]=(\ln a)\,a^x \qquad \int a^x\,dx=\left(\frac{1}{\ln a}\right)a^x+C$$

$$\frac{d}{dx}[\ln x]=\frac{1}{x}\ (x>0) \qquad \int \frac{1}{x}\,dx=\ln|x|+C$$

$$\frac{d}{dx}[\sin x]=\cos x \qquad \int \cos x\,dx=\sin x+C$$

$$\frac{d}{dx}[\cos x]=-\sin x \qquad \int \sin x\,dx=-\cos x+C$$

$$\frac{d}{dx}[\tan x]=\sec^2 x \qquad \int \sec^2 x\,dx=\tan x+C$$

$$\frac{d}{dx}[\sec x]=\sec x\tan x \qquad \int \sec x\tan x\,dx=\sec x+C$$

$$\frac{d}{dx}[\csc x]=-\csc x\tan x \qquad \int \csc x\cot x\,dx=-\csc x+C$$

$$\frac{d}{dx}[\cot x]=-\csc^2 x \qquad \int \csc^2 x\,dx=-\cot x+C$$

$$\frac{d}{dx}[\arcsin x]=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arccos x]=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan x]=\frac{1}{1+x^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} x]=\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{arccsc} x]=\frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{arccot} x]=\frac{-1}{1+x^2}$$

$$\int \frac{du}{\sqrt{a^2-u^2}}=\arcsin\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{a^2+u^2}=\frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C$$

Inverse Trig forms with constants

#1. $\int \frac{1}{4+x^2}dx$ $\quad \int \frac{1}{a^2+u^2}du = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+c$

$u=x \quad a=2$

$du=dx$

$\int \frac{1}{a^2+u^2}du$

$\boxed{\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C}$

Splitting into multiple integrals

Sometimes, u-sub won't work as initially stated, but we can split the integral into multiple integrals:

#2. $\int \frac{x+2}{\sqrt{4-x^2}}dx = \int \frac{x}{\sqrt{4-x^2}}dx + 2\int \frac{1}{\sqrt{4-x^2}}dx$ $\quad \int \frac{1}{\sqrt{a^2-u^2}}du = \arcsin\left(\frac{u}{a}\right)+c$

$u=4-x^2$

$\frac{du}{dx}=-2x$

$du=-2x\,dx$

$x\,dx=-\frac{1}{2}du$

$\int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}du\right) + 2\int \frac{1}{\sqrt{a^2-u^2}}du$ $\quad u=x \quad a=2$, $du=dx$

$-\frac{1}{2}\int u^{-1/2}du + 2\arcsin\left(\frac{u}{a}\right)+c$

$-\frac{1}{2}(2)u^{1/2}$

$\boxed{-\sqrt{4-x^2} + 2\arcsin\left(\frac{x}{2}\right)+C}$

Completing the Square

An old algebra technique is also sometimes useful...completing the square.

If you have a quadratic which is not factorable, like...

$$x^2-4x+7$$

...you can perform a 'complete the square' procedure to make a portion a binomial squared:

$(x^2-4x+4)+7-4$

$(x-2)^2+3$

Polynomial Division

How would we do this one from our extra practice?

#13b. $\int \frac{3x^2-2x+3}{x^2+1}\,dx$

$$\begin{array}{r} 3 \\ x^2+1 \overline{\smash{)}\,3x^2-2x+3} \\ -(3x^2 \quad\quad +3) \\ \hline -2x \end{array}$$

$$\frac{3x^2-2x+3}{x^2+1} = 3 + \frac{-2x}{x^2+1}$$

$$\int \left[3 + \frac{-2x}{x^2+1}\right] dx$$

$$\int 3\,dx - \int \frac{2x}{x^2+1}\,dx \qquad u = x^2+1$$

$$\frac{du}{dx} = 2x$$

$$\int 3\,dx - \int \frac{1}{u}\,du \qquad du = 2x\,dx$$

$$\boxed{3x - \ln|x^2+1| + C}$$

Completing the Square

This can sometimes help with integration:

#3. $\int \frac{1}{x^2-4x+7}\,dx$

$x^2-4x+4+7-4$

$(x-2)^2+3$

$\int \frac{1}{3+(x-2)^2}\,dx$

$u=x-2 \quad a=\sqrt{3}$

$du=dx$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+c$

$\boxed{\frac{1}{\sqrt{3}}\arctan\left(\frac{x-2}{\sqrt{3}}\right)+C}$

Completing the square is only helpful when the denominator is <u>not</u> factorable:

$$\int \frac{1}{x^2-x-6}\,dx$$

(we will learn another technique later which works when the denominator is factorable - partial fraction expansion)

Trigonometry Identities can be used to change the form of the integrand

Sometimes using simple trig identities can help with integration...

#4. $\int \tan(2x)\,dx$

$\int \frac{\sin(2x)}{\cos(2x)}\,dx$

$u=\cos(2x)$

$\frac{du}{dx}=-2\sin(2x)$

$du=-2\sin(2x)dx$

$\sin(2x)dx=-\frac{1}{2}du$

$\int \frac{1}{u}\left(-\frac{1}{2}du\right)$

$-\frac{1}{2}\int \frac{1}{u}\,du$

$-\frac{1}{2}\ln|u|+C$

$\boxed{-\frac{1}{2}\ln|\cos(2x)|+C}$

– or –

$\frac{1}{2}\ln|(\cos(2x))^{-1}|+C$

$\boxed{\frac{1}{2}\ln|\sec(2x)|+C}$

#5. $\int x^3\tan(x^4)\,dx$

$\int \frac{\sin(x^4)}{\cos(x^4)}x^3\,dx$

$u=\cos(x^4)$

$\frac{du}{dx}=-\sin(x^4)4x^3$

$du=-4x^3\sin(x^4)dx$

$x^3\sin(x^4)dx=-\frac{1}{4}du$

$\int \frac{1}{u}\left(-\frac{1}{4}du\right)$

$-\frac{1}{4}\int \frac{1}{u}\,du$

$-\frac{1}{4}\ln|u|+c$

$\boxed{-\frac{1}{4}\ln|\cos(x^4)|+C}$

Here are the trig identities we need you to memorize

Reciprocal identities

$$\sin x = \frac{1}{\csc x} \qquad \csc x = \frac{1}{\sin x}$$
$$\cos x = \frac{1}{\sec x} \qquad \sec x = \frac{1}{\cos x}$$
$$\tan x = \frac{1}{\cot x} \qquad \cot x = \frac{1}{\tan x}$$

Quotient identities

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$

Pythagorean identity forms

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Power Reducing identities

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$
$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

Double-angle identity

$$\sin(2x) = 2\sin(x)\cos(x)$$

One last idea (rarely needed): add and subtract in numerator to enable a split

Sometimes adding and subtracting something in the integral will allow a useful split...

#6. $\int \frac{1}{1+e^x}\,dx = \int \frac{1+e^x-e^x}{1+e^x}\,dx$

$= \int \frac{1+e^x}{1+e^x}\,dx - \int \frac{e^x}{1+e^x}\,dx = \int 1\,dx - \int \frac{e^x}{1+e^x}\,dx$

$= x - \int \frac{1}{u}\,du$

$\boxed{x - \ln|1+e^x| + C}$

$u = 1+e^x$

$\frac{du}{dx} = e^x$

$du = e^x\,dx$

Quotient form suggests maybe Log Rule, so try to make a port of the numerator match the denominator by adding and subtracting the same term...

$$\int \frac{1+e^x-e^x}{1+e^x}\,dx$$

Now split into 2 integrals (remember you can split a numerator but not a denominator - must retain the 'common denominator')...

$$\int \frac{1+e^x}{1+e^x}\,dx - \int \frac{e^x}{1+e^x}\,dx$$

Simplify, then u-sub for 2nd integral...

$$\int 1\,dx - \int \frac{e^x}{1+e^x}\,dx$$

u=1+e^x

$du = e^x\,dx$

$$\int 1\,dx - \int \frac{1}{u}\,du$$

$$x \quad -\ln|u|$$

$$\boxed{x \quad -\ln|1+e^x| + C}$$

Unit 4-7: Integration by Parts

Integration by Parts

Integration by Substitution is one of our main techniques for evaluating integrals. Another very widely used technique is **Integration by Parts**. Where Integration by Substitution is based on the Chain Rule, Integration by Parts is based on the Product Rule:

Derivative using Product Rule

$$\frac{d}{dx}[uv] = u\frac{d}{dx}[v] + v\frac{d}{dx}[u]$$

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx} \longleftarrow$$ integrating this on both sides:

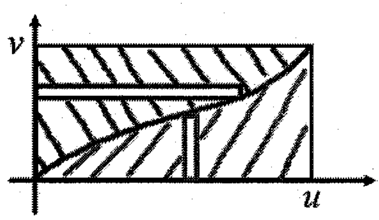
Integration by Parts

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$uv = \int u\,dv + \int v\,du$$

...produces the 'integration by parts formula': $\boxed{\int u\,dv = uv - \int v\,du}$

We can also derive the integration by parts formula by using our ideas of integration representing area:



blue area red area total area

$$\int u\,dv + \int v\,du = uv$$

Now solve for the first term: $\boxed{\int u\,dv = uv - \int v\,du}$

In Integration by Parts, the entire integrand must be divided into two factors which are multiplied together, u and dv

$$\int \underbrace{u}\,\underbrace{dv} = uv - \int v\,du$$

given integral two factors

...then you can instead use the equivalent right side formula. This effectively removes some of the integrand out of the integral into the uv and the remaining integral should be simpler to evaluate than the original integral.

Integration by Parts

1) separate the integrand into two factors, *u* and *dv*. The *dv* factor must contain the differential and be something you can integrate.

2) Take the derivative of *u* and solve for *du* to obtain du.

3) Take the antiderivative of *dv* to obtain *v*.

4) Substitute *u*, *v*, and *dv* into the integration by parts formula.

5) Integrate the remaining (simpler) integral.

Comparing Integration by Part with Integration by Substitution

Integration by Substitution

1) define the 'inside' function to be u.

2) Find du/dx and solve for du to get a 'toolkit' with du and u.

3) Substitution all expressions with x and dx into the original integral to obtain a new integral using u as the variable.

4) Integrate, then resubstitute u to use the original x variable.

Integration by Substitution is used when one function is 'inside' the other (a composition of functions) and we make the 'inside' function *u*. There is no *v*, and we substitute back into the original integral so that the variable changes to *u*.

Integration by Parts

1) separate the integrand into two factors, *u* and *dv*. The *dv* factor must contain the differential and be something you can integrate.

2) Take the derivative of *u* and solve for *du* to obtain du.

3) Take the antiderivative of *dv* to obtain *v*.

4) Substitute *u*, *v*, and *dv* into the integration by parts formula.

5) Integrate the remaining (simpler) integral.

Integration by Parts is used when the integrand is easily separated into two things which are multiplied. There is a *u* and a *v*, and we substitute into a formula (not the original integral). The resulting integral still uses the original integration variable.

#1. $\int e^{(3x)}\,dx$

$u = 3x$

$\frac{du}{dx} = 3$

$du = 3dx$

$dx = \frac{1}{3}du$

$\int e^u(\frac{1}{3}du)$

$\frac{1}{3}\int e^u\,du$

$\frac{1}{3}e^u + C$

$\boxed{\frac{1}{3}e^{(3x)} + C}$

#2. $\int 3xe^x\,dx$

$u = 3x \quad dv = e^x dx$

$\frac{du}{dx} = 3 \quad \int dv = \int e^x dx$

$du = 3dx \quad v = e^x$

$uv - \int v\,du$

$(3x)(e^x) - \int (e^x)(3dx)$

$3xe^x - 3\int e^x dx$

$\boxed{3xe^x - 3e^x + C}$

Integration by Parts - guidelines for choose *u* and *dv*

- **Make sure that *dv* is something you can integrate.** For example, if the integrand contains natural log, we don't have a shortcut to integrate that, so that portion should be part of *u*.

- **If you can, select something for *u* that will get simpler when you take the derivative.** For example, if the integrand contains x^3 that would be best to put into *u*, because when you take the derivative it becomes $2x^2$ (a lower degree and therefore simpler).

- **Try letting *dv* be the more complicated portion of the integrand.**

If you happen to choose wrong, it just means that the resulting remaining integral won't be simpler than the original to integrate. In that case, just choose *u* and *dv* differently and try again (or maybe try a different procedure, like Integration by Substitution).

(show wrong choice)

#3. $\int \ln(x)\,dx$

$u = \ln x \qquad dv = dx$

$\frac{du}{dx} = \frac{1}{x} \qquad \int dv = \int dx$

$du = \frac{1}{x}dx \qquad v = x$

$uv - \int v\,du$

$x\ln x - \int x \frac{1}{x}\,dx$

$x\ln x - \int 1\,dx$

$\boxed{x\ln x - x + C}$

#4. $\int x\sin(4x)\,dx$

$u = x \qquad dv = \sin(4x)dx$

$\frac{du}{dx} = 1 \qquad \int dv = \int \sin(4x)dx$

$du = dx \qquad \int dv = \frac{1}{4}\int 4\sin(4x)$

$v = -\frac{1}{4}\cos(4x)$

$uv - \int v\,du$

$-\frac{1}{4}x\cos(4x) + \frac{1}{4}\int \cos(4x)dx$

$+\frac{1}{4}\frac{1}{4}\int 4\cos(4x)dx$

$\boxed{-\frac{1}{4}x\cos(4x) + \frac{1}{16}\sin(4x) + C}$

#5. $\int 2x^3\cos\left(x^2\right)dx$

(try integration by substitution first)

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\int x^2\cos(x^2)\,2x\,dx$

$\int u\cos(u)\,du$ change letter:

$y = u$

$\int y\cos(y)\,dy$

$u = y \qquad dv = \cos(y)dy$

$\frac{du}{dy} = 1 \qquad \int dv = \int \cos(y)dy$

$du = dy \qquad v = \sin y$

$uv - \int v\,du$

$y\sin y - \int \sin y\,dy$

$y\sin y + \cos y + C \qquad y = u = x^2$

$\boxed{x^2\sin(x^2) + \cos(x^2) + C}$

Sometimes, you need to use integration by parts more than once

#6. $\int x^2\sin x\,dx$

$u = x^2 \qquad dv = \sin x\,dx$

$\frac{du}{dx} = 2x \qquad \int dv = \int \sin x\,dx$

$du = 2x\,dx \qquad v = -\cos x$

$uv - \int v\,du$

$-x^2\cos x - \int(-\cos x)2x\,dx$

$-x^2\cos x + 2\left[\int x\cos x\,dx\right]$

$u = x \qquad dv = \cos x\,dx$

$\frac{du}{dx} = 1 \qquad \int dv = \int \cos x\,dx$

$du = dx \qquad v = \sin x$

$-x^2\cos x + 2\left[uv - \int v\,du\right]$

$-x^2\cos x + 2\left[x\sin x - \int \sin x\,dx\right]$

$-x^2\cos x + 2\left[x\sin x + \cos x\right] + C$

$\boxed{-x^2\cos x + 2x\sin x + 2\cos x + C}$

You may find that after repeated integration by parts, you obtain the original integral...

#1. $\int e^{4x}\cos(2x)\,dx$

$u=\cos(2x)$ $\quad dv=e^{4x}dx$

$\frac{du}{dx}=-2\sin(2x)$ $\quad \int dv=\int e^{4x}dx$

$du=-2\sin(2x)dx$ $\quad v=\frac{1}{4}e^{4x}$

$uv-\int v\,du$

$\frac{1}{4}e^{4x}\cos(2x)-\int(\frac{1}{4}e^{4x})(-2\sin(2x))dx$

$\frac{1}{4}e^{4x}\cos(2x)+\frac{1}{2}\left[\int e^{4x}\sin(2x)\,dx\right]$

$u=\sin(2x)$ $\quad du=e^{4x}dx$

$\frac{du}{dx}=2\cos(2x)$ $\quad \int dv=\int e^{4x}dx$

$du=2\cos(2x)dx$ $\quad v=\frac{1}{4}e^{4x}$

$\frac{1}{4}e^{4x}\cos(2x)+\frac{1}{2}\left(uv-\int v\,du\right)$

$\frac{1}{4}e^{4x}\cos(2x)+\frac{1}{2}\left[\frac{1}{4}e^{4x}\sin(2x)-\int\frac{1}{4}e^{4x}2\cos(2x)dx\right]$

$\frac{1}{4}e^{4x}\cos(2x)+\frac{1}{2}\left[\frac{1}{4}e^{4x}\sin(2x)-\frac{1}{2}\int e^{4x}\cos(2x)dx\right]$

$\int e^{4x}\cos(2x)dx=\frac{1}{4}e^{4x}\cos(2x)+\frac{1}{8}e^{4x}\sin(2x)-\underbrace{\frac{1}{4}\int e^{4x}\cos(2x)dx}_{\text{the original integral!}}$

move to other side

$+\frac{1}{4}\int e^{4x}\cos(2x)dx$ $\qquad +\frac{1}{4}\int e^{4x}\cos(2x)dx$

$\frac{5}{4}\int e^{4x}\cos(2x)dx=\left(\frac{1}{4}e^{4x}\cos(2x)+\frac{1}{8}e^{4x}\sin(2x)\right)$ multiply both sides by $\frac{4}{5}$

$$\boxed{\int e^{4x}\cos(2x)dx=\frac{4}{5}\left[\frac{1}{4}e^{4x}\cos(2x)+\frac{1}{8}e^{4x}\sin(2x)\right]+C}$$

The "tabular method"

If you must use Integration by Parts more than once (and the result doesn't cycle back around to the original integral) this results in a noticable pattern in the terms of the answer. Some textbooks (including ours) point this out and suggest a quicker method call the "tabular method". It isn't anything official, and won't be asked about on the AP Exam, but you could employ this when it is appropriate if you like:

The usual method...

#2. $\int x^2 e^{2x}\,dx$

$u = x^2 \quad dv = e^{2x}dx$

$\frac{du}{dx} = 2x \quad \int dv = \int e^{2x}dx$

$du = 2x\,dx \quad v = \frac{1}{2}e^{2x}$

$uv - \int v\,du$

$\frac{1}{2}x^2e^{2x} - \int \frac{1}{2}e^{2x}(2x)dx$

$u = x \quad dv = e^{2x}dx$

$\frac{du}{dx} = 1 \quad \int dv = \int e^{2x}dx$

$du = dx \quad v = \frac{1}{2}e^{2x}$

$\frac{1}{2}x^2e^{2x} - [\int xe^{2x}dx]$

$\frac{1}{2}x^2e^{2x} - [uv - \int v\,du]$

$\frac{1}{2}x^2e^{2x} - [\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx]$

$\boxed{\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C}$

The "tabular" method...

#3. $\int x^2 e^{2x}\,dx$

sign	u (derivative)	dv (antideriv)
+	x^2	e^{2x}
−	$2x$	$\frac{1}{2}e^{2x}$
+	2	$\frac{1}{4}e^{2x}$
−	0	$\frac{1}{8}e^{2x}$

$+(x^2)(\frac{1}{2}e^{2x}) - (2x)(\frac{1}{4})e^{2x} + (2)(\frac{1}{8}e^{2x})$

$\boxed{\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C}$

#4. $\int x^6 e^x\,dx$

Sign	u	dv
+	x^6	e^x
−	$6x^5$	e^x
+	$30x^4$	e^x
−	$120x^3$	e^x
+	$360x^2$	e^x
−	$720x$	e^x
+	720	e^x
−	0	e^x

$\boxed{x^6e^x - 6x^5e^x + 30x^4e^x - 120x^3e^x + 360x^2e^x - 720xe^x + 720e^x + C}$

Unit 4-8: Trigonometric Integrals

Trigonometric Integrals

Integrals which involve powers of one or more trigonometric functions are referred to as **trigonometric integrals**. For example, the following integrals are trigonometric integrals:

$$\int \sin^3 x \cos^4 x \, dx \qquad\qquad \int \tan^5 x \, dx$$

Integrating such integrals involves changing the form of the integral as given using trigonometric identities in order to produce a form which splits off a portion to form the 'du' for a u-substitution. Often (but not always) the result is that the integral is split into multiple, simpler integrals.

Each case is a little different, and becoming adept at this requires practice, but there are some guiding 'rules of thumb' that suggest techniques which work in different situations.

Things to review that you will need

You should review and memorize some trig identities:

$$\sin^2 u + \cos^2 u = 1 \qquad \sin^2 u = \frac{1-\cos 2u}{2}$$
$$\sec^2 u = \tan^2 u + 1$$
$$\csc^2 u = \cot^2 u + 1 \qquad \cos^2 u = \frac{1+\cos 2u}{2}$$

You should also review memorize the trig derivatives and integrals:

$$\frac{d}{dx}[\sin x] = \cos x \qquad \int \cos x \, dx = \sin x + C$$
$$\frac{d}{dx}[\cos x] = -\sin x \qquad \int \sin x \, dx = -\cos x + C$$
$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \int \sec^2 x \, dx = \tan x + C$$
$$\frac{d}{dx}[\cot x] = -\csc^2 x \qquad \int \csc^2 x \, dx = -\cot x + C$$
$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \int \sec x \tan x \, dx = \sec x + C$$
$$\frac{d}{dx}[\csc x] = -\csc x \cot x \qquad \int \csc x \cot x \, dx = -\csc x + C$$

The general idea behind integrating trigonometric integrals is to try to split the integrand into factors so that one part can become 'u' for a u-substitution and the du needed is present in the integral.

$$\int \sin^3 x \cos^4 x\, dx$$

$$\int \sin^2 x \cos^4 x \sin x\, dx$$ ⟵ split off one sinx to use for 'du'

$$\int (1-\cos^2 x)\cos^4 x \sin x\, dx$$ ⟵ trig indentity to convert other sines to cosines

$$\int (\cos^4 x - \cos^6 x)\sin x\, dx$$ ⟵ ...so that you can split into two 'cosine' integrals

$$\int \cos^4 x \sin x\, dx - \int \cos^6 x \sin x\, dx$$

$$u = \cos x \qquad du = -\sin x\, dx$$

$$\int u^4(-du) \qquad -\int u^6(-du)$$ Now, each integral has the 'du' required to do a u-substitution

$$-\int u^4\, du \qquad +\int u^6\, du$$

$$-\frac{1}{5}u^5 \qquad +\frac{1}{7}u^7 + C$$

$$-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

#1. $\int \sin^3 x \cos^3 x\, dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\, dx$

$\int \sin^3 x \cos^2 x \cos x\, dx$

$\int \sin^3(x)(1-\sin^2 x)\cos x\, dx$

$\int u^3(1-u^2)\, du$

$\int (u^3 - u^5)\, du$

$\frac{1}{4}u^4 - \frac{1}{6}u^6 + C$

$\boxed{\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C}$

#2. $\int \tan^2 x \sec^2 x\, dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x\, dx$

$\int u^2\, du$

$\frac{1}{3}u^3 + C$

$\boxed{\frac{1}{3}\tan^3 x + C}$

Single trig functions are usually just shortcuts or need algebra or trig substitutions...

#3. $\int \cos x\, dx$

$= \boxed{\sin x + C}$

#4. $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x\, dx$

$\sin x\, dx = -du$

$\int \frac{1}{u}(-du)$

$-\int \frac{1}{u}\, du$

$-\ln|u| + C$

$\boxed{-\ln|\cos x| + C}$

...but sec(x) and csc(x) are special cases, and we should add two new shortcuts to handle these...

#5. $\int \sec x \, dx$

$\int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$

$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

$u = \sec x + \tan x$

$\frac{du}{dx} = \sec x \tan x + \sec^2 x$

$du = (\sec^2 x + \sec x \tan x) \, dx$

$\int \frac{1}{u} du$

$\ln|u| + c$

$\boxed{\ln|\sec x + \tan x| + C}$

Similarly:

$$\boxed{\int \csc x \, dx = \ln|\csc x - \cot x| + C}$$

The way I memorize this...

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

so

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

so

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$

$\int \cos^2 x\,dx$ and $\int \sin^2 x\,dx$ are also special cases - we use the power reducing trig identities:

$$\sin^2 u = \frac{1-\cos 2u}{2} \qquad \cos^2 u = \frac{1+\cos 2u}{2}$$

#6. $\int \cos^2 x\,dx$

$\int \frac{1+\cos(2x)}{2}dx$

$\int \frac{1}{2}dx + \frac{1}{2}\int \cos(2x)\,dx$

$\frac{1}{2}x + \frac{1}{2}\,\frac{\sin(2x)}{2} + C$

$\boxed{\frac{1}{2}x + \frac{1}{4}\sin(2x) + C}$

#7. $\int \sin^2 x\,dx$

$\int \frac{1-\cos(2x)}{2}dx$

$\int \frac{1}{2}dx - \frac{1}{2}\int \cos(2x)\,dx$

$\frac{1}{2}x - \frac{1}{2}\,\frac{\sin(2x)}{2} + C$

$\boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x) + C}$

#8. $\int \tan^4 x\,dx$

$\int \tan^2 x \tan^2 x\,dx$

$\int \tan^2 x(\sec^2 x - 1)\,dx$

$\int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$

$\int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx$

$\int \tan^2 x \sec^2 x\,dx - \int \sec^2 x\,dx + \int 1\,dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x\,dx$

$\int u^2\,du - \int \sec^2 x\,dx + \int 1\,dx$

$\frac{1}{3}u^3 - \tan x + x + C$

$\boxed{\frac{1}{3}\tan^3 x - \tan x + x + C}$

#9. $\int \cot^5 x \sin^4 x\,dx$

$\int \frac{\cos^5 x \sin^4 x}{\sin^5 x}dx$

$\int \frac{\cos^5 x}{\sin x}dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$\int \frac{\cos^4 x}{\sin x}\cos x\,dx$

$\int \frac{\cos^2 x \cos^2 x}{\sin x}\cos x\,dx$

$\int \frac{(1-\sin^2 x)(1-\sin^2 x)}{\sin x}\cos x\,dx$

$\int \frac{1 - 2\sin^2 x + \sin^4 x}{\sin x}\cos x\,dx$

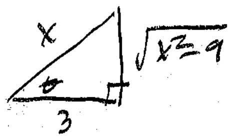
$\int \frac{1 - 2u^2 + u^4}{u}du$

$\int \frac{1}{u}du - 2\int u\,du + \int u^3\,du$

$\ln|u| - u^2 + \frac{1}{4}u^4 + C$

$\boxed{\ln|\sin x| - \sin^2 x + \frac{1}{4}\sin^4 x + C}$

#1. $\int \frac{\sqrt{x^2-9}}{x}\,dx$



$\cos\theta = \frac{3}{x}$

$x = \frac{3}{\cos\theta} = 3\sec\theta$

$\frac{dx}{d\theta} = 3\sec\theta\tan\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$\theta = \arccos\left(\frac{3}{x}\right)$

$\tan\theta = \frac{\sqrt{x^2-9}}{3}$

$\sqrt{x^2-9} = 3\tan\theta$

$\int \frac{3\tan\theta \; 3\sec\theta\tan\theta}{3\sec\theta}\,d\theta$

$3\int \tan^2\theta\,d\theta$

$3\int (\sec^2\theta - 1)\,d\theta$

$3\int \sec^2\theta\,d\theta - \int 3\,d\theta$

$3\tan\theta - 3\theta + C$

$\boxed{3\left(\frac{\sqrt{x^2-9}}{3}\right) - 3\arccos\left(\frac{3}{x}\right) + C}$

#2. $\int \frac{x^3}{\sqrt{16-x^2}}\,dx$

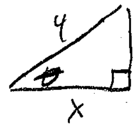
- Build the triangle based upon the form of the radical:

$$\sqrt{16-x^2} = \sqrt{(4)^2-(x)^2}$$

one leg must be x, other leg is the radical

hypotenuse must be 4

- Write two trig ratios, each pairing the constant with one other side.
- Solve each expression for the x-variable and take the derivative to get dx.
- Substitute for x, dx, and the radical into the original integral to convert it to a trigonometric integral which we can solve with the previous section's methods.
- Finally, use the triangle to re-substitute all the θ expressions back to x.



$\sqrt{16-x^2} = \sqrt{(4)^2-(x)^2}$

$\cos\theta = \frac{x}{4}$

$x = 4\cos\theta$

$\frac{dx}{d\theta} = -4\sin\theta$

$dx = -4\sin\theta\,d\theta$

$\sin\theta = \frac{\sqrt{16-x^2}}{4}$

$\sqrt{16-x^2} = 4\sin\theta$

$\int \frac{(4\cos\theta)^3(-4)\sin\theta}{4\sin\theta}\,d\theta$

$\int \frac{64\cos^3\theta\,(-4)\sin\theta}{4\sin\theta}\,d\theta$

$-64\int \cos^3\theta\,d\theta$

$-64\int \cos^2\theta\,\cos\theta\,d\theta$

$u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$du = \cos\theta\,d\theta$

$-64\int (1-\sin^2\theta)\cos\theta\,d\theta$

$-64\int (1-u^2)\,du$

$-64\int 1\,du + 64\int u^2\,du$

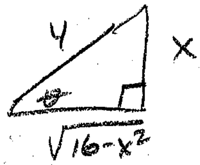
$-64u + \frac{64}{3}u^3 + C$

$-64\sin\theta + \frac{64}{3}\sin^3\theta + C$

$\boxed{-64\left(\frac{\sqrt{16-x^2}}{4}\right) + \frac{64}{3}\left(\frac{\sqrt{16-x^2}}{4}\right)^3 + C}$

#2 w/ legs swapped:

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx$$



$\sin\theta = \frac{x}{4}$

$x = 4\sin\theta$

$\frac{dx}{d\theta} = 4\cos\theta$

$dx = 4\cos\theta\, d\theta$

$\cos\theta = \frac{\sqrt{16-x^2}}{4}$

$\sqrt{16-x^2} = 4\cos\theta$

$$\int \frac{(4\sin\theta)^3\, 4\cos\theta}{4\cos\theta}\,d\theta$$

$$64\int \sin^3\theta\, d\theta$$

$$64\int \sin^2\theta\, \sin\theta\, d\theta$$

$u = \cos\theta$

$\frac{du}{dx} = -\sin\theta$

$du = -\sin\theta\, d\theta$

$\sin\theta\, d\theta = -du$

$$64\int (1-\cos^2\theta)\sin\theta\, d\theta$$

$$64\int (1-u^2)(-du)$$

$$-64\int du + 64\int u^2\, du$$

$$-64u + \frac{64}{3}u^3 + C$$

$$-64\cos\theta + \frac{64}{3}\cos^3\theta + C$$

$$\boxed{-64\left(\frac{\sqrt{16-x^2}}{4}\right) + \frac{64}{3}\left(\frac{\sqrt{16-x^2}}{4}\right)^3 + C}$$

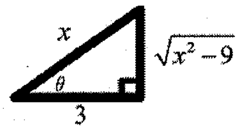
Unit 4-9: Trigonometric Substitution

Trigonometric Substitution

Some integrals have a form in which a part of the integrand can be represented as one side of a right triangle, and then trig functions of angle θ for this triangle can be used to substitute everything in the original integral. This procedure is called **trigonometric substitution**. Let's look at an example to see the concept...

$$\int \frac{\sqrt{x^2-9}}{x}\,dx$$

The part under the radical could be thought of as being a leg of a right triangle:



Then we can use the other 2 sides to form a substitution: $\cos\theta = \dfrac{3}{x}$ $\quad x = \dfrac{3}{\cos\theta}$ $\quad \boxed{x = 3\sec\theta}$

Using the 2 sides with the radical and the constant we get another substitution: $\tan\theta = \dfrac{\sqrt{x^2-9}}{3}$ $\quad \boxed{\sqrt{x^2-9} = 3\tan\theta}$

The last thing we would need to substitute is *dx* so we take the derivative of the x substitution:

$$x = 3\sec\theta$$
$$\frac{dx}{d\theta} = 3\sec\theta\tan\theta$$
$$\boxed{dx = 3\sec\theta\tan\theta\,d\theta}$$

Now we substitute everything back into the original integral and evaluate:

$$\int \frac{\sqrt{x^2-9}}{x}\,dx$$
$$\int \frac{3\tan\theta}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta$$
$$3\int \tan^2\theta\,d\theta$$
$$3\int \left(\sec^2\theta - 1\right)d\theta$$
$$3\int \sec^2\theta\,d\theta + 3\int 1\,d\theta$$
$$3\tan\theta + 3\theta + C$$

Finally, we can substitute out the θ we added:

$$\sqrt{x^2-9} = 3\tan\theta \qquad \theta = \arctan\left(\frac{\sqrt{x^2-9}}{3}\right) \qquad \boxed{\sqrt{x^2-9} + 3\arctan\left(\frac{\sqrt{x^2-9}}{3}\right) + C}$$

We build the triangle to match the form of the radical in the integrand...

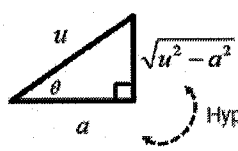
#1. $\int \frac{\sqrt{x^2-9}}{x}\,dx$

In the example we just did, the radical is of the form $\sqrt{u^2 - a^2}$

variable expression

constant

With subtraction, the left quantity must be larger, so this must be the hypotenuse



Hypotenuse is set, but you can put the radical on either leg and the constant is on the remaining leg

...then we write two trig ratios, each pairing the constant with one other side: $\cos\theta = \dfrac{3}{x}$ $\qquad \tan\theta = \dfrac{\sqrt{x^2-9}}{3}$

We then solve each expression for the x-variable and take the derivative of x...

$$x = \frac{3}{\cos\theta} = 3\sec\theta \qquad \sqrt{x^2-9} = 3\tan\theta$$

...and substitute for x, dx, and the radical into the original integral to convert it to a trigonometric integral which we can solve with the previous section's methods.

$$\frac{dx}{d\theta} = 3\sec\theta\tan\theta$$
$$dx = 3\sec\theta\tan\theta\,d\theta$$

Finally, use the triangle to re-substitute all the θ expressions back to *x*.

#3. $\int x^3\sqrt{x^2+4}\,dx$

$\int (2\tan\theta)^3\, 2\sec\theta\, 2\sec^2\theta\, d\theta$

$32\int \tan^3\theta\sec^3\theta\, d\theta$

$32\int \tan^2\theta\sec^2\theta\,\sec\theta\tan\theta\, d\theta$

$32\int (\sec^2\theta-1)\sec^2\theta\,\sec\theta\tan\theta\, d\theta$

$u=\sec\theta$

$\frac{du}{d\theta}=\sec\theta\tan\theta$

$du=\sec\theta\tan\theta\, d\theta$

$32\int (u^2-1)u^2\,du$

$32\int (u^4-u^2)\,du$

$\frac{32}{5}u^5-\frac{32}{3}u^3+C$

$\frac{32}{5}\sec^5\theta-\frac{32}{3}\sec^3\theta+C = \boxed{\frac{32}{5}\left(\frac{\sqrt{x^2+4}}{2}\right)^5-\frac{32}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3+C}$

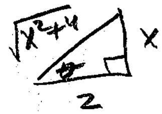
- Build the triangle based upon the form of the radical:

$$\sqrt{16-x^2}=\sqrt{(4)^2-(x)^2}$$

one leg must be x, other leg is the radical

hypotenuse must be 4

- Write two trig ratios, each pairing the constant with one other side.
- Solve each expression for the x-variable and take the derivative to get dx.
- Substitute for x, dx, and the radical into the original integral to convert it to a trigonometric integral which we can solve with the previous section's methods.
- Finally, use the triangle to re-substitute all the θ expressions back to x.



$\tan\theta=\frac{x}{2}$

$x=2\tan\theta$

$\frac{dx}{d\theta}=2\sec^2\theta$

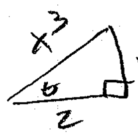
$dx=2\sec^2\theta\, d\theta$

$\cos\theta=\frac{2}{\sqrt{x^2+4}}$

$\sqrt{x^2+4}=\frac{2}{\cos\theta}=2\sec\theta$

$\sec\theta=\frac{\sqrt{x^2+4}}{2}$

#4. $\int \frac{\sqrt{x^6-4}}{x}\,dx$



$\sqrt{x^6-4}=\sqrt{(x^3)^2-(2)^2}$

$\tan\theta=\frac{\sqrt{x^6-4}}{2}$

$\sqrt{x^6-4}=2\tan\theta$

$\cos\theta=\frac{2}{x^3}$

$x^3=\frac{2}{\cos\theta}=2\sec\theta$

$x=(2\sec\theta)^{1/3}$

$\frac{dx}{d\theta}=\frac{1}{3}(2\sec\theta)^{-2/3}(2\sec\theta\tan\theta)$

$dx=\frac{2}{3}(2\sec\theta)^{-2/3}\sec\theta\tan\theta\, d\theta$

$\theta=\arccos\left(\frac{2}{x^3}\right)$

$\int \frac{2\tan\theta}{(2\sec\theta)^{1/3}}\,\frac{2(2\sec\theta)^{-2/3}\sec\theta\tan\theta}{3}\,d\theta$

$\frac{4}{3}\int \frac{\tan^2\theta\sec\theta}{(2\sec\theta)^{1/3}(2\sec\theta)^{2/3}}\,d\theta$

$\frac{4}{3}\int \frac{\tan^2\theta\sec\theta}{2\sec\theta}\,d\theta$

$\frac{2}{3}\int \tan^2\theta\, d\theta$

$\frac{2}{3}\int (\sec^2\theta-1)\,d\theta$

$\frac{2}{3}\int \sec^2\theta\, d\theta-\int\frac{2}{3}\,d\theta$

$\frac{2}{3}\tan\theta-\frac{2}{3}\theta+C$

$\boxed{\frac{2}{3}\left(\frac{\sqrt{x^6-4}}{2}\right)-\frac{2}{3}\left(\arccos\left(\frac{2}{x^3}\right)\right)+C}$

Unit 4-10: Partial Fractions

Partial Fraction Expansion

Another technique we can use to evaluate integrals is **Partial Fraction Expansion**. This technique works when we have a rational function with degree higher in the denominator, and we can factor the denominator into multiple linear or quadratic factors. We then expand the integral into multiple smaller integrals, one for each factor to evaluate.

Partial Fractions Procedure

1) First, if degree in numerator is not lower than denominator, use polynomial division to divide, producing a polynomial plus a remainder with lower degree in numerator.

2) Factor the denominator into linear and quadratic factors of the form: $(px+q)^m$ and $(ax^2+bx+c)^n$

3) For each (possibly multiple) linear factor, create terms of the form:

$$(px+q)^m \to \frac{A}{(px+q)}+\frac{B}{(px+q)^2}+\dots+\frac{F}{(px+q)^m}$$ ...where A,B,...F are (currently unknown) constants.

4) For each (possibly multiple) quadratic factor, create terms of the form:

$$(ax^2+bx+c)^n \to \frac{Gx+H}{(ax^2+bx+c)}+\frac{Ix+J}{(ax^2+bx+c)^2}+\dots+\frac{Mx+N}{(ax^2+bx+c)^n}$$ ...where G,H,...N are (currently unknown) constants.

5) Add all these new terms and set equal to the original integrand. Then multiply each new numerator by whatever is needed so that all terms have the same common denominator.

6) Equate the sum of all the numerators with the original integrand's numerator to form the 'basic equation'. The gather all the x^2 terms together, x terms together, and constants together on each side, and form the equations of a system (which have the unknown constants as 'variables') by matching coefficients.

7) Solve the resulting system for the constants A, B.... then fill these in to form the partial fraction expansion.

Partial Fraction Expansion - Most common case - repeated linear factors

#1. $\int \frac{3x-5}{x^2+6x-7}\,dx$

x^2+6x-7
$(x+7)(x-1)$

$$\frac{3x-5}{(x+7)(x-1)} = \frac{A}{x+7}+\frac{B}{x-1} = \frac{A(x-1)}{(x+7)(x-1)}+\frac{B(x+7)}{(x+7)(x-1)}$$

$$Ax-A+Bx+7B = 3x-5$$
$$(A+B)x+(-A+7B) = (3)x+(-5)$$

$$\begin{cases} A+B=3 \\ -A+7B=-5 \end{cases} \quad 8B=-2 \quad B=-1/4 \quad A-\tfrac{1}{4}=3 \quad 4A-1=12 \quad 4A=13,\ A=13/4$$

$$\int\left[\frac{(13/4)}{x+7}+\frac{(-1/4)}{x-1}\right]dx$$

$$\frac{13}{4}\int\frac{1}{x+7}\,dx-\frac{1}{4}\int\frac{1}{x-1}\,dx$$

$$\boxed{\frac{13}{4}\ln|x+7|-\frac{1}{4}\ln|x-1|+C}$$

#2. $\int \frac{1}{x^2-5x+6}\,dx$

x^2-5x+6
$(x-2)(x-3)$

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2}+\frac{B}{x-3} = \frac{A(x-3)}{(x-2)(x-3)}+\frac{B(x-2)}{(x-2)(x-3)}$$

$$Ax-3A+Bx-2B = 1$$
$$(A+B)x+(-3A-2B) = (0)x+(1)$$

$$\begin{cases} A+B=0 \\ -3A-2B=1 \end{cases} \quad \begin{array}{r} 3A+3B=0 \\ -3A-2B=1 \\ \hline B=1 \end{array} \quad A+1=0 \quad A=-1$$

$$\int\left[\frac{-1}{x-2}+\frac{1}{x-3}\right]dx$$

$$-\int\frac{1}{x-2}\,dx+\int\frac{1}{x-3}\,dx$$

$$\boxed{-\ln|x-2|+\ln|x-3|+C}$$

$$\boxed{=\ln\left|\frac{x-3}{x-2}\right|+C}$$

Partial Fractions works when the denominator is factorable...when it isn't, complete the square

$$\int \frac{1}{x^2+2x+7}dx$$ not factorable

$$\int \frac{1}{x^2+2x+7}dx \qquad (x^2+2x+1)+7-1$$

$$\int \frac{1}{(x+1)^2+6}dx \qquad (x+1)^2+6$$

$$\int \frac{1}{(x+1)^2+(\sqrt{6})^2}dx \qquad u=x+1, du=dx \quad a=\sqrt{6}$$

$$\int \frac{1}{u^2+a^2}du = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$

$$\frac{1}{\sqrt{6}}\arctan\left(\frac{x+1}{\sqrt{6}}\right)+C$$

What if numerator degree is equal to or higher than denominator?

You use polynomial division to divide out a polynomial, leaving a remainder to do the partial fractions procedure on...

#3. $\int \frac{x^3-x+3}{x^2+x-2}dx$

$$\begin{array}{r} x-1 \\ x^2+x-2 \overline{)\, x^3+0x^2-x+3} \\ -(x^3+x^2-2x) \\ \hline -x^2+x+3 \\ -(-x^2-x+2) \\ \hline 2x+1 \end{array}$$

$$\frac{x^3-x+3}{x^2+x-2} = x-1+\frac{2x+1}{x^2+x-2}$$

$$\int\left[x-1+\frac{2x+1}{x^2+x-2}\right]dx$$

$$\int(x-1)dx+\int\frac{2x+1}{x^2+x-2}dx$$

x^2+x-2
$(x+2)(x-1)$

$$\frac{2x+1}{x^2+x-2} = \frac{A}{x+2}+\frac{B}{x-1} = \frac{A(x-1)}{(x+2)(x-1)}+\frac{B(x+2)}{(x+2)(x-1)} = \frac{1}{x+2}+\frac{1}{x-1}$$

$$Ax-A+Bx+2B = 2x+1$$

$$(A+B)x+(-A+2B) = (2)x+(1)$$

$$\begin{cases} A+B=2 \\ -A+2B=1 \end{cases} \qquad 3B=3 \quad B=1 \qquad A+1=2 \quad A=1$$

$$\int(x-1)dx+\int\frac{1}{x+2}dx+\int\frac{1}{x-1}dx$$

$$\boxed{\tfrac{1}{2}x^2-x+\ln|x+2|+\ln|x-1|+C}$$

What if there are multiple copies of factors?

You include a term for each 'multiplicity' with its own constants...

#4. $\frac{1}{(x-5)^3(x^2+x+1)^2} = \frac{A}{x-5}+\frac{B}{(x-5)^2}+\frac{C}{(x-5)^3}+\frac{Dx+E}{x^2+x+1}+\frac{Fx+G}{(x^2+x+1)^2}$

#5. $\int \frac{5x+3}{x^2+x-2}\,dx$

x^2+x-2

$(x+2)(x-1)$

$\frac{5x+3}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

$A(x-1)+B(x+2) = 5x+3$

$Ax - A + Bx + 2B = 5x+3$

$(A+B)x + (-A+2B) = (\ )x + (3)$

$\begin{cases} A+B=5 \\ -A+2B=3 \end{cases}$

$3B = 8$

$B = \frac{8}{3}$

$A + \frac{8}{3} = 5$

$3A + 8 = 15$

$3A = 7$

$A = 7/3$

$\frac{7}{3}\int \frac{1}{x+2}dx + \frac{8}{3}\int \frac{1}{x-1}dx$

$\boxed{\frac{7}{3}\ln|x+2| + \frac{8}{3}\ln|x-1| + C}$

#6. $\int \frac{3x-5}{x^2+2x-8}\,dx$

x^2+2x-8

$(x+4)(x-2)$

$\frac{3x-5}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}$

$A(x-2) + B(x+4) = 3x-5$

$Ax - 2A + Bx + 4B = 3x-5$

$(A+B)x + (-2A+4B) = (3)x + (-5)$

$\begin{cases} A+B=3 \\ -2A+4B=-5 \end{cases}$

$2A+2B=6$

$-2A+4B=-5$

$6B = 1$

$B = \frac{1}{6}$

$A + \frac{1}{6} = 3$

$6A + 1 = 18$

$6A = 17$

$A = \frac{17}{6}$

$\frac{17}{6}\int \frac{1}{x+4}dx + \frac{1}{6}\int \frac{1}{x-2}dx$

$\boxed{\frac{17}{6}\ln|x+4| + \frac{1}{6}\ln|x-2| + C}$

#7. $\int \frac{6x-7}{x^2-3x-4}\,dx$

x^2-3x-4

$(x+1)(x-4)$

$\frac{6x-7}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$

$A(x-4)+B(x+1) = 6x-7$

$Ax-4A+Bx+B = 6x-7$

$(A+B)x+(-4A+B) = (6)x+(-7)$

$\begin{cases} A+B=6 \\ -4A+B=-7 \end{cases}$

$-A-B=-6$
$-4A+B=-7$
$-5A=-13$
$A=\frac{13}{5}$

$\frac{13}{5}+B=6$
$13+5B=30$
$5B=17$
$B=\frac{17}{5}$

$\frac{13}{5}\int \frac{1}{x+1}dx + \frac{17}{5}\int \frac{1}{x-4}dx$

$\boxed{\frac{13}{5}\ln|x+1| + \frac{17}{5}\ln|x-4|+C}$

A very complex example

$$\int \frac{1}{x^3-1}\,dx$$

synthetic division to start factoring:

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & -1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

$$x^3-1=(x-1)(x^2+x+1) \qquad \frac{x^3-1}{x-1}=x^2+x+1$$

(doesn't factor further)

$$\int \frac{1}{x^3-1}\,dx$$

$$\int \frac{1}{(x-1)(x^2+x+1)}\,dx$$

$$\frac{1}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1} \qquad \frac{1}{(x-1)(x^2+x+1)}=\frac{\left(\frac{1}{3}\right)}{x-1}+\frac{\left(-\frac{1}{3}\right)x+\left(-\frac{2}{3}\right)}{x^2+x+1}$$

$$\frac{1}{(x-1)(x^2+x+1)}=\frac{A(x^2+x+1)}{(x^2+x+1)(x-1)}+\frac{(Bx+C)(x-1)}{(x^2+x+1)(x-1)}$$

$$Ax^2+Ax+A+Bx^2-Bx+Cx-C=1$$

$$(A+B)x^2+(A-B+C)x+(A-C)=(0)x^2+(0)x+(1)$$

$$\begin{cases} A+B \quad\;\; =0 \\ A-B+C=0 \\ A \quad\;\; -C=1 \end{cases} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix} \; rref \; \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{-1}{3} \\ 0 & 0 & 1 & \frac{-2}{3} \end{bmatrix}$$

$$A=\frac{1}{3},\ B=-\frac{1}{3},\ C=-\frac{2}{3}$$

$$\int\left[\frac{\left(\frac{1}{3}\right)}{x-1}+\frac{\left(-\frac{1}{3}\right)x+\left(-\frac{2}{3}\right)}{x^2+x+1}\right]dx$$

$$\frac{1}{3}\int\frac{1}{x-1}\,dx-\frac{1}{3}\int\frac{x+2}{x^2+x+1}\,dx$$

add/subtract something to make u-sub work for denominator:

$$u=x^2+x+1$$

$$du=(2x+1)dx$$

$$du=2\left(x+\frac{1}{2}\right)dx \quad \textit{need } x+\frac{1}{2} \textit{ in numerator}$$

$$\frac{1}{3}\int\frac{1}{x-1}\,dx-\frac{1}{3}\int\frac{x+\frac{1}{2}-\frac{1}{2}+2}{x^2+x+1}\,dx$$

$$\frac{1}{3}\int\frac{1}{x-1}\,dx-\frac{1}{3}\int\frac{x+\frac{1}{2}+\frac{3}{2}}{x^2+x+1}\,dx$$

$$\frac{1}{3}\int\frac{1}{x-1}\,dx-\frac{1}{3}\int\frac{x+\frac{1}{2}}{x^2+x+1}\,dx-\frac{1}{3}\int\frac{\frac{3}{2}}{x^2+x+1}\,dx$$

$$\frac{1}{3}\int\frac{1}{x-1}\,dx-\frac{1}{3}\int\frac{x+\frac{1}{2}}{x^2+x+1}\,dx-\frac{1}{2}\int\frac{1}{x^2+x+1}\,dx$$

A very complex example

$$\int \frac{1}{x^3-1}\,dx$$

$$\frac{1}{3}\int \frac{1}{x-1}\,dx-\frac{1}{3}\int \frac{x+\frac{1}{2}}{x^2+x+1}\,dx-\frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$$

At this point, the first two integrals can be done by u-substitution. For the last one, the denominator isn't factorable, so let's complete the square and hope for an arctan form:

$$\frac{1}{3}\int \frac{1}{x-1}\,dx-\frac{1}{3}\int \frac{x+\frac{1}{2}}{x^2+x+1}\,dx-\frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$x^2+x+\frac{1}{4}+1-\frac{1}{4}$$

$$\left(x-\frac{1}{2}\right)^2+\frac{3}{4}$$

$u=x-1$ $\qquad u=x^2+x+1$ $\qquad u=x-\frac{1}{2},\ a=\sqrt{\frac{3}{4}}$

$du=dx$ $\qquad du=(2x+1)\,dx$ $\qquad du=dx$

$$\frac{1}{3}\int \frac{1}{u}\,du-\frac{1}{3}\int \frac{1}{u}\,du-\frac{1}{2}\int \frac{1}{u^2+a^2}\,du$$

$$\boxed{\frac{1}{3}\ln|x-1|-\frac{1}{3}\ln\left|x^2+x+1\right|-\frac{1}{2}\frac{1}{\sqrt{\frac{3}{4}}}\arctan\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)+C}$$

We won't encounter anything nearly this complex in our course or on the AP Exam. But it is good to know how flexible and powerful these tools are :)

Unit 4-11: Improper Integrals

There are some conditions required in order to use the Fund. Theorem for definite integrals

When we evaluate definite integrals using the Fundamental Theorem of Calculus $\int_a^b f(x)\,dx = F(b) - F(a)$ there are a few conditions that must be met:

- The interval [a,b] must be finite.
- *f(x)* must be continuous over the entire interval [a,b].

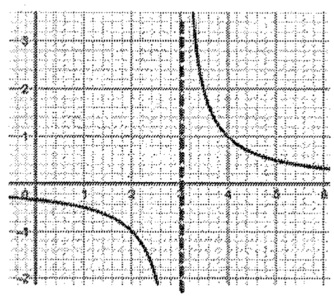
Integrals which do not meet these conditions are called **Improper Integrals**.

Examples of improper integrals:

$$\int_2^{\infty} \frac{1}{2-x^2}\,dx$$

improper because the interval is not finite

$$\int_2^4 \frac{1}{x-3}\,dx$$



improper because the function is not continuous over [2,4]

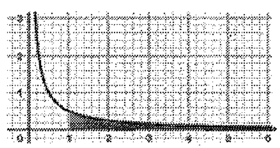
Convergence and Divergence of Improper Integrals

In this section, we will learn techniques for attempting to evaluate improper integrals. Some improper integrals will evaluate to a finite, numerical value and these integrals are said to **converge** to this value. Other improper integrals will evaluate to an infinite value and these integrals are said to **diverge**.

Evaluate by replacing infinite limits or x value at a discontinuity with a constant

There are a couple of variations on the theme, but the main idea for evaluating improper integrals is to replace any limit of integration x-value with a constant, evaluate the integral using the constant, then take the limit as that constant approaches the problem value.

A first example: $\int_1^{\infty} \frac{1}{x^2}\,dx$



This is equivalent to finding the area under this curve from 1 out to infinity. Seemingly, this area might be infinite, but on the other hand, the curve is rapidly approaching zero.

First, change to proper integral form: $= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$

Then, we'll evaluate the indefinite integral... $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \left[-x^{-1}\right] = \left[-\frac{1}{x}\right]$

Now, when we plug in the limits of integration we will use a constant for the infinity and use a limit to evaluate:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b\to\infty}\left[-\frac{1}{x}\right]^b - \left[-\frac{1}{x}\right]_1$$

$$= \lim_{b\to\infty}\left[-\frac{1}{b}\right] - \left[-\frac{1}{(1)}\right]$$

$$= 0 - [-1] = \boxed{1}$$

Interestingly, even though we are integrating forever in x, the area converges to a finite area of 1.

Another example

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx$$

First, we'll evaluate the indefinite integral... $\int \frac{1}{x}\,dx = \ln|x|$

Now, when we plug in the limits of integration we will use a constant for the infinity and use a limit to evaluate:

$$\lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx = \lim_{b\to\infty} \left[\ln|x|\right]_1^b$$

$$= \lim_{b\to\infty} \left[\ln|x|\right]^b - \left[\ln|x|\right]_1$$

$$= \lim_{b\to\infty} \left[\ln|b|\right] - \left[\ln|1|\right]$$

∞ **diverges**

This improper integral **diverges**. The 1/x curve doesn't approach zero as fast as the $1/x^2$ curve, so as x increases to infinity, the area does not stay finite, but grows infinitely.

$\cancel{\infty - 0 = \infty}$

We shouldn't write down things like this because you can't do arithmetic with infinity (may be penalized on an AP exam)

Forms with infinite integration limits that produce improper integrals

$\int_a^{\infty} f(x)\,dx$

to evaluate...

$\lim_{b\to\infty} \left[F(b)\right] - \left[F(a)\right]$

$\int_{-\infty}^{a} f(x)\,dx$

to evaluate...

$\left[F(a)\right] - \lim_{b\to\infty} \left[F(b)\right]$

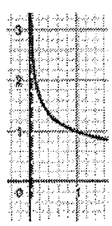
$\int_{-\infty}^{\infty} f(x)\,dx$

to evaluate, split the integral at any convenient x − value, c...

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$

If there is any x-value where f(x) is discontinuous, replace that x value with a constant, use limit

#3. $\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$



$$\int \frac{1}{\sqrt[3]{x}}\,dx = \int x^{-1/3}\,dx = \left[\frac{3}{2}x^{2/3}\right]_0^1$$

When we plug in the zero, we must use a limit as x approaches zero from the right side...

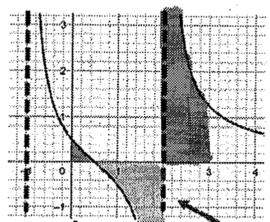
$$\left[\frac{3}{2}(1)^{2/3}\right] - \lim_{b\to 0^+} \left(\frac{3}{2}(b)^{2/3}\right)$$

$$\frac{3}{2}(1) - \frac{3}{2}(0) = \boxed{\frac{3}{2}}$$

This area *converges* to 3/2.

Always check to make sure the function is continuous over the integral of integration

#4. $\int_0^3 \frac{2x-1}{x^2-x-2}\,dx = \int_0^3 \frac{2x-1}{(x+1)(x-2)}\,dx$



at x=2 is a problem, so we must split the integration here and use limits approaching 2

We must split the integral at x=2...

$$\int_0^2 \frac{2x-1}{x^2-x-2}\,dx + \int_2^3 \frac{2x-1}{x^2-x-2}\,dx$$

Evaluate the indefinite integral (using u-substitution):

$$\int \frac{2x-1}{x^2-x-2}\,dx \quad u = x^2-x-2 \quad du = (2x-1)dx$$

$$\int \frac{1}{u}\,du = \ln|u| = \ln|x^2-x-2|$$

Now plug in limits of integration and use limits for the x=2 values:

$$\int_0^2 \frac{2x-1}{x^2-x-2}\,dx + \int_2^3 \frac{2x-1}{x^2-x-2}\,dx$$

$$\left[\ln|x^2-x-2|\right]_0^2 + \left[\ln|x^2-x-2|\right]_2^3$$

$$\lim_{b\to 2^-}\left(\ln|b^2-b-2|\right) - \ln|0| + \left[\ln|3^2-3-2|\right]^3 - \lim_{b\to 2^+}\left(\ln|b^2-b-2|\right)$$

$$-\infty - undef + \ln|3| - (-\infty)$$

$$-\infty - \infty + \ln|3| + \infty$$

The ln0 is undefined but we could use a limit to evaluate this as well because we approaching x from the right:

$$\ln|0| = \lim_{x\to 0^+} \ln|x| = -\infty$$

Positive and negative infinities are fighting for control, but we just say that this improper integral **diverges**.

We say an integral diverges if, for any reason, it doesn't resolve to a single, finite, number

#5. $\int_0^\infty \cos(x)\,dx = \lim_{b\to\infty} \int_0^b \cos(x)dx$

$\lim_{b\to\infty} \left[\sin(x)\right]_0^b$

$\lim_{b\to\infty} \left[\sin(b) - \sin(0)\right]$

oscillates $\quad - 0$

$\boxed{\text{diverges}}$

We can use all of our previous techiques when evaluating the indefinite integral...

We may still need any of our previous methods in order to evaluate the indefinite integral. With an improper integral, what changes is that at any problematic x values, we evaluate using a limit as x is approaching those values.

#6. $\int_{-\infty}^{1} xe^{2x}\,dx = \lim_{b \to -\infty^+} \int_b^1 xe^{2x}\,dx$ by parts:

$u = x \quad dv = e^{2x}dx$

$\frac{du}{dx} = 1 \quad \int dv = \int e^{2x}dx$

$du = dx \quad v = \frac{1}{2}e^{2x}$

$\lim_{b \to -\infty^+} \left[uv - \int v\,du\right]_b^1$

$\lim_{b \to -\infty^+} \left[\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx\right]_b^1$

$\lim_{b \to -\infty^+} \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_b^1$

$\left[\frac{1}{2}(1)e^{2(1)} - \frac{1}{4}(1)e^{2(1)}\right] - \lim_{b \to -\infty^+}\left[\frac{1}{2}be^{2b} - \frac{1}{4}e^{2b}\right]$

$\frac{1}{2}e^2 - \frac{1}{4}e^2 - \lim_{b \to -\infty^+}\left(\frac{1}{2}be^{2b}\right) + \lim_{b \to -\infty^+}\left(\frac{1}{4}e^{2b}\right)$

$\to (-\infty)(0)$

$\frac{1}{2}e^2 - \frac{1}{4}e^2 - \frac{1}{2}\lim_{b \to -\infty^+}\frac{b}{e^{-2b}} + 0$

$\lim_{b \to -\infty^+} b = -\infty$

$\lim_{b \to -\infty^+} e^{-2b} = \infty$

$\left(-\frac{\infty}{\infty}\right)$ ~~indeterminant form~~ use L'Hopital's Rule

$-\frac{1}{2}\lim_{b \to -\infty^+}\frac{1}{-2e^{-2b}}$

$\frac{1}{2}e^2 - \frac{1}{4}e^2 - 0 + 0$

$= \boxed{\frac{1}{4}e^2 \quad \text{(converges)}}$

Unit 4-12: Strategies for Integration

Summary of Integration Strategies

We've now learned a number of strategies for integration, so let's consider these and when they are applicable.

1) Simplify algebraically / Use Antiderivative Shortcuts

We should always first consider if we can split the integrand into multiple terms, each of which can be integrated using a shortcut:

$$\int \frac{x^3-2x}{\sqrt{x}}\,dx = \int \frac{x^3-2x}{x^{1/2}}\,dx = \int\left(\frac{x^3}{x^{1/2}}-\frac{2x}{x^{1/2}}\right)dx = \int x^{5/2}\,dx - 2\int x^{1/2}\,dx$$

2) Use trig identities

When trig functions are involved, sometimes trig identities will simplify things:

$$\int\left(1+\tan^2 x\right)dx$$

$$\int \sec^2 x\,dx$$

$$\tan x + C$$

$$\frac{\cos^2 x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}$$

$$1+\tan^2 x = \sec^2 x$$

3) u-substitution

When one function is 'inside' another (or considering the entire denominator as 'inside') <u>and the derivative of the inside function appears in the integral</u>, try u-substitution:

$$\int 3e^{-x^2}x\,dx$$

$$u=-x^2,\ du=-2x\,dx,\ x\,dx=-\frac{1}{2}du$$

take derivative to find du, substitute back into original integral:

$$\int 3e^{-x^2}x\,dx = -\frac{3}{2}\int e^u\,du$$

4) Trig functions raised to powers - trigonometric integrals

If there are trig functions raised to powers, use the rules for trigonometric integrals. The main idea is to reserve some of the trig function(s) to build the 'du' and write everything else as powers of one trig function.

$$\int \tan^5 x\sec^7 x\,dx = \int \tan^4 x\sec^6 x\sec x\tan x\,dx$$

$$=\int\left(\tan^2 x\right)^2\sec^6 x\sec x\tan x\,dx = \int\left(\sec^2 x-1\right)^2\sec^6 x\sec x\tan x\,dx$$

$$u=\sec x,\ du=\sec x\tan x\,dx$$

$$=\int\left(u^2-1\right)^2u^6\,du = \int\left(u^{10}-2u^8+u^6\right)du$$

5) Rational functions with factorable denominator - use partial fractions

For rational functions, try partial fraction expansion:

$$\int \frac{x^3+x}{x-1}\,dx = \int\left(x^2+x+2+\frac{2}{x-1}\right)dx$$

<u>Partial Fractions Procedure</u>

1) First, if degree in numerator is not lower than denominator, use polynomial division to divide, producing a polynomial plus a remainder with lower degree in numerator.

2) Factor the denominator into linear and quadratic factors of the form: $(px+q)^m$ and $(ax^2+bx+c)^n$

3) For each (possibly multiple) linear factor, create terms of the form:

$$(px+q)^m \rightarrow \frac{A}{(px+q)}+\frac{B}{(px+q)^2}+\ldots+\frac{F}{(px+q)^m}$$ ...where A,B,...F are (currently unknown) constants.

4) For each (possibly multiple) quadratic factor, create terms of the form:

$$(ax^2+bx+c)^n \rightarrow \frac{Gx+H}{(ax^2+bx+c)}+\frac{Ix+J}{(ax^2+bx+c)^2}+\ldots+\frac{Mx+N}{(ax^2+bx+c)^n}$$ ...where G,H,...N are (currently unknown) constants.

5) Add all these new terms and set equal to the original integrand. Then multiply each new numerator by whatever is needed so that all terms have the same common denominator.

6) Equate the sum of all the numerators with the original integrand's numerator to form the 'basic equation'. The gather all the x^2 terms together, x terms together, and constants together on each side, and form the equations of a system (which have the unknown constants as 'variables') by matching coefficients.

7) Solve the resulting system for the constants A, B,... then fill these in to form the partial fraction expansion.

6) Rational functions with non-factorable denominator - complete the square to try for arctan form

When you have a quadratic polynomial in the denominator of a rational function, you can try completing the square to obtain a form which matches the arctan integration shortcut (this sometimes happens for one part of a partial fraction expansion):

$$\int \frac{1}{x^2+x+1}\,dx$$

complete the square...

$$x^2+x+\underline{\underline{1/4}}+1-\underline{\underline{1/4}}$$

$$\left(x+1/2\right)^2+3/4$$

$$\int \frac{1}{\left(x+1/2\right)^2+3/4}\,dx$$

...now matches the arctan shortcut:

$$\int \frac{du}{u^2+a^2}=\frac{1}{a}\arctan\left(\frac{u}{a}\right)$$

with $u = x+1/2,\ a=\sqrt{3}/2$

$$\frac{2}{\sqrt{3}}\arctan\left(\frac{x+1/2}{\left(\sqrt{3}/2\right)}\right)+C$$

7) Integrand includes two factors multiplied - use integration by parts

If the integrand can be split into two factors multiplied together, try integration by parts:

$$\int \underbrace{x}\,\underbrace{e^x\,dx}$$

deriv. to find du			antiderivative to find dv
	$u=x,$	$dv=e^x dx$	
	$\frac{du}{dx}=1$	$\int dv=\int e^x dx$	
	$du=1dx$	$v=e^x$	

substitute into

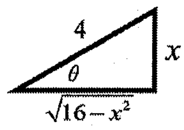
$$uv-\int v\,du$$

$$xe^x-\int e^x dx$$

8) Integral includes radicals - try trigonometric substitution to produce a trigonometric form

When integrand contains radicals, sometimes you can use <u>trigonometric substitution</u>:

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx$$



$$\sin\theta = \frac{x}{4},\quad x = 4\sin\theta,\quad dx = 4\cos\theta\, d\theta$$

$$\cos\theta = \frac{\sqrt{16-x^2}}{4};\quad \sqrt{16-x^2} = 4\cos\theta$$

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = \int \frac{(4\sin\theta)^3}{4\cos\theta} 4\cos\theta\, d\theta = 64\int \sin^3\theta\, d\theta$$

then...

$$64\int \sin^3\theta\, d\theta = 64\int \sin^2\theta \sin\theta\, d\theta = 64\int \left(1-\cos^2\theta\right)\sin\theta\, d\theta$$

$$= 64\int \sin\theta\, d\theta - 64\int \cos^2\theta \sin\theta\, d\theta$$

(u-sub)

9) Improper integrals (applies to definite integrals only)

For definite integrals, be on the lookout for two things which can cause an definite integral to not meet conditions for evaluation:

- The interval [a,b] must be finite. $\int_2^{\infty} \frac{1}{2-x^2}\,dx$ an infinite limit of integration
- *f(x)* must be continuous over the entire interval [a,b]. $\int_2^4 \frac{1}{x-3}\,dx$ a vertical asymptote within the integration interval

Both are handled by replacing any 'bad' values of *x* with constants and taking the limit as *x* approaches the bad value:

$$\int_x^{\infty} \frac{1}{2-x^2}\,dx = \lim_{b\to\infty} \int_x^{b} \frac{1}{2-x^2}\,dx$$

$$\int_2^4 \frac{1}{x-3}\,dx = \lim_{b\to 3^-} \int_2^{b} \frac{1}{x-3}\,dx + \lim_{d\to 3^+} \int_d^{4} \frac{1}{x-3}\,dx$$

If your first strategy doesn't work, try other strategies until you find one that works!

Sometimes, more than one method is required...the first produces multiple integrals which each require further, different, strategies.