

AP Calc BC – Lesson Notes – Unit 4: Applications of Derivatives

Unit 4-1: Extrema on an Interval

Larsen: 3.1 (Stewart: 4.1)

Terminology of Extrema

Maximum = The highest numerical value.

Minimum = The lowest numerical value.

Extrema = (plural) The set of all maximum and minimum values.

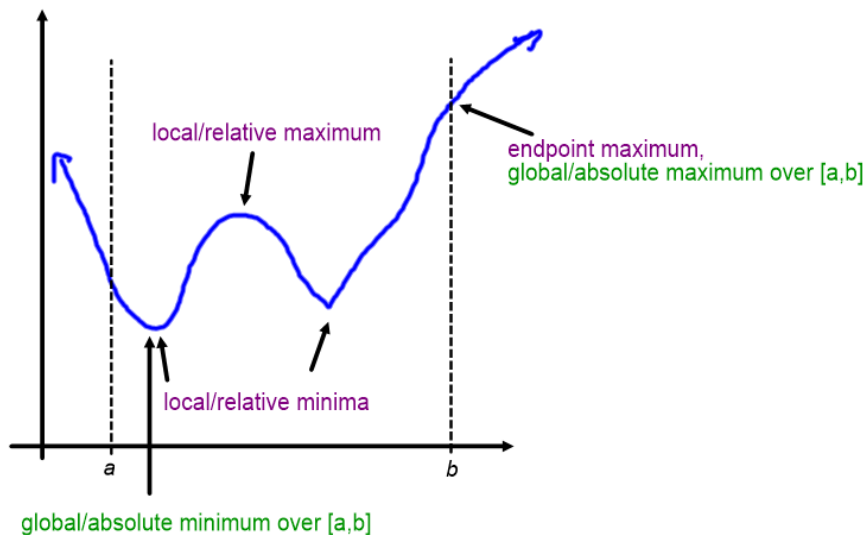
Local/Relative Maximum = The highest numerical value in a local region where a curve changes from increasing to decreasing.

Local/Relative Minimum = The lowest numerical value in a local region where a curve changes from decreasing to increasing.

Endpoint Maximum/Minimum = The highest (or lowest) numerical value over a defined interval if it occurs at a point which exists at the upper or lower end of the interval.

Absolute/Global Maximum/Minimum over an interval = The highest (or lowest) numerical value over a defined interval regardless of whether it occurs within or at the endpoints of the interval.

Critical Number = An x value where $f'(x)$ is either zero or undefined.



Local/Relative Extrema only occur at critical numbers:

- When first derivative = 0.
- When first derivative is undefined.
- Make a 'slope map'/'wiggle graph' to verify changing direction.

Absolute/Global Extrema may occur at Local/Relative Extrema or at endpoints:

- Find $f(x)$ at both endpoints.
- Compare endpoint values with local extrema $f(x)$ values to find absolute/global extrema.

Ex) Find the absolute maximum & absolute minimum for:

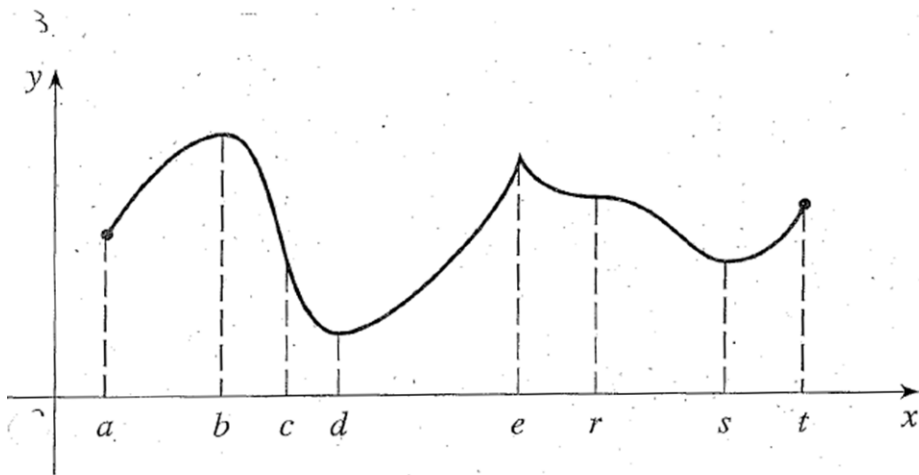
$$f(x) = \frac{3}{5}x^5 - \frac{2}{3}x^3 - x + 2 \quad \text{on the interval } [-2, 1]$$

Remember...

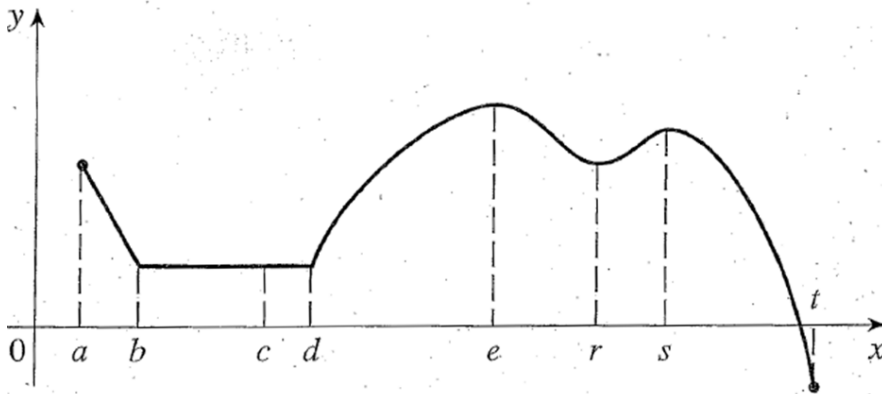
Three places to check for a max or a min:

- **derivative = 0**
- **f' does not exist**
- **endpoints**

Ex) Find all local and absolute extrema over the interval $[a, t]$



Ex) Find all local and absolute extrema over the interval $[a, t]$



Ex) Sketch the graph of a function f that is continuous on $[0, 3]$ and has the given properties:

2 is a critical number, but f has no local maximum or minimum

- Ex) a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
- b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
- c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.

Ex) Find the absolute and local max and min values of f . Begin by sketching its graph by hand.

$$f(x) = 2 - x^4$$

Ex) Find the critical numbers of the function:

$$g(t) = \sqrt{t}(1-t)$$

Ex) Find the absolute max and absolute min values of

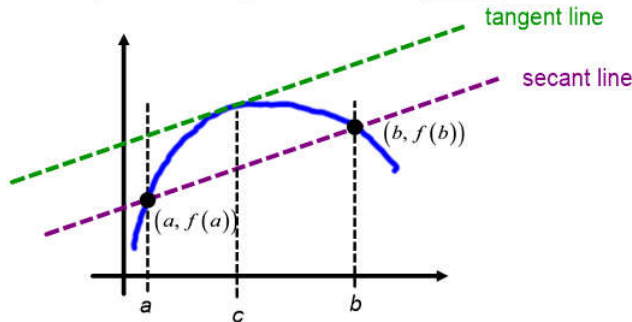
$$f(x) = \frac{\ln x}{x}, \text{ on the interval } [1, 3]$$

Mean Value Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists a number c in (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The Mean Value Theorem says that if a curve is continuous and differentiable then if you make a secant line with the points at the ends on an interval there must exist an input value c in the interval where the derivative (slope of the tangent line to the curve at c) is equal to the slope of the secant line. At this point, the tangent line will be parallel to the secant line.



In other words, you can find a mean (average) rate of change across an interval, and there is some input value where the instantaneous rate of change equals the mean rate of change.

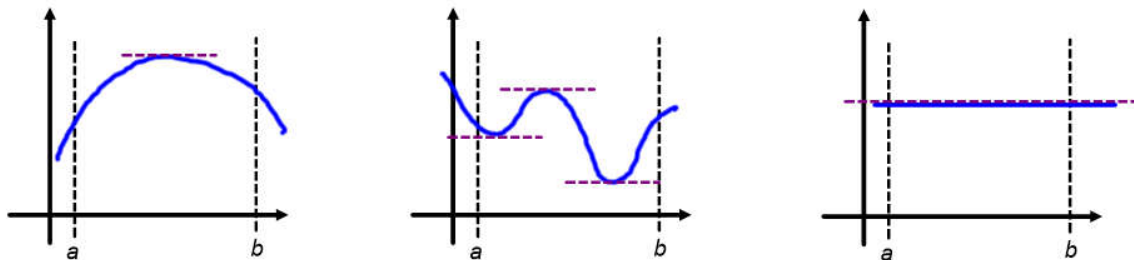
If, on a 2-hour car trip, you average 50 miles per hour, then (according to the Mean Value Theorem) at least once during the trip, your speedometer actually read 50 mph.

Rolle's Theorem

Rolle's Theorem is a specific case of the Mean Value Theorem (with horizontal secant line)...

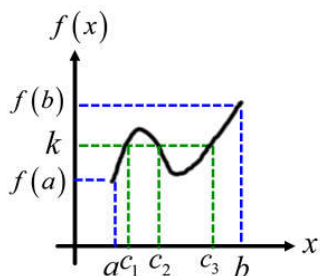
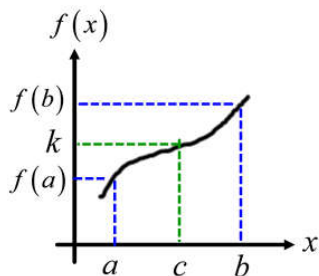
Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) . If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

(as long as a function curve is continuous and differentiable over an interval, there must be at least one place where the first derivative is zero = horizontal tangent)



Reminder: Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

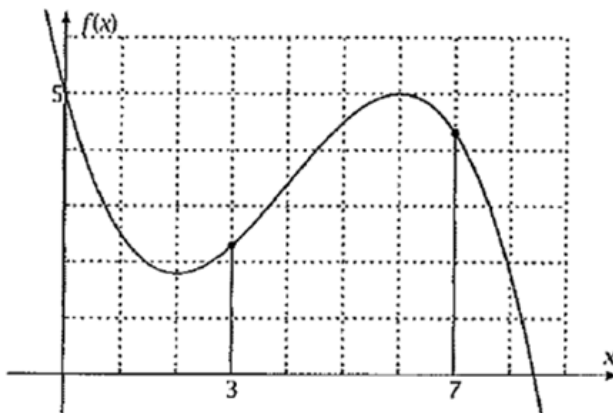


If a function is continuous, to get from one value to another you must go through all the values in-between, and for a value in an x -interval, there must be a value c in the x -interval for which $f(c)$ is between the endpoint y -values.

Examples

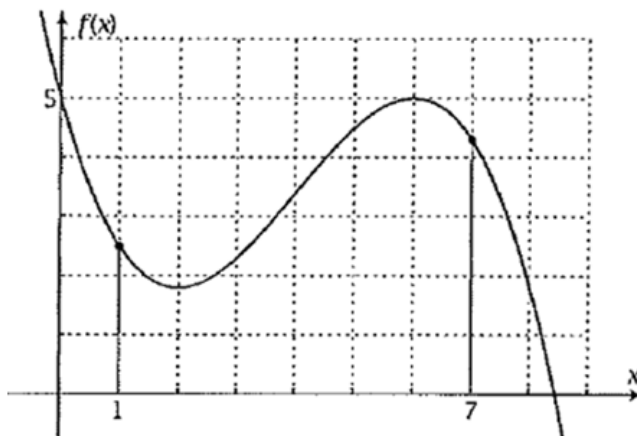
1. For $f(x) = -0.1x^3 + 1.2x^2 - 3.6x + 5$, graphed below, there is a value of $x = c$ between 3 and 7 at which the tangent to the graph is parallel to the secant line through $(3, f(3))$ and $(7, f(7))$.

- Draw the secant line and the tangent line.
- From the graph, $c \approx$ _____
- Is f differentiable on $(3, 7)$? _____
- Is f continuous on $[3, 7]$? _____

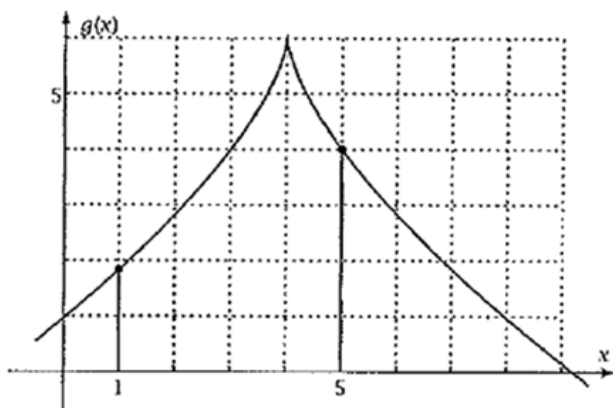


2. Function f from Problem 1 has two values of $x = c$ between $x = 1$ and $x = 7$ at which $f'(c)$ equals the slope of the corresponding secant line. (That is, the tangent line parallels the secant line.)

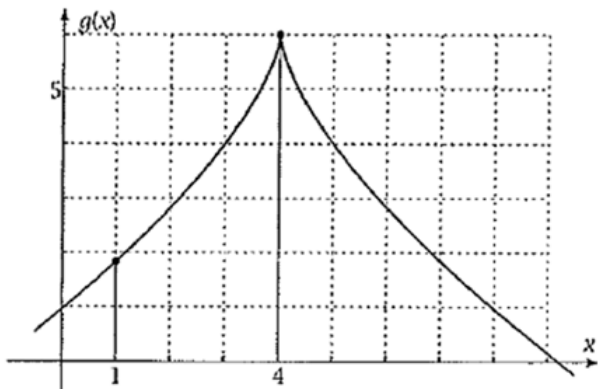
- Draw the secant and tangents on the graph below.
- From the graph, $c \approx$ _____ and $c \approx$ _____
- Is f differentiable on $(1, 7)$? _____
- Is f continuous on $[1, 7]$? _____



3. For $g(x) = 6 - 2(x - 4)^{2/3}$, graphed below,
- Draw a secant line through $(1, g(1))$ and $(5, g(5))$.
 - Is g differentiable on $(1, 5)$? _____
 - Is g continuous on $[1, 5]$? _____
 - Tell why there is *no* value of $x = c$ between $x = 1$ and $x = 5$ at which $g'(c)$ equals the slope of the secant line.



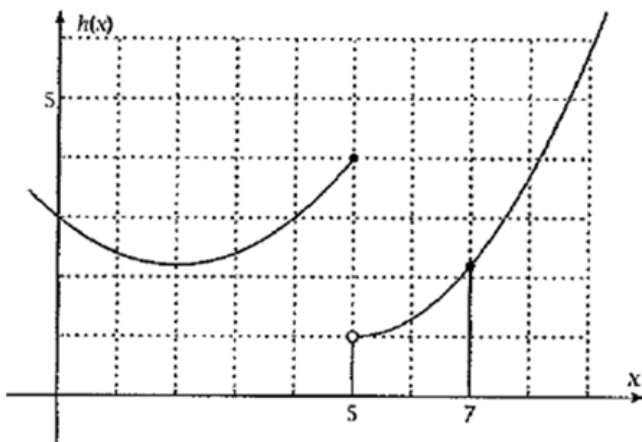
4. Function g from Problem 3 *does* have a value $x = c$ in $(1, 4)$ for which $g'(c)$ equals the slope of the secant line through $(1, g(1))$ and $(4, g(4))$.
- Draw the secant line and tangent line, below.
 - From the graph, $c \approx$ _____
 - Is g differentiable on $(1, 4)$? _____
 - Is g continuous on $[1, 4]$? _____



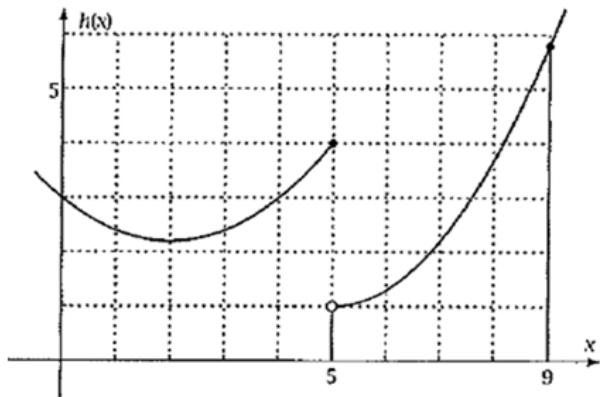
5. Piecewise function h is defined by

$$h(x) = \begin{cases} 0.2(x - 2)^2 + 2.2, & \text{if } x \leq 5 \\ 0.3(x - 5)^2 + 1, & \text{if } x > 5 \end{cases}$$

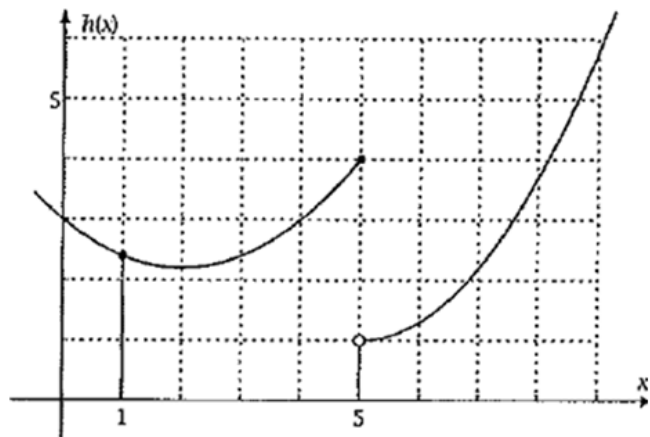
- Draw a secant line through $(5, h(5))$ and $(7, h(7))$.
- Is h differentiable on $(5, 7)$? _____
- Is h continuous on $[5, 7]$? _____
- Why is there *no* value $x = c$ in $(5, 7)$ for which $h'(c)$ equals the slope of the secant line?



6. The graph below is function h from Problem 5.
- Draw a secant line through $(5, h(5))$ and $(9, h(9))$.
 - Is h differentiable on $(5, 9)$? _____
 - Is h continuous on $[5, 9]$? _____
 - There is a point $x = c$ in $(5, 9)$ where $h'(c)$ equals the slope of the secant line. Draw the tangent line. Estimate the value of c . _____



7. The graph below is function h from Problem 5.
- Draw a secant line through $(1, h(1))$ and $(5, h(5))$.
 - Show that there is a point $x = c$ in $(1, 5)$ where $h'(c)$ equals the slope of the secant line.
 - Is h differentiable on $(1, 5)$? _____
 - Explain why h is continuous on $[1, 5]$, even though there is a step discontinuity at $x = 5$.



8. The mean value theorem states:
 If f is differentiable on (a, b) and
 f is continuous on $[a, b]$,
 then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{i.e., the secant's slope}$$

For which problem(s) are

- the hypotheses and conclusion true? _____
 - the hypotheses and conclusion not true? _____
 - the conclusion true, but not the hypotheses? _____
9. The number $x = c$ in the above problems is the "mean" value referred to in the name "mean value theorem." Explain why the hypotheses are sufficient conditions for the conclusion, but *not* necessary conditions.

Ex) Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

$$f(x) = x^2 - 4x + 1, \quad [0, 4]$$

5. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$.

Why does this not contradict Rolle's Theorem?

11. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f(x) = 3x^2 + 2x + 5, \quad [-1, 1]$$

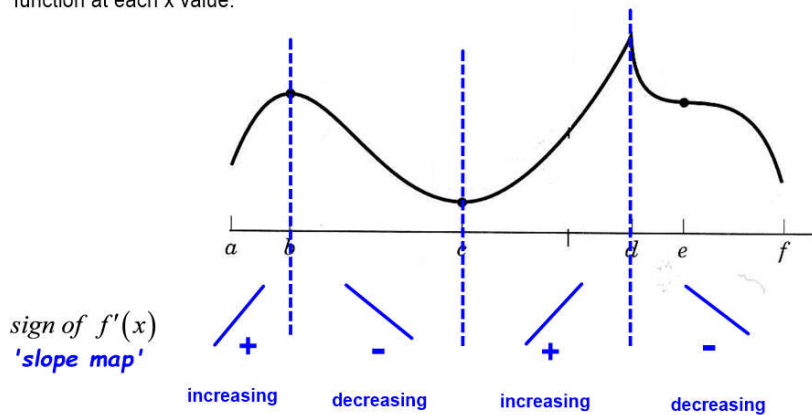
19. Show that the equation $x^5 - 6x + c = 0$ has at most one root in the interval $[-1, 1]$.

Unit 4-3: First Derivative Test

Larsen: 3.3 (Stewart: 4.3)

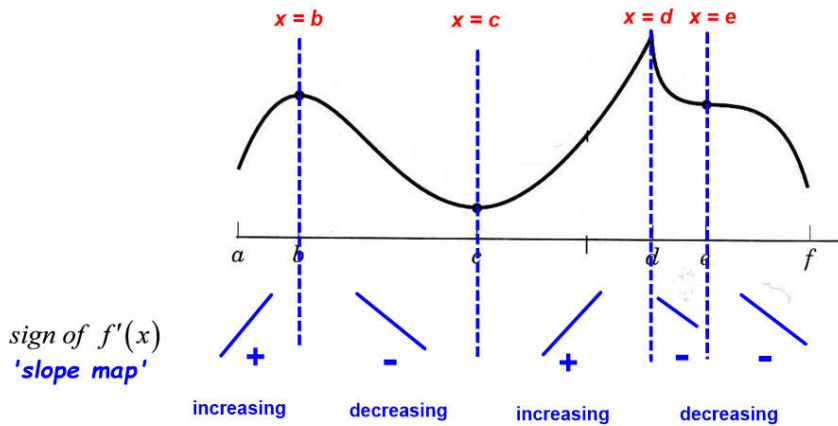
The First Derivative Test

The **first derivative** of a function gives the instantaneous rate of change, or '**slope**', of the function at each x value:



...by checking the *sign of $f'(x)$* we can determine if the function is increasing or decreasing.

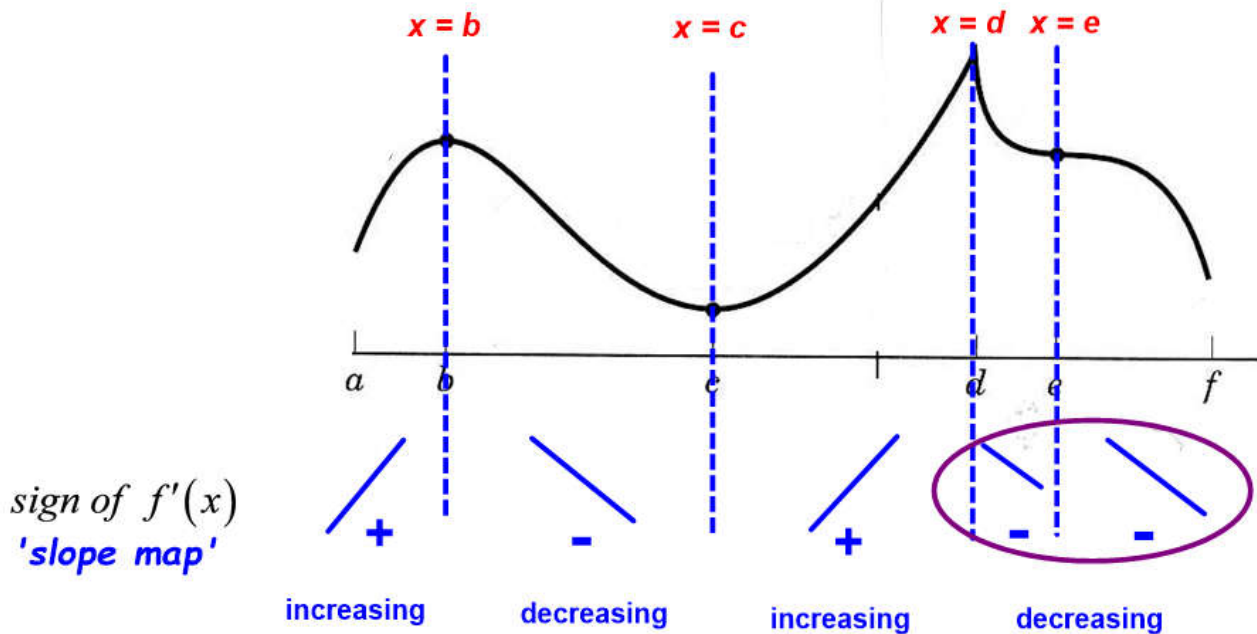
The sign of $f'(x)$ can only change at certain x -values, which are called **critical numbers**



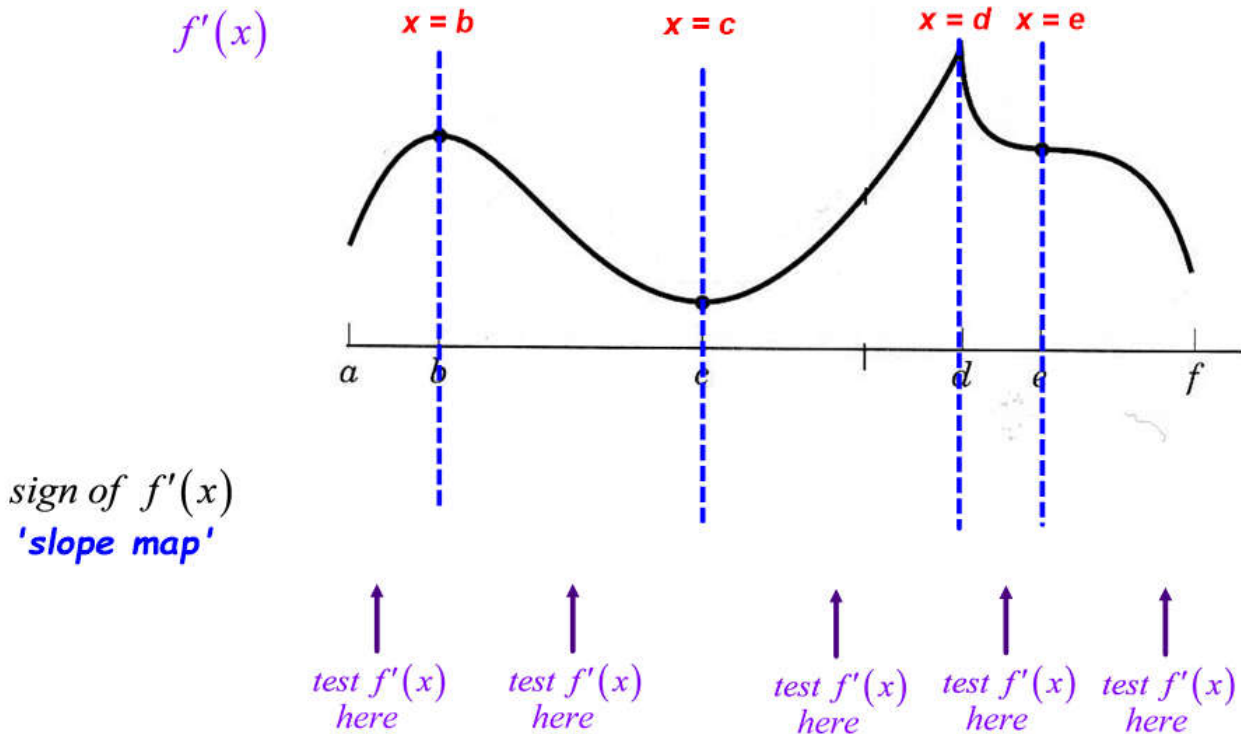
A **critical number** occurs at any x -value where $f'(x)=0$ or $f'(x)$ DNE

The First Derivative Test

Note that the sign of $f'(x)$ usually, but not always, changes as x crosses a critical number...



To conduct a first derivative test on a function, we first identify all the critical numbers, then we select a single test x -value in each region between critical numbers, and find the sign of the first derivative to determine if the function is increasing or decreasing in this region.



Where we use the First Derivative Test

There are two primary reasons we use the First Derivative Test:

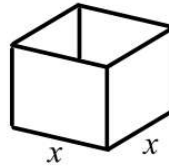
- 1) To quickly determine where the relative maximum or minimum values of a function occur.
- 2) To sketch a function curve without using a calculator (we also use some precalculus procedures, and sometimes also the Second Derivative test).

Using the First Derivative Test to determine a function's relative extrema

Ex) A manufacturer wants to design an open top, square base, box using 108 sq. in. of material.

If x is the side length of the base, the volume of the box is given by: $V(x) = 27x - \frac{1}{4}x^3$

Determine the side length which will give the largest volume.



Using the First Derivative Test to sketch a function curve

Ex) Find the critical numbers of the function: $f(x) = 6x^4 + 12x^3 + 20$

Find the domain and intercepts of the function.

Find the open interval(s) on which the function is increasing or decreasing.

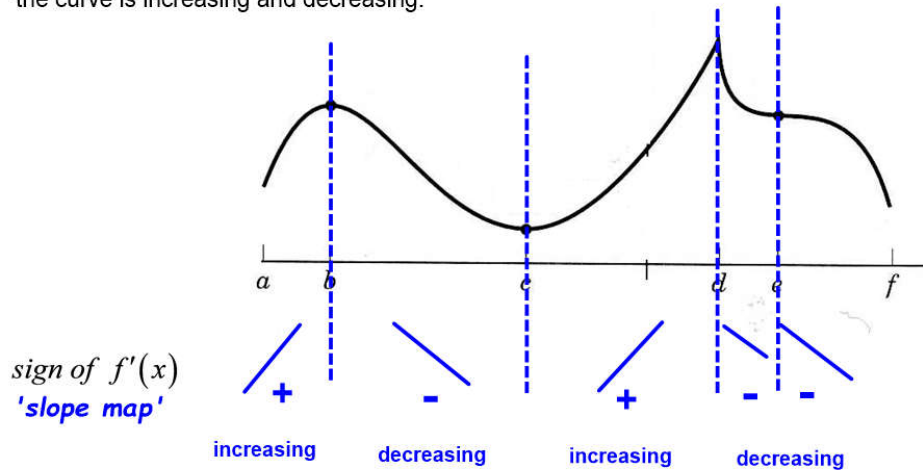
Find any relative extrema, and sketch the function *without using a calculator*.

Unit 4-4: Second Derivative Test

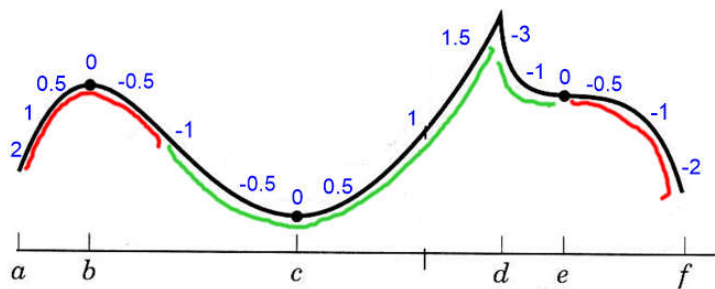
Larsen: 3.4 (Stewart: 4.3)

The Second Derivative Test

The **first derivative** of a function gives us information on the 'slope' of a curve and where the curve is increasing and decreasing.



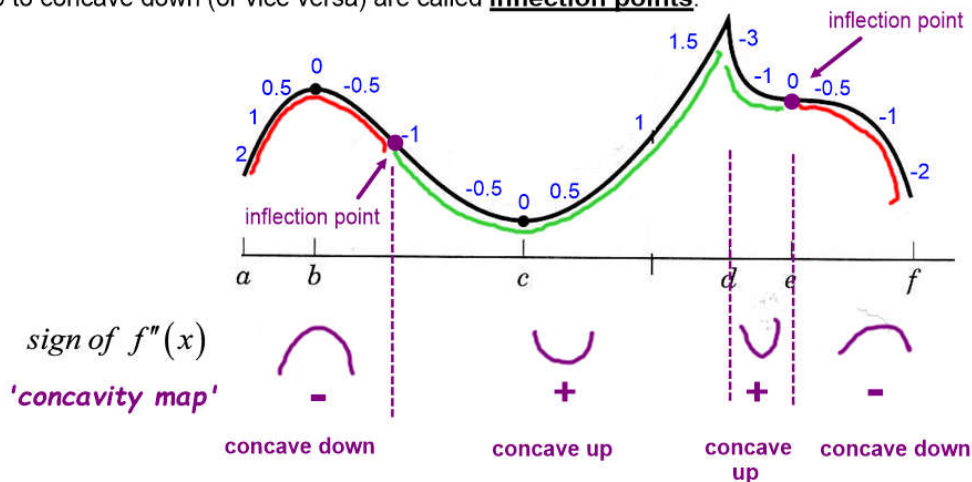
The **second derivative** of a function gives us information on how fast the first derivative of a function is changing. If we estimate the slope of the tangent line at each x ...



...in some regions, the first derivative is increasing numerically (which means the second derivative would be positive). This produces a curve that curves upward and is called concave up.

...in other regions, the first derivative is decreasing numerically (which means the second derivative would be negative). This produces a curve that curves downward and is called concave down.

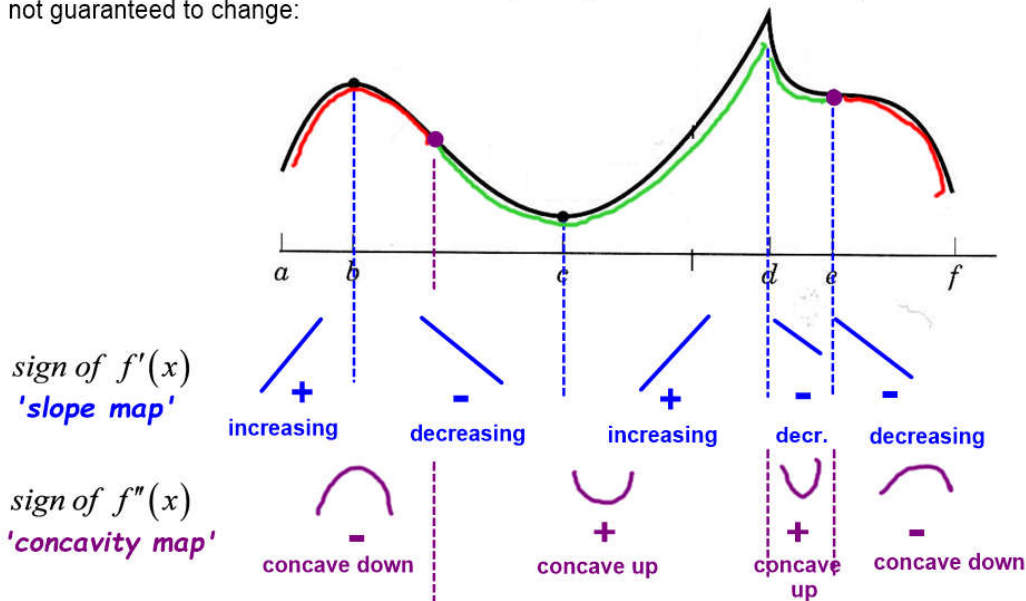
For the first derivative, the points where the function changed from increasing to decreasing were called critical points. Similarly, the points where the concavity changes from concave up to concave down (or vice versa) are called inflection points.



...and these are found by finding the x -values where the second derivative is either zero or undefined.

The Second Derivative Test

Note that the x-values of the critical points and the inflection points are typically not the same (although they can be), and also that you need to check each region because the sign of either the first or second derivative can only change at critical or inflection points, but is not guaranteed to change:



Where we use the Second Derivative Test

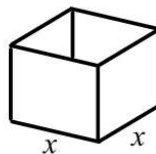
There are two primary reasons we use the Second Derivative Test:

1) When we have used the First Derivative Test to determine where a relative maximum or minimum is located, we can plug that same x-value into the second derivative. If the second derivative is positive, the function is concave up here so this extrema is a relative minimum. If the second derivative is negative, the function is concave down here so this extrema is a relative maximum.

2) To sketch a function curve without using a calculator, once we have made a 'slope' map, we can also find the second derivative, find where it is zero or undefined to locate the inflection points, then test values in each region to find the sign of the second derivative to determine where the function's concavity in each region. This gives us more detailed information to sketch the curve shape properly in each region.

Using the Second Derivative Test to determine a function's relative extrema

Ex) A manufacturer wants to design an open top, square base, box using 108 sq. in. of material. If x is the side length of the base, the volume of the box is given by: $V(x) = 27x - \frac{1}{4}x^3$
Determine the side length which will give the largest volume.



Using the Second Derivative Test to sketch a function curve

Ex) Find the critical numbers of the function: $f(x) = 6x^4 + 12x^3 + 20$

Find the domain and intercepts of the function.

Find the open interval(s) on which the function is increasing or decreasing.

Find any relative extrema, and sketch the function *without using a calculator*

Unit 4-5: Indeterminant Forms, L'Hopital's Rule, Asymptotes
 Larsen: 7.7 (Stewart: 4.4)

Curve sketching is assisted by finding any asymptotes

Horizontal asymptotes (possibly) occur if the limit as x becomes large approaches a constant as x approaches infinity.

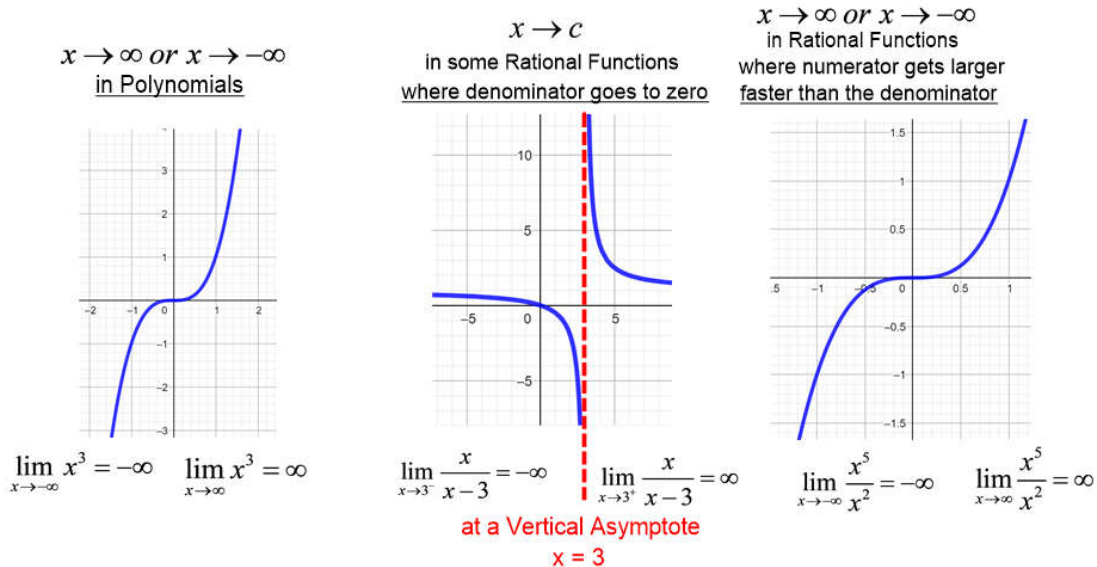
Vertical asymptotes (possibly) occur if the limit becomes large as x approaches the vertical asymptote (an infinite limit).

A reminder from earlier in this course...

Infinite Limits

An **infinite limit** is a limit that, when evaluated, is increasing without bound to $+\infty$ or decreasing without bound to $-\infty$.

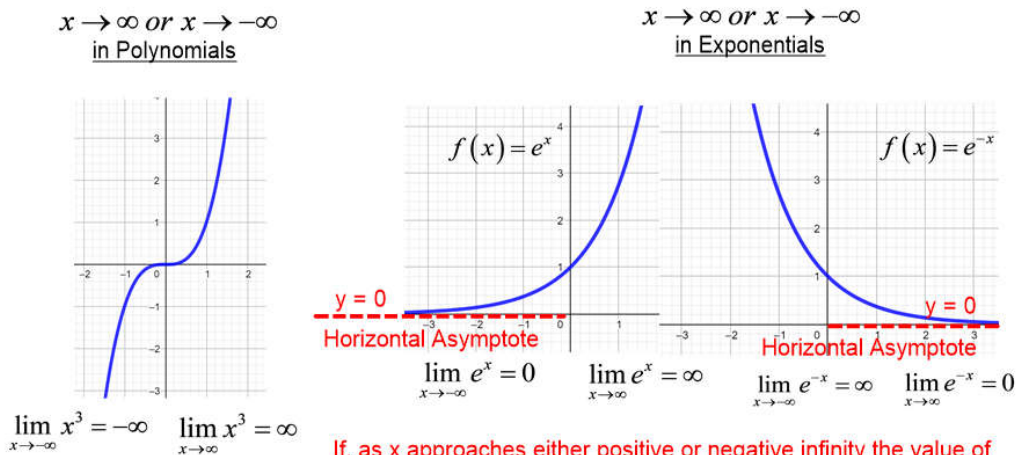
There are three general situations where we see infinite limits:



Limits at Infinity

A **limit at infinity** is a limit where x is approaching either $+\infty$ or $-\infty$.

When evaluate limits at infinity by imagining what will happen as x gets very large (either in the positive or negative direction). The resulting limit value may be zero, a constant, or an infinite limit, depending upon the function:



you get an infinite limit

If, as x approaches either positive or negative infinity the value of the limit approaches zero or any other constant, then there is a **horizontal asymptote** at this y value.

Evaluating such limits analytically

To evaluate, first consider what happens as x approaches the target value.

$$\lim_{x \rightarrow -\infty} x^2(x-1)$$

Here, two number which are multiplied are both getting very large, but because x is negative and one is squared but the other is not, we have...

$$\infty(-\infty)$$

...so:

$$\lim_{x \rightarrow -\infty} x^2(x-1) = -\infty$$

$$\lim_{x \rightarrow \infty} \frac{x^5}{e^x}$$

The numerator and denominator are both getting very large:

$\frac{\infty}{\infty}$ this is an **indeterminant form** (the numerator and denominator are 'fighting for control')

...in this case, the exponential increases faster than the power, so the denominator will eventually be much larger than the numerator:

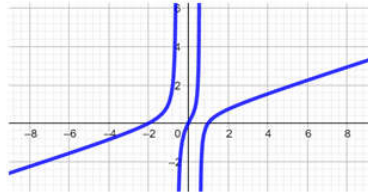
$$\lim_{x \rightarrow \infty} \frac{x^5}{e^x} = 0$$

Evaluating Rational Function limits

If you have a limit of a rational function with polynomials for numerator and denominator, three things can happen:

numerator degree is larger:

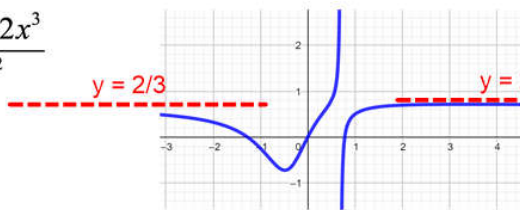
$$\lim_{x \rightarrow -\infty} \frac{x^5 + x^4 - 2x^3}{3x^4 - x^2}$$



$$\lim_{x \rightarrow \infty} \frac{x^5 + x^4 - 2x^3}{3x^4 - x^2} = \infty$$

numerator and denominator degrees are the same:

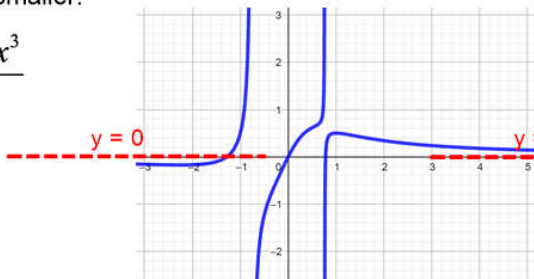
$$\lim_{x \rightarrow \infty} \frac{2x^5 + x^4 - 2x^3}{3x^5 - x^2}$$



$$\lim_{x \rightarrow \infty} \frac{2x^5 + x^4 - 2x^3}{3x^5 - x^2} = \frac{2}{3}$$

numerator degree is smaller:

$$\lim_{x \rightarrow \infty} \frac{2x^5 + x^4 - 2x^3}{3x^6 - x^2}$$



$$\lim_{x \rightarrow \infty} \frac{2x^5 + x^4 - 2x^3}{3x^6 - x^2} = 0$$

$$\lim_{x \rightarrow \infty} \frac{2x^5 + x^4 - 2x^3}{3x^5 - x^2}$$

divide everything by x^5

$$\lim_{x \rightarrow \infty} \frac{\frac{2x^5}{x^5} + \frac{x^4}{x^5} - \frac{2x^3}{x^5}}{\frac{3x^5}{x^5} - \frac{x^2}{x^5}}$$

$$\lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x} - \frac{2}{x^2}}{3 - \frac{1}{x^3}}$$

$$\lim_{x \rightarrow \infty} \frac{2+0-0}{3-0} = \frac{2}{3}$$

Evaluating Rational Function limits without graphing

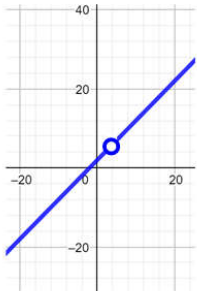
There is a procedure you can use to evaluate rational function limits without graphing.

- If numerator degree is higher than denominator, then the limit is an infinite limit (be careful of the sign).
- If denominator degree is higher than numerator, then the limit is 0.
- If the degrees of the numerator and denominator are the same, divide every term in both the numerator and denominator by $x^{\text{degree of denominator}}$. Then cancel within each term, and all terms with a constant over a power of x go to zero.

Vertical Asymptotes occur at uncanceled zeros in the denominator

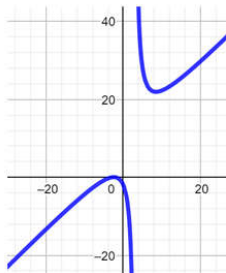
Consider these two limits:

$$\begin{aligned} \lim_{x \rightarrow 3^-} \frac{x^2 - x - 6}{x - 3} \\ &= \lim_{x \rightarrow 3^-} \frac{(x-3)(x+2)}{x-3} \\ &= \lim_{x \rightarrow 3^-} (x+2) = 5 \end{aligned}$$



No Vertical Asymptote because the x-3 could be factored and cancelled

$$\begin{aligned} \lim_{x \rightarrow 3^-} \frac{x^2 + 5x + 6}{x - 3} \\ &= \lim_{x \rightarrow 3^-} \frac{(x+3)(x+2)}{x-3} \\ &= \frac{45}{0} \end{aligned}$$



Vertical Asymptote because the x-3 could not be factored and cancelled

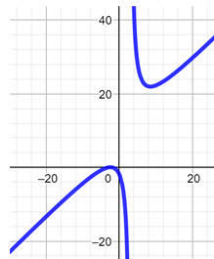
Evaluating Rational Function limits near a vertical asymptote

When direct substitution results in divide by zero and no way factoring removes this, you know the limit evaluates to either positive or negative infinity. To determine which, you consider the **signs** of the numerator and denominator:

$$\begin{aligned} \lim_{x \rightarrow 3^-} \frac{x^2 + 5x + 6}{x - 3} \\ &= \lim_{x \rightarrow 3^-} \frac{(x+3)(x+2)}{x-3} \\ &= \frac{45}{0} = +\infty \text{ or } -\infty \end{aligned}$$

$$\begin{array}{l} (+) \\ (-) \\ \boxed{= -\infty} \end{array}$$

negative because as x approach 3 from the left x is less than 3, so x-3 is negative



Horizontal Asymptotes require evaluating limits as x approaches +/- infinity...

$$\lim_{x \rightarrow \infty} \frac{2x^5 + x^4 - 2x^3}{3x^5 - x^2} = \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2x^5}{x^5} + \frac{x^4}{x^5} - \frac{2}{x^5}}{\frac{3x^5}{x^5} - \frac{x^2}{x^5}}$$

$$= \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x^4} - \frac{2}{x^5}}{3 - \frac{1}{x^3}}$$

$$= \lim_{x \rightarrow \infty} \frac{2+0-0}{3-0} = \boxed{\frac{2}{3}}$$

L'Hôpital's Rule gives us another alternative evaluation technique

When you directly evaluate, if you get one of the following indeterminate forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}, \frac{\infty}{-\infty}, \frac{-\infty}{\infty}$$

L'Hôpital's Rule: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$

$$\lim_{x \rightarrow \infty} \frac{2x^5 + x^4 - 2x^3}{3x^5 - x^2}$$

Note: separate derivatives, not quotient rule.

by L'Hôpital's Rule:

$$= \lim_{x \rightarrow \infty} \frac{10x^4 + 4x^3 - 6x^2}{15x^4 - 2x} = \frac{\infty}{\infty}$$

by L'Hôpital's Rule:

$$= \lim_{x \rightarrow \infty} \frac{40x^3 + 12x^2 - 12x}{60x^3 - 2} = \frac{\infty}{\infty}$$

by L'Hôpital's Rule:

$$= \lim_{x \rightarrow \infty} \frac{120x^2 + 24x - 12}{180x^2} = \frac{\infty}{\infty}$$

by L'Hôpital's Rule:

$$= \lim_{x \rightarrow \infty} \frac{240x + 24}{360x} = \frac{\infty}{\infty}$$

by L'Hôpital's Rule:

$$= \lim_{x \rightarrow \infty} \frac{240}{360} = \frac{2}{3}$$

Conditions to apply L'Hopital's Rule

L'Hopital's rule can be applied repeatedly until the limit either evaluates to a constant or an infinity, but it is important that you check that all conditions for applying L'Hopital's Rule are met first and at each stage:

1) The numerator and denominator functions must both be differentiable on the open interval of interest, and the denominator must not be zero (except possibly at the target x-value).

2) Must be a valid indeterminate form: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{-\infty}{-\infty}$, $\frac{\infty}{-\infty}$, $\frac{-\infty}{\infty}$

$$\lim_{x \rightarrow 0} \frac{x^2 - 4x}{2x - 1}$$

$$\lim_{x \rightarrow 1} \frac{\cos(\pi x)}{\ln(x)}$$

$$\lim_{\theta \rightarrow 0} \frac{\sin(2\theta)}{\theta}$$

Some other indeterminate forms can be converted to a form where L'Hopital applies

Some limits evaluate to other indeterminate forms where it isn't clear if the value is a constant or an infinity, and three specific other indeterminate forms can be converted into a form where L'Hopital's Rule can be applied. These are:

Product indeterminate forms: $0 \cdot \infty, 0 \cdot -\infty$ algebraically manipulate to force into a rational function.

Difference indeterminate forms: $\infty - \infty$ algebraically manipulate to force into a rational function.

Power indeterminate forms: $1^\infty, 0^\infty, \infty^0$ set expression equal to y, take ln of both sides, then convert right side into a L'Hopital indeterminate form.

Product indeterminate forms:

$$0 \cdot \infty, 0 \cdot -\infty$$

algebraically manipulate to force into a rational function.

$$\lim_{x \rightarrow \infty} e^{-x} \sqrt{x}$$

Difference indeterminate forms:

$$\infty - \infty$$

algebraically manipulate to force into a rational function.

$$\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$$

Power indeterminate forms:

$$1^\infty, 0^\infty, \infty^0$$

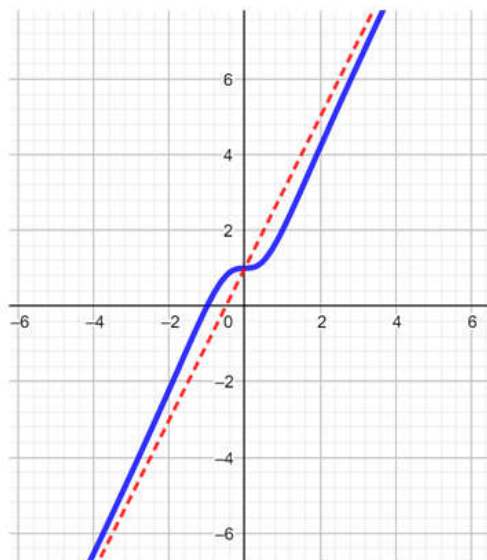
set expression equal to y, take ln of both sides, then convert right side into a L'Hopital indeterminate form.

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x$$

Slant Asymptotes

Sometimes, if you evaluate a rational function limit at infinity you get an infinite limit, but even though the limit is approaching an infinity, it is also aligning with a straight line (but not a vertical or horizontal asymptote). This occurs with rational functions where numerator and denominator are polynomials and the degree of the numerator is exactly one higher than the degree of the denominator:

$$f(x) = \frac{2x^3 + x^2 + 1}{x^2 + 1}$$



Summary Curve Sketching Procedure

With this, admittedly long, procedure, you can find everything you need to answer any question about a given function and sketch the function without a calculator.

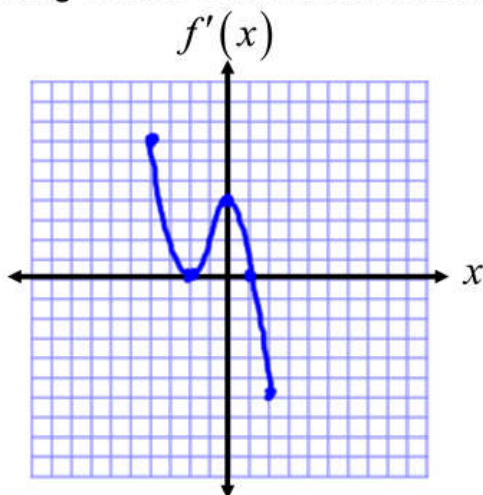
<u>Precalc</u>	<u>1st derivative</u>	<u>2nd derivative</u>
1) Find the domain.	6) Compute $f'(x)$ and find critical values where $f'(x) = 0$ or $f'(x)$ is undefined	9) Compute $f''(x)$ and find inflection point x-values where $f''(x) = 0$ or $f''(x)$ is undefined
2) Find the x- and y-intercepts (skip x-intercepts if too hard to factor)	7) Divide the domain into intervals split at the critical values and test the sign of $f'(x)$ in each interval to make a 'slope' map.	10) Divide the domain into intervals split at the inflection points and test the sign of $f''(x)$ in each interval to make a concavity map.
3) Check for vertical asymptotes (any uncancelled zeros in denominator)?	8) Plug critical value x's into orig. function and list local max and min points.	11) Plug inflection x's into original function to get inflection points, then sketch.
4) Check for horizontal asymptotes by $\lim_{x \rightarrow \infty} f(x), \lim_{x \rightarrow -\infty} f(x)$		
5) Check for slant asymptote if rational function w/degree different by one.		

Fully analyze and sketch without a calculator:

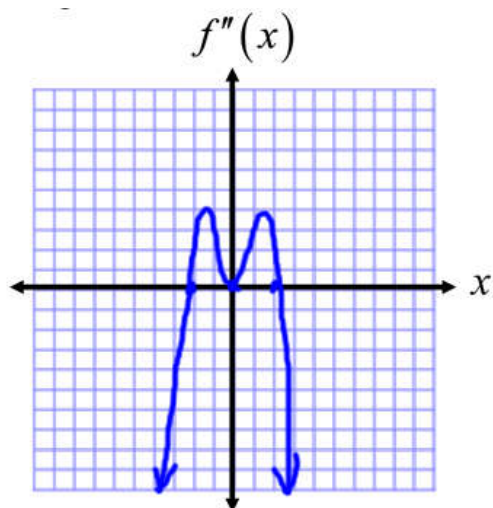
$$f(x) = \frac{-x^2 - 4x - 7}{x + 3}$$

Deducing things about $f(x)$ from a graph of $f'(x)$ or $f''(x)$

Making a table of information in all intervals of the given graph can provide clues...



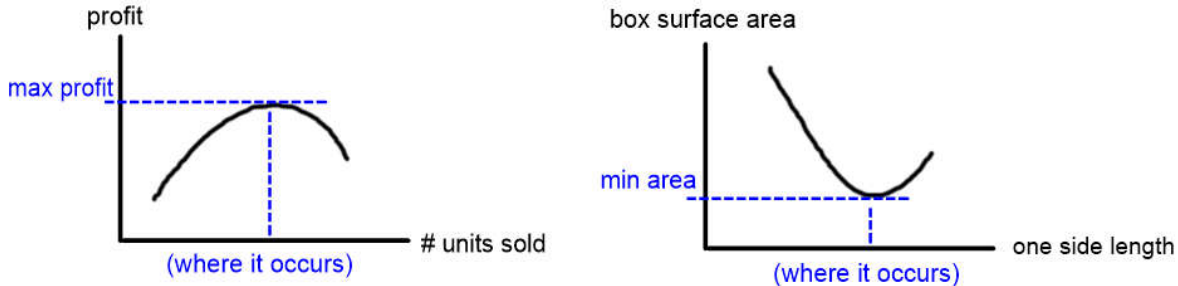
- On what intervals is f increasing?
- On what intervals is the graph of f concave up?
- At what x -values does f have relative extrema?
- At what x -values does the graph of f have a point of inflection?



- On what intervals is f concave up and concave down?
- On what intervals is the graph of f' increasing or decreasing?
- At what x -values does f have relative extrema?

Optimization

Optimization refers to finding the optimum value for a real-world problem - finding the values which make a value either a maximum or a minimum.



- Step 1:** Identify the quantity to be maximized or minimized.
- Step 2:** Assign symbols to represent other variables in the problem.
Use an illustration to assist you.
- Step 3:** Determine the relationships among these variables.
- Step 4:** Express the quantity to be optimized as a function of **one** of these variables.
(Be sure to state the domain).
- Step 5:** Use first-derivative test to locate relative extrema (where optimum value occurs).
- Step 6:** Use second-derivative test to verify you have min or max as appropriate.
- Step 7:** Re-read the problem to make sure you are answering the question
(may want the optimum value, where it occurs, or something else).

Terminology to know regarding business problems

Cost: $C(x)$ represents the cost to produce x units.

Demand: $p(x)$ represents the price that is associated with a specific number of units sold.

Revenue: $R(x)$ represents the sales (money brought in) from selling x units. If you have a demand (price) function, $R(x) = (\text{\#units sold})(\text{price}) = x * p(x)$.

Profit: Is the net money made selling x units, revenue - cost: $P(x) = R(x) - C(x)$.

'Marginal': Is the amount to add one additional unit, so it represents the 'derivative of'...marginal cost = $C'(x)$, marginal revenue = $R'(x)$.

Maximum Profit: Since $P(x) = R(x) - C(x)$, $P'(x) = R'(x) - C'(x)$ and max profit occurs when $P'(x) = 0$, so max profit occurs when $R'(x) = C'(x)$.

Ex) Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

Ex) A rectangular storage container with an open top is to have a volume of 10 m^3 . The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest container.

Ex) The cost function is $C(x) = 680 + 4x + 0.01x^2$.

If a company charges a price p for each unit x , it will be able to sell $x(p) = 500(12 - p)$ units.

Find the production level that will maximize profit.

A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

Ex) Find the point on the parabola $x + y^2 = 0$ that is closest to the point $(0, -3)$.

Ex) During the summer months Terry makes and sells necklaces on the beach. Last summer he sold the necklaces for \$10 each and his sales averaged 20 per day. When he increased the price by \$1, he found that he lost two sales per day.

- Find the demand function, assuming that it is linear.
- If the material for each necklace costs Terry \$6, what should the selling price be to maximize his profit?

Ex) Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r .

ex. Suppose you run a small independent furniture business. Your assistant signs a deal with a customer to deliver up to 400 chairs, the exact number to be determined by the customer later. The price will be \$90 per chair up to 300 chairs, and above 300, the price will be reduced by \$0.25 per chair (on the whole order) for every additional chair over 300 order. What are the largest and smallest revenues your company can make under this deal?

ex. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

46. The graph shows the fuel consumption c of a car (measured in gallons per hour) as a function of the speed v of the car. At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that $c(v)$ is minimized for this car when $v \approx 30$ mi/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in gallons *per mile*. Let's call this consumption G . Using the graph, estimate the speed at which G has its minimum value.

