

AP Calc BC – Lesson Notes – Unit 3: Derivative-Applications

Unit 3-1: Indeterminant Forms and L'Hopital's Rule

Our main method to evaluate limits is analytically (plug-in the x-value)

Depending on the form and the particular limit, the result can have a variety of forms which resolve to a single, non-infinite value...

$$(a \text{ number}) \quad \frac{(a \text{ number})}{(a \text{ number})} \quad \frac{0}{(a \text{ number})} = 0 \quad \frac{(a \text{ number})}{\pm\infty} = 0$$

But if any of the following forms occur, we need to investigate further...

$$\frac{(a \text{ number})}{0} \quad (0)(\pm\infty) \quad \frac{0}{0} \quad \frac{\pm\infty}{\pm\infty}$$

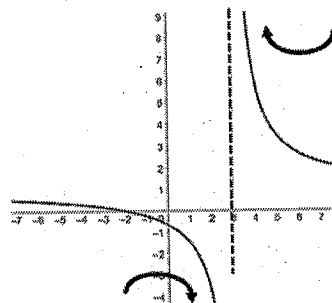
Limits which evaluate to $\frac{(a \text{ number})}{0}$

Dividing any constant by zero means the function we are taking a limit of has an uncanceled zero in the denominator at the target x-value, and therefore has a **vertical asymptote** here.

$$\lim_{x \rightarrow 3} \frac{x+2}{x-3}$$

If the limit is one-sided, the result is either ∞ or $-\infty$ (or can say DNE)

If the limit is two-sided, the result is always DNE.



#1. $\lim_{x \rightarrow 3^-} \frac{x^2+5x+6}{x-3}$

$$\Rightarrow \lim_{x \rightarrow 3^-} \frac{(x+2)(x+3)}{(x-3)}$$

(vertical asymptote)

$$x \rightarrow 3^- \quad \frac{(+)(+)}{(-)}$$

$$\therefore \lim_{x \rightarrow 3^-} \frac{x^2+5x+6}{x-3} \rightarrow -\infty$$

(or DNE)

#2. $\lim_{x \rightarrow 3^+} \frac{x^2+5x+6}{x-3}$

$$= \lim_{x \rightarrow 3^+} \frac{(x+2)(x+3)}{(x-3)}$$

(vertical asymptote)

$$x \rightarrow 3^+ \quad \frac{(+)(+)}{(+)}$$

$$\therefore \lim_{x \rightarrow 3^+} \frac{x^2+5x+6}{x-3} \rightarrow \infty$$

(or DNE)

#3. $\lim_{x \rightarrow 3} \frac{x^2+5x+6}{x-3}$

$$= \lim_{x \rightarrow 3} \frac{(x+2)(x+3)}{(x-3)}$$

(vertical asymptote)

$$\therefore \lim_{x \rightarrow 3} \frac{x^2+5x+6}{x-3} \text{ DNE}$$

Limits which evaluate to $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

These particular forms are called **indeterminant forms** and in these cases we can use **L'Hopital's Rule**:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

If the resulting limit is also an indeterminant form, L'Hopital's Rule can be applied multiple times until a non-indeterminant form is obtained.

On the AP Exam, College Board has a specific way they prefer to see work shown...

#4. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$

$$\begin{aligned} \lim_{x \rightarrow 0} (e^{2x} - 1) &= 0 \\ \lim_{x \rightarrow 0} x &= 0 \quad \left(\frac{0}{0}\right) \therefore \text{L'Hopital's Rule applies} \\ &= \lim_{x \rightarrow 0} \frac{e^{2x}(2)}{1} \\ &= \frac{e^0(2)}{1} = 2 \end{aligned}$$

#5. $\lim_{x \rightarrow -\infty} \frac{x^2}{e^{-x}}$

$$\begin{aligned} \lim_{x \rightarrow -\infty} x^2 &\rightarrow \infty \\ \lim_{x \rightarrow -\infty} e^{-x} &\rightarrow \infty \\ \left(\frac{\infty}{\infty}\right) \therefore \text{L'Hopital's Rule applies} \\ &= \lim_{x \rightarrow -\infty} \frac{2x}{e^{-x}(-1)} \quad \left(\frac{-\infty}{-\infty}\right) \\ &= \lim_{x \rightarrow -\infty} \frac{2}{e^{-x}} \rightarrow \frac{2}{\infty} \\ \therefore \lim_{x \rightarrow -\infty} \frac{x^2}{e^{-x}} &= 0 \end{aligned}$$

#6. $\lim_{x \rightarrow \infty} \frac{3x^2 + x}{4x - 7}$

$$\begin{aligned} \lim_{x \rightarrow \infty} (3x^2 + x) &\rightarrow \infty \\ \lim_{x \rightarrow \infty} (4x - 7) &\rightarrow \infty \\ \left(\frac{\infty}{\infty}\right) \therefore \text{L'Hopital's Rule applies} \\ &= \lim_{x \rightarrow \infty} \frac{6x + 1}{4} \rightarrow \infty \quad \left(\text{or PNE}\right) \end{aligned}$$

Limits which evaluate to $(0)(\pm\infty)$

Limits of the form $(0)(\infty)$ or $(0)(-\infty)$ can usually be rearranged using algebra into one of the indeterminant forms where L'Hopital's Rule applies:

#7. $\lim_{x \rightarrow \infty} e^{-x} \sqrt{x} \rightarrow (0)(\infty)$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{e^x} \quad \lim_{x \rightarrow \infty} \sqrt{x} \rightarrow \infty \quad \left(\frac{\infty}{\infty}\right) \therefore \text{L'Hopital's Rule applies} \\ &\quad \lim_{x \rightarrow \infty} e^x \rightarrow \infty \\ &= \lim_{x \rightarrow \infty} \frac{\frac{1}{2}x^{-1/2}}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}e^x} \rightarrow \frac{1}{\infty} \\ \therefore \lim_{x \rightarrow \infty} e^{-x} \sqrt{x} &= 0 \end{aligned}$$

Limits Involving a function raised to a function

There is one last special limit case: taking a limit of a function raised to a function. In these cases we must employ a logarithm to generate a form where L'Hopital's Rule applies:

#8. $\lim_{x \rightarrow 0^+} (\sin x)^x$

$$y = \lim_{x \rightarrow 0^+} (\sin x)^x$$

$$\ln y = \ln \left[\lim_{x \rightarrow 0^+} (\sin x)^x \right]$$

$$\ln y = \lim_{x \rightarrow 0^+} (x) \cdot \lim_{x \rightarrow 0^+} \ln(\sin x)$$

$$\ln y = 0 \cdot (-\infty)$$

$x \rightarrow 0^+ \quad y = \ln x$
 \nearrow
 $-\infty$

$$\ln y = \lim_{x \rightarrow 0^+} x \ln(\sin x)$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{(\frac{1}{x})} \quad \lim_{x \rightarrow 0^+} \ln(\sin x) \rightarrow -\infty$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} \rightarrow \infty$$

$(\frac{-\infty}{\infty}) \therefore$ L'Hopital's Rule applies

$$\ln y = \lim_{x \rightarrow 0^+} \frac{\frac{1}{\sin x} (\cos x)}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{\cot x}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{-x^2}{\tan x} \left(\frac{0}{0} \right)$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{-2x}{\sec^2 x} = \lim_{x \rightarrow 0^+} (-2x (\cos x)^2)$$

$$\ln y = (-2)(0)(1)^2 = 0$$

$$y = e^0 = 1 \therefore \lim_{x \rightarrow 0^+} (\sin x)^x = 1$$

More examples...

#9. $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = y$

$$\ln y = \ln \left(\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \right)$$

$$\ln y = \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right) \rightarrow (\infty)(0)$$

$$\ln y = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{(\frac{1}{x})} \quad \lim_{x \rightarrow \infty} \ln \left(1 + \frac{1}{x}\right) = 0$$

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\ln y = \lim_{x \rightarrow \infty} \frac{\frac{1}{1+\frac{1}{x}} (-\frac{1}{x^2})}{(-\frac{1}{x^2})} \quad \left(\frac{0}{0} \right) \therefore \text{L'Hopital's Rule applies}$$

$$\ln y = \lim_{x \rightarrow \infty} \frac{1}{1+\frac{1}{x}} = 1$$

$$y = e^1$$

$$\therefore \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

#10. $\lim_{x \rightarrow 0} \frac{x^2 - y}{2x - 1}$

$$= \frac{(0)^2 - y}{2(0) - 1}$$

$$= \underline{y}$$

#11. $\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$

$$\lim_{x \rightarrow 0} \sin(3x) = 0$$

$$\lim_{x \rightarrow 0} x = 0 \quad \left(\frac{0}{0} \right)$$

\therefore L'Hopital's Rule applies

$$= \lim_{x \rightarrow 0} \frac{3 \cos(3x)}{1}$$

$$= \frac{3(1)}{1}$$

$$= 3$$

More examples...

#12. $\lim_{x \rightarrow 0} \frac{\sin(3x)}{x^2}$

$\lim_{x \rightarrow 0} \sin(3x) = 0$ $(\frac{0}{0})$ \therefore L'Hopital's Rule applies
 $\lim_{x \rightarrow 0} x^2 = 0$

$= \lim_{x \rightarrow 0} \frac{3 \cos(3x)}{2x}$

$\lim_{x \rightarrow 0} 3 \cos(3x) = 3$ $(\frac{3}{0})$ vertical asymptote
 $\lim_{x \rightarrow 0} 2x = 0$

$\therefore \lim_{x \rightarrow 0} \frac{\sin(3x)}{x^2}$ DNE

#13. $\lim_{x \rightarrow 1} \frac{\cos(\pi x)}{\ln(x)}$

$\lim_{x \rightarrow 1} \cos(\pi x) = -1$ $(\frac{-1}{0})$
 $\lim_{x \rightarrow 1} \ln(x) = 0$ vertical asymptote

$\therefore \lim_{x \rightarrow 1} \frac{\cos(\pi x)}{\ln(x)}$ DNE

#14. $\lim_{x \rightarrow \infty} 3^{-x} x^2 \rightarrow (0)(\infty)$

$= \lim_{x \rightarrow \infty} \frac{x^2}{3^x}$

$\lim_{x \rightarrow \infty} x^2 \rightarrow \infty$ $(\frac{\infty}{\infty})$ L'Hopital's Rule applies
 $\lim_{x \rightarrow \infty} 3^x \rightarrow \infty$

$= \lim_{x \rightarrow \infty} \frac{2x}{3^x \ln(3)}$ $(\frac{\infty}{\infty})$

$= \lim_{x \rightarrow \infty} \frac{2}{3^x (\ln(3))^2} \rightarrow \frac{2}{\infty}$

$= 0$

#15. $\lim_{x \rightarrow \infty} \frac{2x^3 + x^2 + 1}{5x^2 - 3x}$

$\lim_{x \rightarrow \infty} (2x^3 + x^2 + 1) \rightarrow \infty$ $(\frac{\infty}{\infty})$
 $\lim_{x \rightarrow \infty} (5x^2 - 3x) \rightarrow \infty$
 \therefore L'Hopital's Rule applies

$= \lim_{x \rightarrow \infty} \frac{6x^2 + 2x}{10x - 3}$ $(\frac{\infty}{\infty})$

$= \lim_{x \rightarrow \infty} \frac{12x + 2}{10} \rightarrow \infty$
 or DNE

#16. $\lim_{x \rightarrow \infty} \frac{2x^3 + x^2 + 1}{5x^3 - 3x}$

$\lim_{x \rightarrow \infty} (2x^3 + x^2 + 1) \rightarrow \infty$

$\lim_{x \rightarrow \infty} (5x^3 - 3x) \rightarrow \infty$

$(\frac{\infty}{\infty})$ \therefore L'Hopital's Rule applies

$= \lim_{x \rightarrow \infty} \frac{6x^2 + 2x}{15x^2 - 3}$ $(\frac{\infty}{\infty})$

$= \lim_{x \rightarrow \infty} \frac{12x + 2}{30x}$ $(\frac{\infty}{\infty})$

$= \lim_{x \rightarrow \infty} \frac{12}{30} = \frac{12}{30}$
 $= \frac{2}{5}$

#17. $\lim_{x \rightarrow \infty} \frac{2x^3 + x^2 + 1}{5x^4 - 3x}$

$\lim_{x \rightarrow \infty} (2x^3 + x^2 + 1) \rightarrow \infty$

$\lim_{x \rightarrow \infty} (5x^4 - 3x) \rightarrow \infty$

$(\frac{\infty}{\infty})$ \therefore L'Hopital's Rule applies

$= \lim_{x \rightarrow \infty} \frac{6x^2 + 2x}{20x^3 - 3}$ $(\frac{\infty}{\infty})$

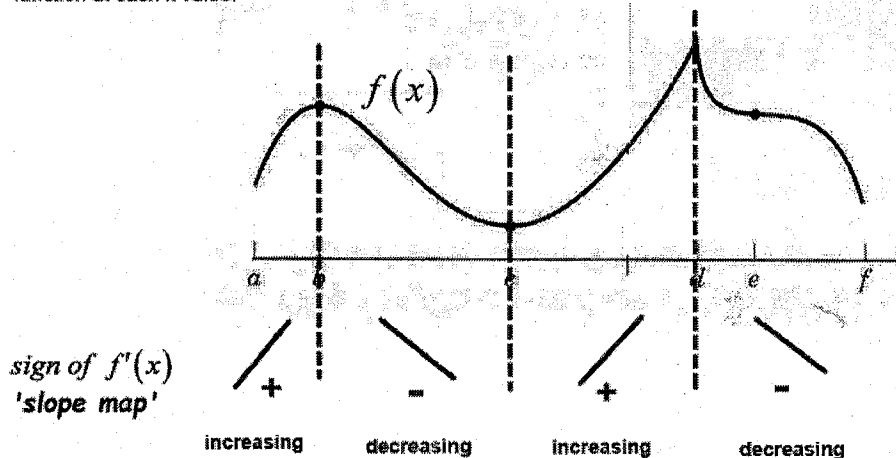
$= \lim_{x \rightarrow \infty} \frac{12x + 2}{60x^2}$ $(\frac{\infty}{\infty})$

$= \lim_{x \rightarrow \infty} \frac{12}{120x} = 0$

Unit 3-2: The First and Second Derivative Tests

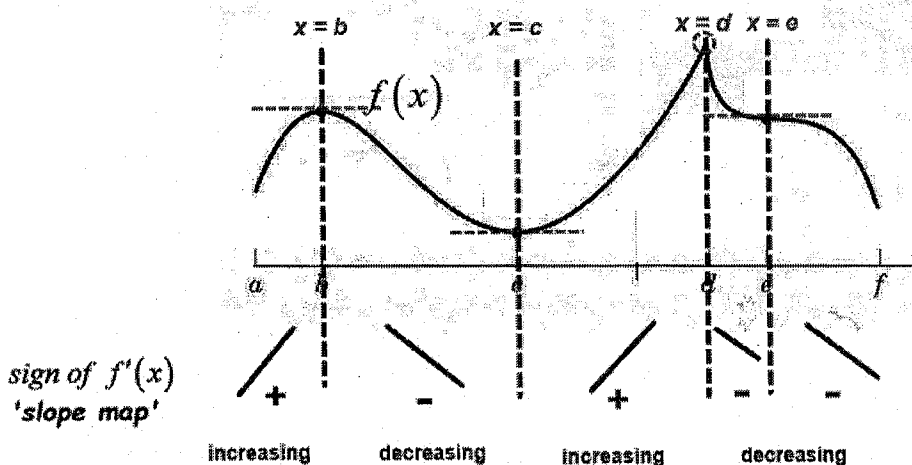
The First Derivative Test

The **first derivative** of a function gives the instantaneous rate of change, or '**slope**', of the function at each x value:



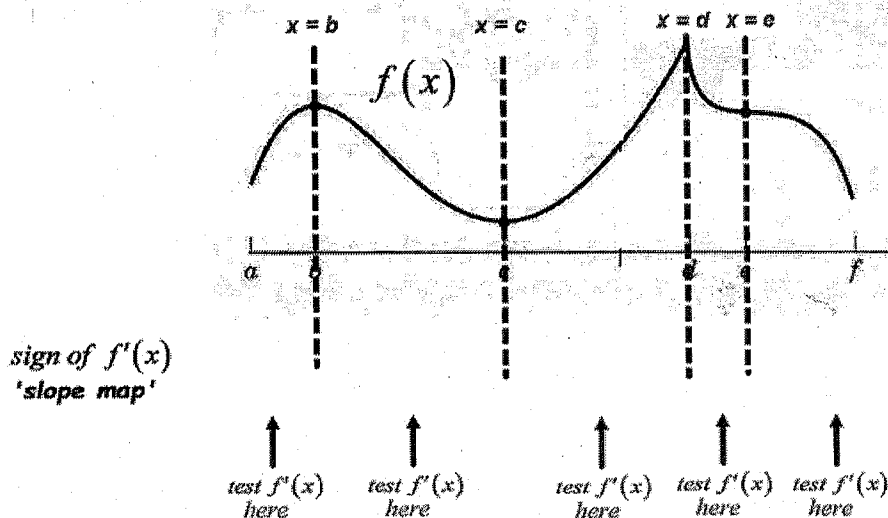
...by checking the **sign of $f'(x)$** we can determine if the function is increasing or decreasing.

The sign of $f'(x)$ can only change at certain x -values, which are called **critical numbers**



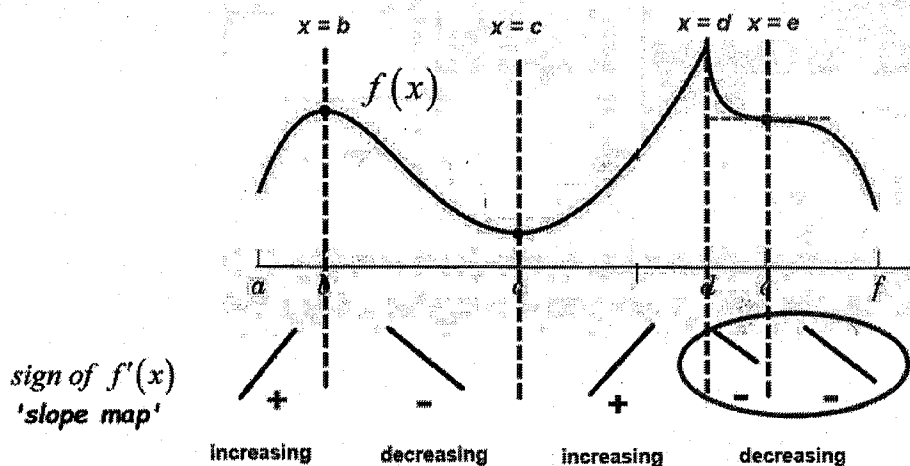
A critical number occurs at any x -value where $f'(x) = 0$ or $f'(x)$ DNE

To conduct a first derivative test on a function, we first identify all the critical numbers, then we select a single test x -value in each region between critical numbers, and find the sign of the first derivative to determine if the function is increasing or decreasing in this region.

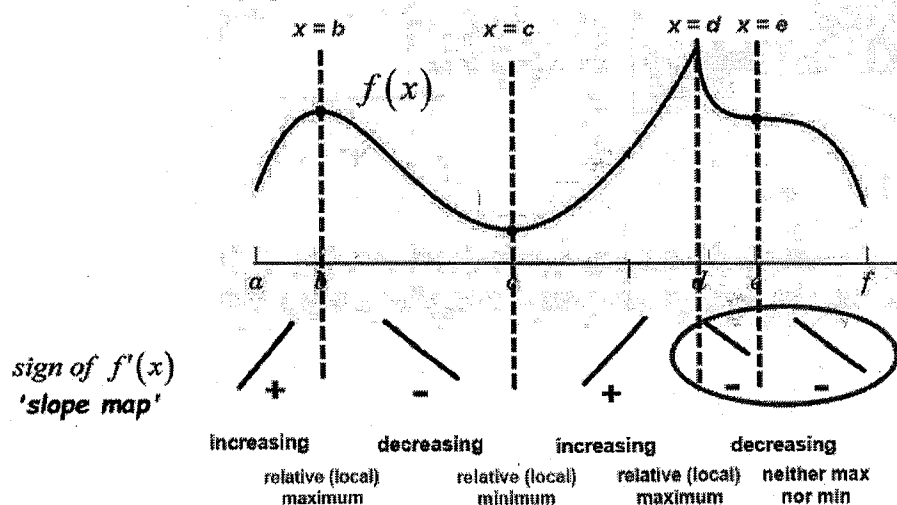


The First Derivative Test

Note that the sign of $f'(x)$ usually, but not always, changes as x crosses a critical number...

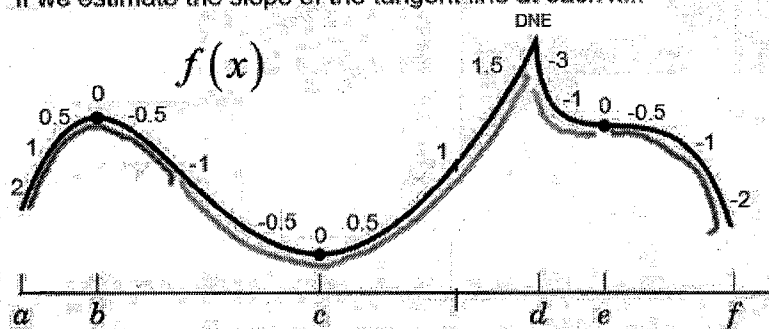


Relative (local) maxima and minima can only occur at critical points, but not all critical points are max/mins:



The Second Derivative Test

The **second derivative** of a function gives us information on how fast the first derivative of a function is changing. If we estimate the slope of the tangent line at each x ...

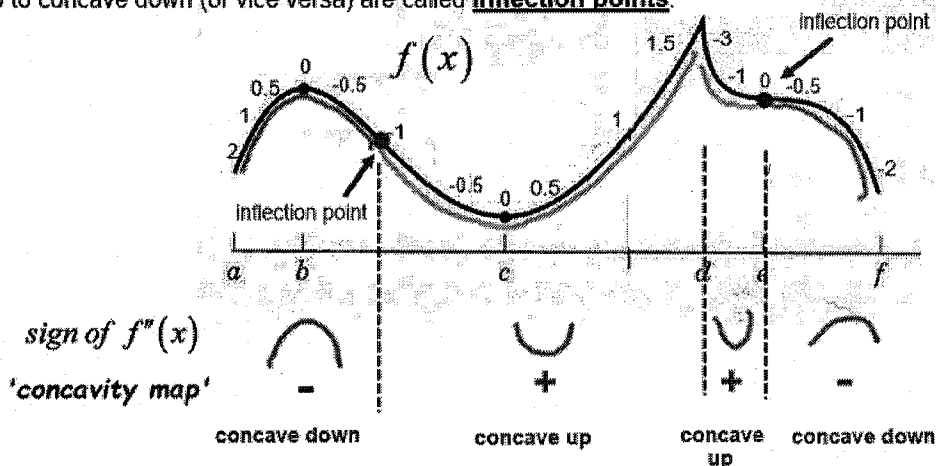


...in some regions, the first derivative is increasing numerically (which means the second derivative would be positive). This produces a curve that curves upward and is called **concave up**.

...in other regions, the first derivative is decreasing numerically (which means the second derivative would be negative). This produces a curve that curves downward and is called **concave down**.

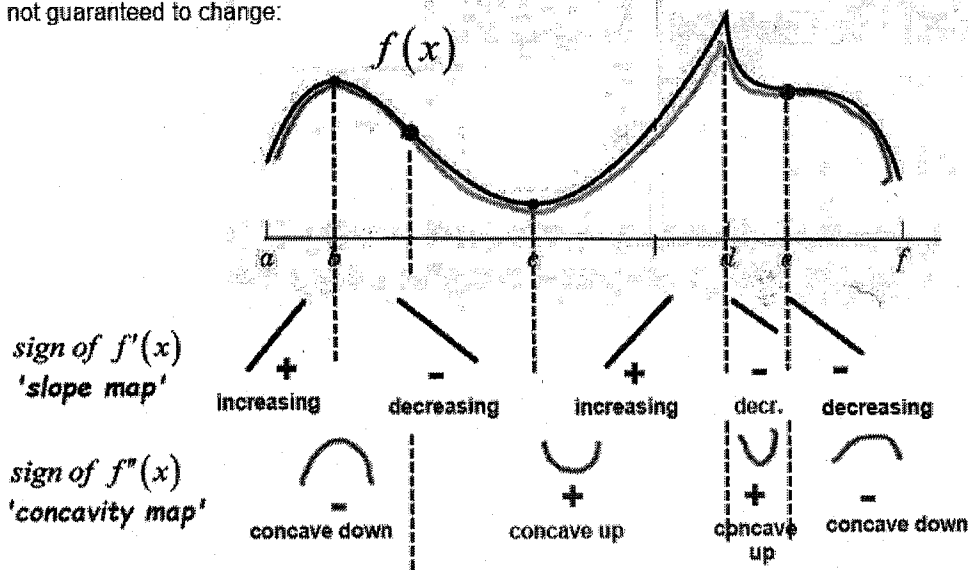
The Second Derivative Test

For the first derivative, the points where the function changed from increasing to decreasing were called critical points. Similarly, the points where the concavity changes from concave up to concave down (or vice versa) are called inflection points.



...and these are found by finding the x-values where the second derivative is either zero or undefined.

Note that the x-values of the critical points and the inflection points are typically not the same (although they can be), and also that you need to check each region because the sign of either the first or second derivative can only change at critical or inflection points, but is not guaranteed to change:



Ways we can use the First and Second Derivative Tests

We are given information about a function...

- We are given the function's expression, e.g: $f(x) = 2x^2 - 3x + 5$
- We are given the graph of the function, $f(x)$
- We are given the graph of a derivative of the function: $f'(x)$ (or, rarely, $f''(x)$)
- In a word problem, we are given information we can use to build a function.

We are then asked to find things like the following:

- Critical points and x-intervals where $f(x)$ is increasing/decreasing
- Inflection points and x-intervals where $f(x)$ is concave up/down
- The location of relative (local) extrema (maxima and minima)
- The absolute maximum/minimum of a function over a given interval.
- The 'optimum' value (maximum or minimum) under some constraint.

Working with a function expression (without a calculator)

Find the following for $f(x)$:

- Critical points and x-intervals where $f(x)$ is increasing/decreasing
- Inflection points and x-intervals where $f(x)$ is concave up/down
- The location of relative (local) extrema (maxima and minima)
- Horizontal and vertical asymptotes

...then sketch the function

#1. $f(x) = \frac{-x^2 - 4x - 7}{x+3}$

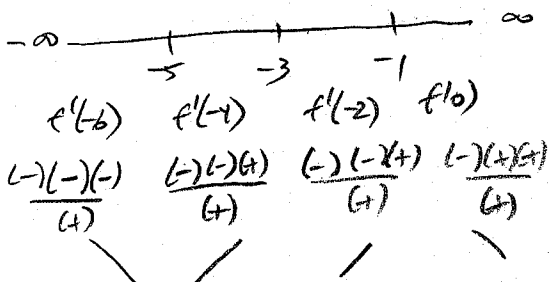
$$\begin{aligned} f'(x) &= \frac{(x+3)(-2x-4) - (-x^2-4x-7)(1)}{(x+3)^2} \\ &= \frac{-2x^2-10x-12+x^2+4x+7}{(x+3)^2} \\ &= \frac{-x^2-6x-5}{(x+3)^2} = \frac{-(x+6x+5)}{(x+3)^2} \\ &= \frac{-(x+1)(x+5)}{(x+3)^2} \end{aligned}$$

critical pts when...

$f'(x)=0$ or $f'(x)$ DNE

$-(x+1)(x+5)=0$ $f'(x)$ DNE

$x=-1, x=-5$ $x=-3$



$f(x)$ increasing: $(-5, -3) \cup (-1, \infty)$
 $f(x)$ decreasing: $(-\infty, -5) \cup (-3, -1)$
 $f(x)$ relative minimum at $(-5, 6)$
 $f(x)$ relative maximum at $(-1, -1)$

$f(-5) = \frac{-25 + 20 - 7}{-5+3} = \frac{-12}{-2} = 6$

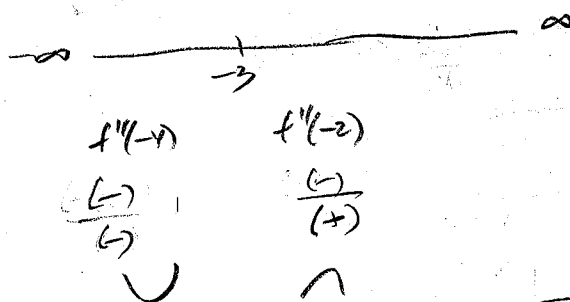
$f(-1) = \frac{-1 + 4 - 7}{-1+3} = \frac{-4}{2} = -2$

$$\begin{aligned} f''(x) &= \frac{(x+3)^2(-2x-6) - (-x^2-6x-5)(2(x+3)(1))}{(x+3)^4} \\ &= \frac{(x^2+6x+9)(-2x-6) + (x^2+6x+5)2(x+3)}{(x+3)^4} \\ &= \frac{(x^2+6x+9)(-2x-6) + (x^2+6x+5)2(x+3)}{(x+3)^4} \\ &= \frac{-2(x+3)(x^2+6x+9) + 2(x+3)(x^2+6x+5)}{(x+3)^4} \\ &= \frac{-2(x+3)(x^2+6x+9 - x^2-6x-5)}{(x+3)^4} = \frac{-8}{(x+3)^3} \end{aligned}$$

inflection when

$f''(x)=0$

$f''(x)$ DNE
 $x=-3$

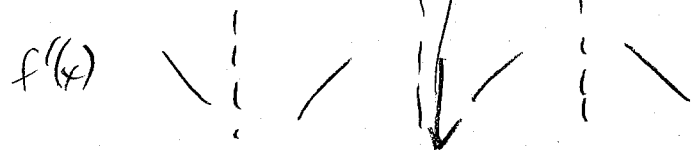
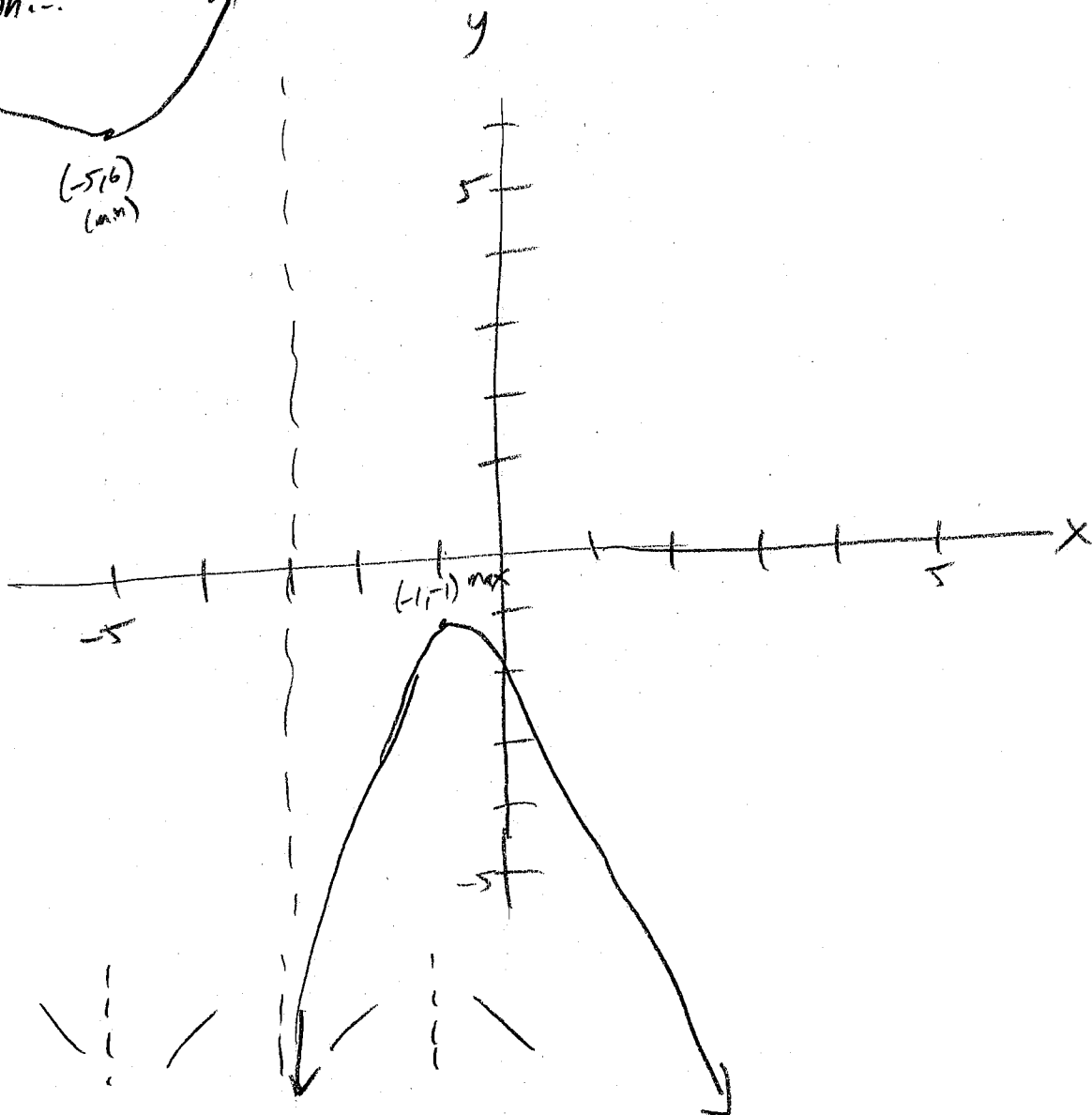
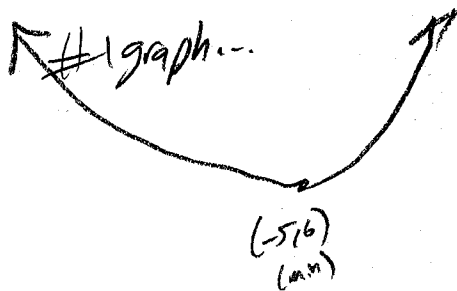


$f(x)$ concave up $(-\infty, -3)$
 $f(x)$ concave down $(-3, \infty)$
 inflection at $x=-3$ (no point)

vertical asymptote: $x=-3$

horiz. asymptote:

$\lim_{x \rightarrow \pm\infty} \frac{-x^2-4x-7}{x+3} \rightarrow \pm\infty$
 (no horizontal asymptote)



Working with a function expression (without a calculator)

Find the following for $f(x)$:

- Critical points and x-intervals where $f(x)$ is increasing/decreasing
- Inflection points and x-intervals where $f(x)$ is concave up/down
- The location of relative (local) extrema (maxima and minima)
- Horizontal and vertical asymptotes

...then sketch the function

#2. $f(x) = 6x^4 + 12x^3 + 20$

$$f'(x) = 24x^3 + 36x^2$$

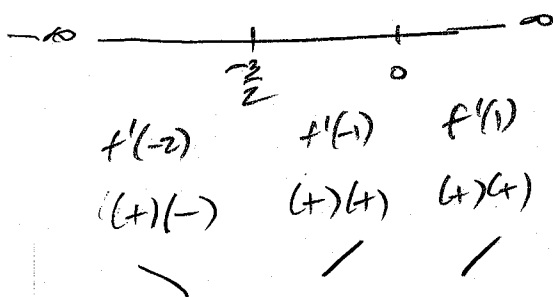
$$= 12x^2(2x + 3)$$

critical pts

$$f'(x) = 0$$

$f'(x) \text{ DNE}$

$$x = 0, x = -\frac{3}{2}$$



f increasing $(-\frac{3}{2}, 0) \cup (0, \infty)$
 f decreasing $(-\infty, -\frac{3}{2})$
 f relative min at $(-\frac{3}{2}, 9.875)$
 (no relative max)

no vertical asymptote

no horizontal asymptotes (polynomial)

$$f''(x) = 72x^2 + 72x$$

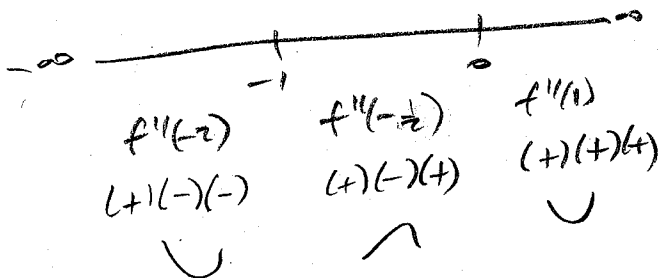
$$= 72x(x + 1)$$

inflection pts

$$f''(x) = 0$$

$$x = 0, x = -1$$

$f''(x) \text{ DNE}$



f concave up $(-\infty, -1) \cup (0, \infty)$
 f concave down $(-1, 0)$

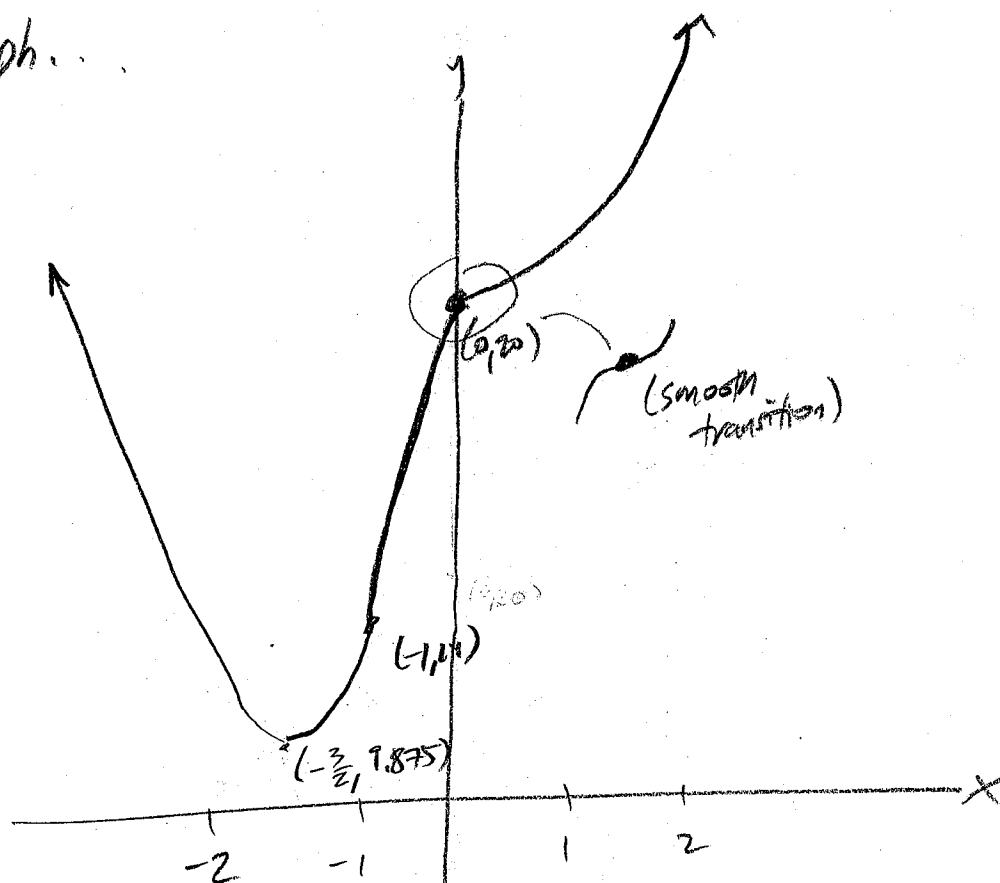
inflection pts at $x = -1, x = 0$

$$f(-1) = 6 - 12 + 20 = 14$$

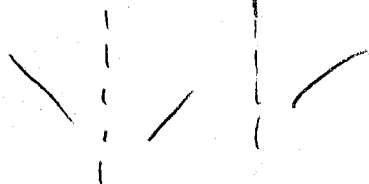
$$f(0) = 20$$

$$(-1, 14) (0, 20)$$

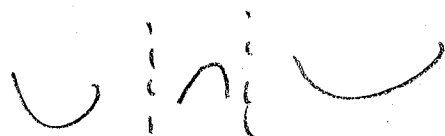
#2 graph...



$f'(x)$



$f''(x)$



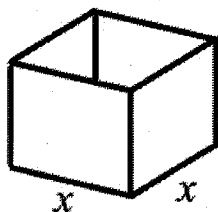
Using derivatives to solve an optimization problem

#3. A manufacturer wants to design an open top, square base, box using 108 sq. in. of material. If x is the side length of the base, the volume of the box is given by:

Determine the side length which will give the largest volume.

$$V(x) = 27x - \frac{1}{4}x^3$$

(In Unit 3 part 2 we will practice finding functions like this from word problems)



$$V(x) = 27x - \frac{1}{4}x^3$$

relative max when $V'(x) = 0$

$$V'(x) = 27 - \frac{3}{4}x^2 = 0$$

$$\frac{3}{4}x^2 = 27$$

$$x^2 = 27\left(\frac{4}{3}\right) = 36$$

$$x = 6$$

verify maximum w/ $V''(x)$:

$$V''(x) = -\frac{3}{2}x$$

$$V''(6) = -\frac{3}{2}(6) < 0$$

V is concave down at $x=6$

ii $x=6$ is where volume is a maximum

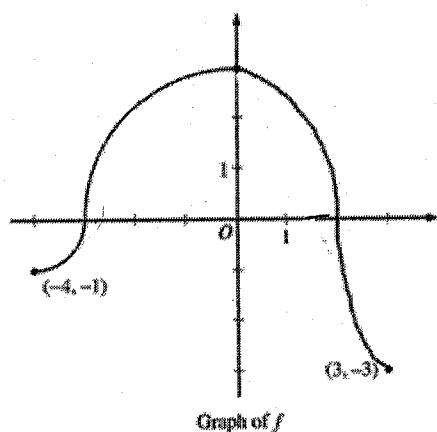


Working with a graph of a function, $f(x)$

Using the graph, find the following:

- Critical points and intervals where $f(x)$ is increasing/decreasing
- Inflection points and intervals where $f(x)$ is concave up/down
- Relative extrema

#4.



critical pts when $f'(x) = 0$ (or DNE)
which occurs at $(-4, -1)$, $(0, 3)$ and $(3, -3)$

f is increasing over $(-4, 0)$

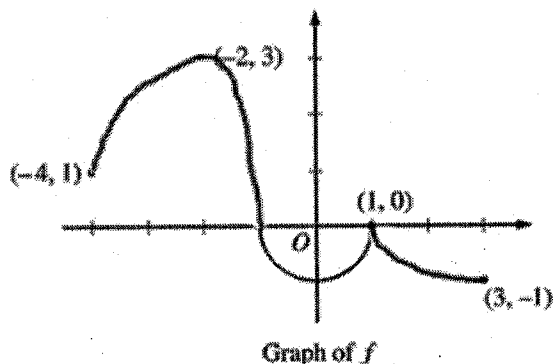
f is decreasing over $(0, 3)$

f is concave up over $(-4, -3) \cup (2, 3)$

f is concave down over $(-3, 2)$

inflection points at $(-3, 0)$ and $(2, 0)$

#5.



critical pts when $f'(x) = 0$ (or DNE)
which occurs at $(-2, 3)$, $(0, -1)$ and $(1, 0)$

f is increasing over $(-4, -2) \cup (0, 1)$

f is decreasing over $(-2, 0) \cup (1, 3)$

f is concave up over $(-1, 1) \cup (1, 3)$

f is concave down over $(-4, -1)$

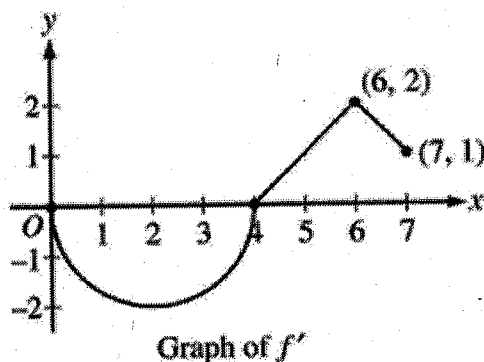
inflection point at $(-1, 0)$

Working with a graph of a derivative of a function, $f'(x)$

Using the graph, find the following:

- Critical points and intervals where $f(x)$ is increasing/decreasing
- Inflection points and intervals where $f(x)$ is concave up/down
- Relative extrema

#6.



Critical pts when $f'(x) = 0$ (or DNE)
which occurs at $x=0, x=4$

$f(x)$ is increasing when $f'(x) > 0$ which
occurs over $(4, 7)$

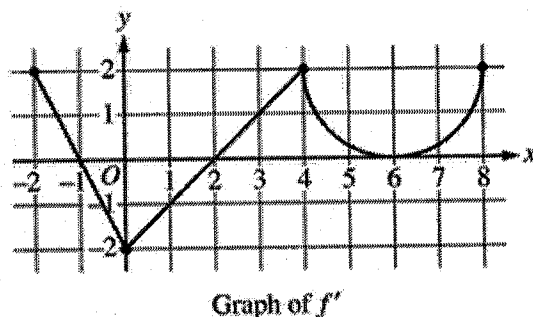
$f(x)$ is decreasing when $f'(x) < 0$ which
occurs over $(0, 4)$

$f(x)$ is concave up when $f''(x) > 0$
which means $f'(x)$ is increasing
which occurs over $(2, 6)$

$f(x)$ is concave down when $f''(x) < 0$
which means $f'(x)$ is decreasing
which occurs over $(0, 2) \cup (6, 7)$

inflection points at $x=2$ and $x=6$

#7.



critical pts when $f'(x) = 0$ (or DNE)
which occurs at $x=-1, x=2$, and $x=6$

$f(x)$ is increasing when $f'(x) > 0$ which
occurs over $(-2, -1) \cup (2, 6) \cup (6, 8)$

$f(x)$ is decreasing when $f'(x) < 0$ which
occurs over $(-1, 2)$

$f(x)$ is concave up when $f''(x) > 0$
which means $f'(x)$ is increasing
which occurs over $(0, 4) \cup (6, 8)$

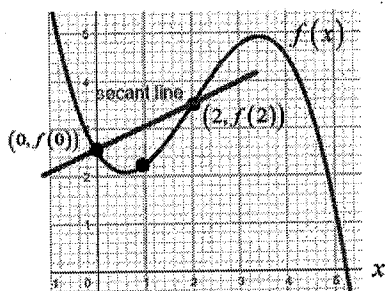
$f(x)$ is concave down when $f''(x) < 0$
which means $f'(x)$ is decreasing
which occurs over $(-2, 0) \cup (4, 6)$

inflection points at $x=0, x=4$ and $x=6$

Unit 3-3: Average vs. Instantaneous Rate of Change, and Mean Value/Rolle's Theorem

Average vs Instantaneous Rate of Change

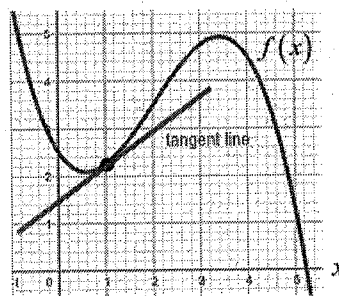
Average Rate of Change



Average Rate of Change is always evaluated over an interval.
Average Rate of Change uses the Algebra slope expression:

$$\text{Average Rate of Change of } f(x) \text{ near } x = 1 = \frac{\Delta y}{\Delta x} = \frac{f(2) - f(0)}{2 - 0}$$

Instantaneous Rate of Change



Instantaneous Rate of Change is always evaluated at a single x-value.
Instantaneous Rate of Change is the derivative:

$$\text{Instantaneous Rate of Change of } f(x) \text{ at } x = 1 = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- #1. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t (in minutes) is given by a differentiable function C (in ounces). Selected values of $C(t)$ are given in the table:

t (minutes)	0	1	4	9	16
$C(t)$ (ounces)	0	2	4	6	8

- a) What is the average rate of change of the amount of coffee over the interval $1 \leq t \leq 9$?

$$\begin{aligned} \text{ARD, } C &= \frac{C(9) - C(1)}{9 - 1} \\ &= \frac{6 - 2}{8} \\ &= \frac{4}{8} \\ &= \frac{1}{2} \frac{\text{ounces}}{\text{min}} \\ &= 0.5 \frac{\text{ounces}}{\text{min}} \end{aligned}$$

- b) Estimate the instantaneous rate of change in the amount of coffee at $t = 10$ minutes.

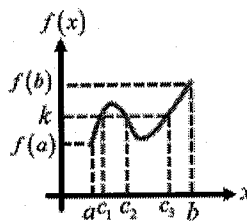
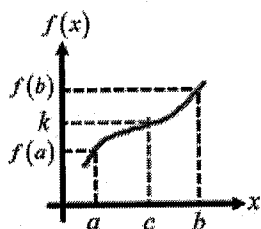
$$\begin{aligned} f'(10) &\approx \frac{C(16) - C(4)}{16 - 4} \\ &\approx \frac{8 - 4}{16 - 4} \\ &\approx \frac{4}{12} \\ &\approx \frac{1}{3} \\ &\approx 0.286 \frac{\text{ounces}}{\text{min}} \end{aligned}$$

- c) If the function is found to be $C(t) = 2\sqrt{t}$, find the instantaneous rate of change in the amount of coffee at $t = 10$ minutes.

$$\begin{aligned} C(t) &= 2t^{1/2} \\ C'(t) &= t^{-1/2} = \frac{1}{\sqrt{t}} \\ C'(10) &= \frac{1}{\sqrt{10}} \\ &= 0.316 \frac{\text{ounces}}{\text{min}} \end{aligned}$$

Reminder: Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.



Note: This theorem doesn't provide a method for finding the value(s) c , and doesn't indicate the number of c values which map to k , it only guarantees the existence of at least one number c such that $f(c) = k$.

- #2. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t (in minutes) is given by a differentiable function C (in ounces). Selected values of $C(t)$ are given in the table:

t (minutes)	0	1	4	9	16
$C(t)$ (ounces)	0	2	4	6	8

Is there a time, $t = k$, in the interval $0 \leq t \leq 16$ when the coffee cup definitely contains 5 ounces of coffee? Justify your answer.

$C(t)$ is differentiable and therefore continuous over $[0, 16]$

Since $0 \leq 5 \leq 8$, the Intermediate Value Theorem guarantees k , $0 \leq k \leq 16$,

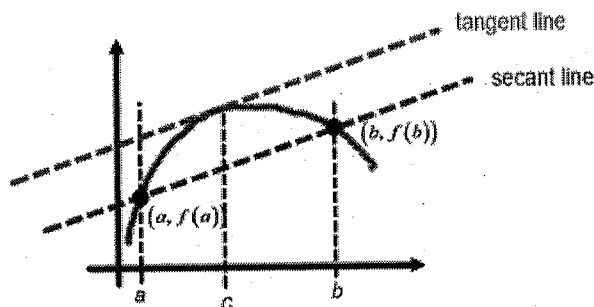
such that $C(k) = 5$ ounces.

Mean Value Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists a number c in (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The Mean Value Theorem says that if a curve is continuous and differentiable over an interval, then there must be at least one x -value in the interval at which the derivative is the same as the average rate of change over the interval. In other words, there must be an x -value where the slope of the tangent line is the same as the slope of the secant line through the endpoints.



- #3. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t (in minutes) is given by a differentiable function C (in ounces). Selected values of $C(t)$ are given in the table:

t (minutes)	0	1	4	9	16
$C(t)$ (ounces)	0	2	4	6	8

- a) Is there a time, $t = k$, in the interval $0 \leq t \leq 16$ when the rate of change of the amount of coffee equals the average rate of change of the amount of coffee? Justify your answer.

- b) If you are further told that $C(t) = 2\sqrt{t}$, find k .

a) $C(t)$ is differentiable, and therefore continuous over $[0, 16]$
 \therefore the Mean Value Theorem does guarantee k , $0 \leq k \leq 16$, such that $C'(k)$ equals the average rate of change of $C(t)$ over $[0, 16]$.

b) avg rate of change = $\frac{C(16) - C(0)}{16 - 0} = \frac{8 - 0}{16} = \frac{1}{2} \frac{\text{ounces}}{\text{min}}$

$$C(t) = 2t^{1/2}$$

$$C'(t) = t^{-1/2} = \frac{1}{\sqrt{t}}$$

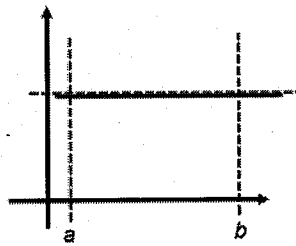
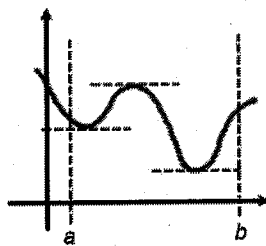
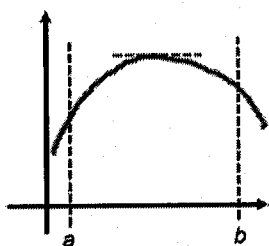
$$C'(k) = \frac{1}{\sqrt{k}} = \frac{1}{2} \text{ when } \sqrt{k} = 2$$

$k = 4 \text{ minutes}$

Rolle's Theorem

If the function has the same value at both ends of the interval (so that the average rate of change is zero), this special case of the Mean Value Theorem is called Rolle's Theorem:

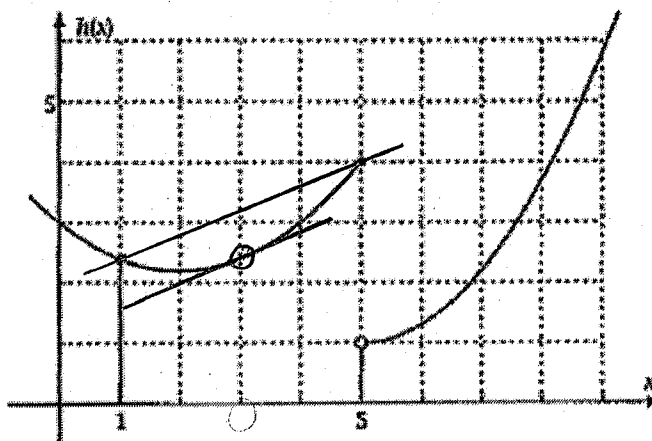
Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) . If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.



#4. The graph below is function h from Problem 5.

- Draw a secant line through $(1, h(1))$ and $(5, h(5))$.
- Show that there is a point $x = c$ in $(1, 5)$ where $h'(c)$ equals the slope of the secant line.
- Is h differentiable on $(1, 5)$? yes
- Explain why h is continuous on $[1, 5]$, even though there is a step discontinuity at $x = 5$.

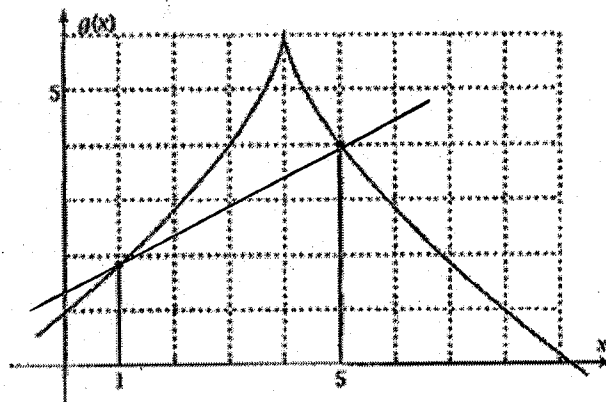
Because the discontinuity occurs at the end of the interval.



#5. For $g(x) = 6 - 2(x - 4)^{2/3}$, graphed below,

- Draw a secant line through $(1, g(1))$ and $(5, g(5))$.
- Is g differentiable on $(1, 5)$? no
- Is g continuous on $[1, 5]$? yes
- Tell why there is no value of $x = c$ between $x = 1$ and $x = 5$ at which $g'(c)$ equals the slope of the secant line.

*Does not meet the MVT condition
 $g(x)$ differentiable on $(1, 5)$*



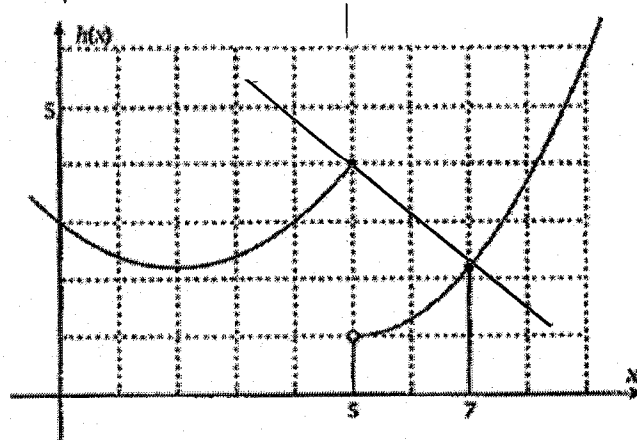
#6. Piecewise function h is defined by

$$h(x) = \begin{cases} 0.2(x - 2)^2 + 2.2, & \text{if } x \leq 5 \\ 0.3(x - 5)^2 + 1, & \text{if } x > 5 \end{cases}$$

- Draw a secant line through $(5, h(5))$ and $(7, h(7))$.
- Is h differentiable on $(5, 7)$? yes
- Is h continuous on $[5, 7]$? no
- Why is there no value $x = c$ in $(5, 7)$ for which $h'(c)$ equals the slope of the secant line?

Does not meet the MVT condition

$g(x)$ continuous on $[1, 5]$



On the AP Exam, the Intermediate and Mean Value Theorems are often found in problems involving a table of selected values of a function:

#7.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

The height of a tree at time t is given by a twice-differentiable function H , where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

- a) Explain why there must be at least one time, t , for $2 < t < 10$, such that $H'(t) = 2$.

$H(5) = 6$ and $H(3) = 2$ so the average rate of change over $[3, 5]$

$$\text{is } \frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = \frac{4}{2} = 2.$$

Since $H(t)$ is differentiable, it is also continuous over $[3, 5]$
the Mean Value Theorem guarantees t , $3 < t < 5$, such that $H'(t) = 2$.

∴ There must be at least one time, t , for $2 < t < 10$, such that $H'(t) = 2$.

- b) Explain why there must be at least one time, t , for $2 < t < 10$, such that $H(t) = 4$.

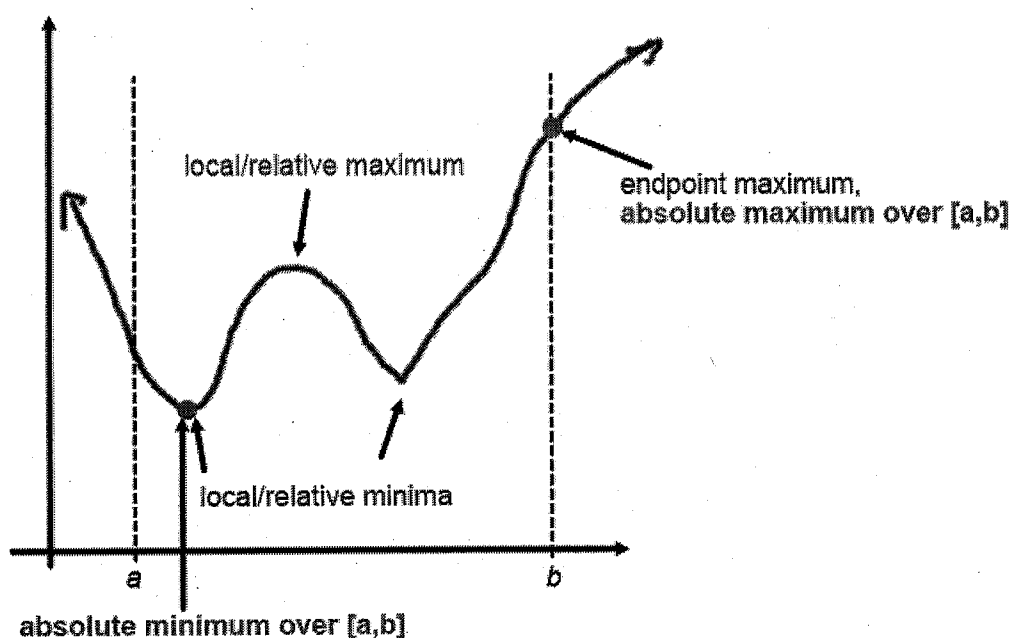
Since $H(t)$ is differentiable, it is also continuous over $[2, 10]$.

$$H(10) = 15 \text{ and } H(2) = 1.5 \text{ and since } 1.5 \leq 4 \leq 15$$

the Intermediate Value Theorem guarantees at least one time, t ,
 $2 < t < 10$, such that $H(t) = 4$.

Unit 3-4: Extrema on an Interval

Terminology of Extrema



Finding Extrema on an Interval

**Absolute Extrema may occur at
Local/Relative Extrema or at endpoints.**

**To find absolute max / min over an interval
(without a calculator) perform a 'candidates test':**

- 1) Find all relative extrema and include these x-values in a list of x candidates where absolute max / min may occur.
- 2) Add the two endpoint x-values to the list (if not already there).
- 3) Find the value of the function at these candidate x-values by simply plugging each into the function, and identify the absolute maximum and minimum value.

Examples

#1. Find the absolute maximum and absolute minimum value of

$$f(x) = \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 \text{ over } [-4, 3]$$

$$\begin{aligned} f'(x) &= x^3 + 2x^2 - 3x \\ &= x(x^2 + 2x - 3) \\ &= x(x+3)(x-1) \end{aligned}$$

critical pts at

$$x=0, x=-3, x=1$$

x	$f(x) = \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2$
0	0
-3	-11.25 ← abs min
1	-1.583
-4	-2.667
3	24.75 ← abs max

over $[-4, 3]$:

$f(x)$ has abs max of 24.75 at $x=3$

$f(x)$ has abs min of -11.25 at $x=-3$

#2. Find the absolute maximum and absolute minimum value of

$$f(x) = x^2 - 4x + 1 \text{ over } [-3, 5]$$

$$f'(x) = 2x - 4$$

critical pts at

$$2x - 4 = 0$$

$$x = 2$$

x	$f(x) = x^2 - 4x + 1$
2	$4 - 8 + 1 = -3$
-3	$9 + 12 + 1 = 22$
5	$25 - 20 + 1 = 6$

over $[-3, 5]$:

$f(x)$ has abs max of 22 at $x=-3$

$f(x)$ has abs min of -3 at $x=2$

Unit 3-5: Optimization

Remember this problem?

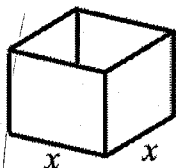
A manufacturer wants to design an open top, square base, box using 108 sq. in. of material.

If x is the side length of the base, the volume of the box is given by:

Determine the side length which will give the largest volume.

$$V(x) = 27x - \frac{1}{4}x^3$$

(In Unit 3 part 2 we will practice finding functions like this from word problems)



objective function

$$\text{max } V(x) = 27x - \frac{1}{4}x^3$$

$$V'(x) = 27 - \frac{3}{4}x^2 = 0$$

$$\frac{3}{4}x^2 = 27$$

$$x^2 = 27\left(\frac{4}{3}\right) = 36$$

$$\underline{\underline{x = 6 \text{ in}}}$$

verify max:

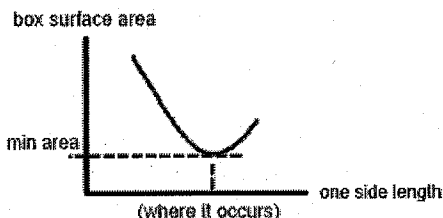
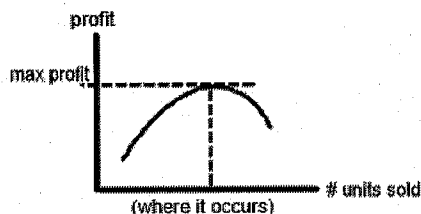
$$V'' = -\frac{3}{2}x < 0$$

V is concave down

∴ this is a max

Optimization

Optimization refers to finding the optimum value for a real-world problem - finding the values which make a value either a maximum or a minimum.



Step 1: Identify the quantity to be maximized or minimized - we need a function for this called the 'objective function'.

Step 2: Define a variable for the object, and anything else varying in the problem. Use an illustration to assist you.

Step 3: Use the variables to build the objective function.

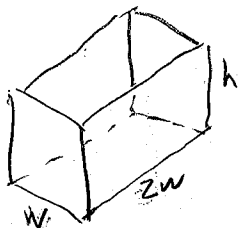
Step 4: If the objective function has more than two variables, make a second equation, the constraint, that also relates the variables (this may involve using a constant from the problem).

Step 5: Solve the constraint for a variable and substitute into the objective function so that the objective is a function of only one variable.

Step 6: Use the First and Second Derivative Tests to find the max or min and verify it is a max or min.

Step 7: Re-read the problem to make sure you are answering the question (may want the optimum value, where it occurs, or something else).

- #1. A rectangular storage container with an open top is to have a volume of 10 m^3 . The length of its base is twice the width. Material for the base costs \$10 per m^2 and material for the sides costs \$6 per m^2 . Find the cost of materials for the cheapest container.



objective function

$$\begin{aligned} \min_{\text{cost}} C &= 2(wh)(6) + 2(2wh)(6) + (2w^2)(10) \\ C &= 36wh + 20w^2 \\ C &= 36w\left(\frac{5}{w^2}\right) + 20w^2 = 180w^{-1} + 20w^2 \\ C' &= -180w^{-2} + 40w = 0 \end{aligned}$$

constraint

$$\begin{aligned} V &= 10 \text{ m}^3 \\ 2w^2h &= 10 \\ h &= \frac{10}{2w^2} = \frac{5}{w^2} \end{aligned}$$

verify min:

$$C'' = 360w^{-3} + 40 > 0$$

so C is concave up \checkmark

\therefore this is a minimum cost

$$40w = \frac{180}{w^2}$$

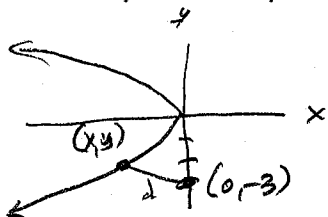
$$w^3 = \frac{180}{40} = 4.5$$

$$w = \sqrt[3]{4.5} = (4.5)^{1/3} = 1.650963624 \text{ m}$$

$$h = \frac{5}{w^2} = 1.834404028$$

$$C = \$163.541$$

- #2. Find the point on the parabola $x + y^2 = 0$ that is closest to the point $(0, -3)$.



objective

$$\min d = \sqrt{(x-0)^2 + (y+3)^2}$$

define $f = d^2$

$$\min f = x^2 + (y+3)^2$$

$$f = (-y^2)^2 + (y+3)^2$$

$$f = y^4 + y^2 + 6y + 9$$

$$f' = 4y^3 + 2y + 6 = 0$$

when $y = -1$ (calculator graph)

$$x = -y^2 = -(-1)^2 = -1$$

point on parabola closest to $(0, -3)$

is $\underline{\underline{(-1, -1)}}$

constraint

$$x + y^2 = 0$$

$$x = -y^2$$

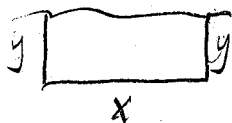
verify min:

$$f'' = 12y^2 + 2 > 0$$

f is concave up \checkmark

\therefore this is a minimum distance

- #3. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 245,000 square meters, and no fencing is needed along the one side which is along the river. What dimension will require the least amount of fencing?



objective
min $F = x + 2y$

$$F = x + 2\left(\frac{245000}{x}\right)$$

$$F = x + 490000x^{-1}$$

$$F' = 1 - 490000x^{-2} = 0$$

$$1 = \frac{490000}{x^2}$$

$$x^2 = 490000$$

$$x = 700 \text{ m}, y = 350 \text{ m}$$

constraint

$$A = 245000 \text{ m}^2$$

$$xy = 245000$$

$$y = \frac{245000}{x}$$

verify min

$$F'' = 2(490000)x^{-3} > 0$$

F is concave up \checkmark

so this is a min. Fencing

- #4. Find two positive numbers where the sum of the first number plus three times the second number is a minimum, if the product of the numbers must be 147.

objective

min $S = x + 3y$

$$S = x + 3\left(\frac{147}{x}\right)$$

$$S = x + 441x^{-1}$$

$$S' = 1 - 441x^{-2} = 0$$

$$1 = \frac{441}{x^2}$$

$$x^2 = 441$$

$$x = \sqrt{441} = 21, y = 7$$

constraint

$$xy = 147$$

$$y = \frac{147}{x}$$

verify min

$$S'' = 2(441)x^{-3} > 0$$

S is concave up \checkmark

so this is a minimum sum

- #5. The rate (in mg carbon / m³/hr) at which photosynthesis takes place for a species of phytoplankton is modeled by the function:

$$P = \frac{100I}{I^2 + I + 4}$$

Where I is the light intensity measured in thousands of foot-candles and P is the photosynthesis rate. For what light intensity is P a maximum?

objective

(no constraint)

max $P = \frac{100I}{I^2 + I + 4}$

$$P' = \frac{(I^2 + I + 4)(100) - 100I(2I + 1)}{(I^2 + I + 4)^2} = 0$$

$$= \frac{100I^2 + 100I + 400 - 200I^2 - 100I}{(I^2 + I + 4)^2} = 0$$

$$\frac{-100I^2 + 400}{(I^2 + I + 4)^2} = 0$$

when $100I^2 = 400$

$$I^2 = \frac{400}{100} = 4$$

$$I = 2 \text{ thousand foot-candles}$$

verify max (test pts)

$-\infty$ $\frac{1}{2}$ ∞

$$P'(0)$$

$$\frac{400}{400}$$

/

$$P'(3)$$

$$\frac{60}{40}$$

\

this is a max rate

Unit 3-6: Derivatives in Physics applications

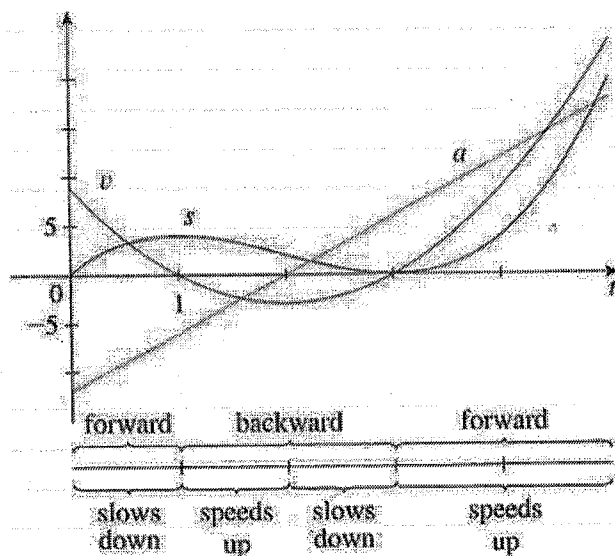
Derivative Terminology in Physics

If the original function is a position function (position as a function of time), then the first and higher-order derivatives have specific meanings and names...

original function	$s(t)$	position at a specific time	"position"
first derivative	$v(t) = s'(t)$	how fast position is changing at a specific time	"velocity"
second derivative	$a(t) = v'(t) = s''(t)$	how fast velocity is changing at a specific time	"acceleration"
third derivative	although we don't see it often, the derivative of acceleration, how fast acceleration changes at a specific time (the 3rd derivative of position) is called "jerk".		

We also say an object or particle **"speeds up"** when the velocity is positive and increasing or when the velocity is negative and decreasing (velocity and acceleration are both positive or both negative).

We also say an object or particle **"slows down"** when the velocity and acceleration have opposite signs.



#1. A particle moves along the x-axis so that its velocity at time t , $0 \leq t \leq 5$ is given by $v(t) = -6t^2 + 30t - 36$. At time $t = 1$, the position of the particle is $x(1) = 4$.

- Find the acceleration of the particle at any time, t .
- Find the minimum acceleration of the particle.
- When is the particle at rest?
- When is the particle moving to the left?
- Is the particle speeding up or slowing down at $t = 1$?

a) $a(t) = -12t + 30$

b) abs min $a(t)$ may occur at critical pt) or interval ends:

$a'(t) = -12 \neq 0$, DNE so no relative extrema

t	$a(t) = -12t + 30$	the minimum acceleration of the particle is -30 which occurs at $t = 5$.
0	30	
5	$-12(5) + 30 = -30$	

c) At rest when $v(t) = 0$

$$-6t^2 + 30t - 36 = 0$$

no calculator?

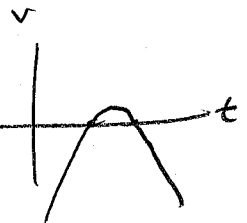
$$-6(t^2 - 5t + 6) = 0$$

$$-6(t-2)(t-3) = 0$$

at $t = 2$ and $t = 3$.

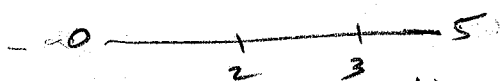
Calculator allowed?

at $t = 2$ and $t = 3$.



d) particle moves to the left when $v(t) < 0$

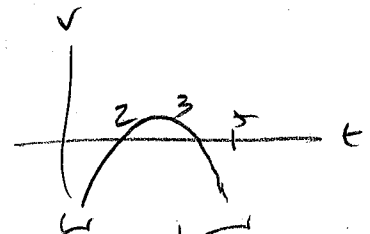
no calculator?



$v(1)$	$v(2.5)$	$v(4)$
$(-)(-)(-)$	$(-)(+)(-)$	$(-)(+)(+)$
$(-)$	$(+)$	$(-)$

from $0 \leq t < 2$ and $3 < t \leq 5$

calculator?



from $0 \leq t < 2$ and $3 < t \leq 5$.

e) $v(1) = -6(1)^2 + 30(1) - 36 = -42 < 0$

$$a(1) = -12(1) + 30 > 0$$

since the signs of $v(1)$ and $a(1)$ are different, the particle is slowing down at $t = 1$.

#2. A particle moves along the x-axis so that its velocity at time t , $0 \leq t \leq 5$, is given by $v(t) = 3(t-1)(t-3)$. At time $t = 2$, the position of the particle is $x(2) = 0$.

a) Find the acceleration of the particle at any time, t .

b) Find the minimum acceleration of the particle.

c) When is the particle at rest?

d) When is the particle moving to the left?

e) Is the particle speeding up or slowing down at $t = 2$?

a) $v(t) = 3(t^2 - 4t + 3) = 3t^2 - 12t + 9$
 $a(t) = 6t - 12$

b) abs min $a(t)$ may occur at critical pts. or interval ends.

$a'(t) = 6 \neq 0$ or PNE so no relative extrema

t	$a(t) = 6t - 12$
0	-12
5	$30 - 12 = 18$

The minimum acceleration of the particle is -12 which occurs at $t = 0$.

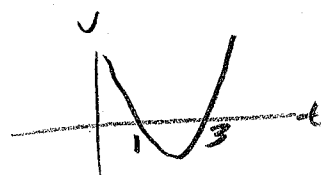
c) particle is at rest when $v(t) = 0$

no calculator?

$3(t-1)(t-3) = 0$

at $t = 1$ and $t = 3$

calculator?



at $t = 1$ and $t = 3$.

d) particle moves to the left when $v(t) < 0$

no calculator?



$(-)(-)(-) \quad (+)(+)(-)$

$(-)(-)(-) \quad (+)(+)(-)$

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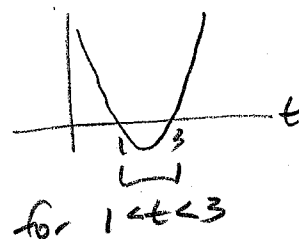
$(-)(-)(-) \quad (+)(+)(-)$

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$(-)(-)(-) \quad (+)(+)(-)$

$(-)(-)(-) \quad (+)(+)(-)$

calculator?



e) $v(2) = 3(2-1)(2-3) = 3(1)(-1) = -3$

$a(2) = 6(2) - 12 = 0$

No acceleration, so the particle is neither speeding up or slowing down

at $t = 2$.

- #3. The velocity of a particle moving along the x-axis is modeled by a differentiable function v , where the position x is measured in meters, and the time t is measured in seconds. Selected values of $v(t)$ are given in the table.

t (seconds)	0	8	20	25	32	40
$v(t)$ (meters per second)	3	5	-10	-8	-4	7

Estimate the acceleration of the particle at $t = 36$ seconds.

$$a(36) = v'(36) \approx \frac{v(40) - v(32)}{40 - 32} = \frac{7 - (-4)}{8} = \frac{12}{8} = \frac{3}{2} \text{ m/sec/sec}$$

- #4. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is given by $s(t) = 80t - 15t^2$.

- Find the velocity of the ball, $v(t)$.
- What is the maximum height reached by the ball?
- What is the velocity of the ball when it is 96 ft. above the ground, on its way up? ...on its way down?

a) $v(t) = 80 - 30t$

b) at max, $v(t) = 0$, which occurs at $80 - 30t = 0$
 $30t = 80$
 $t = \frac{8}{3}$ seconds

The height at this time is $s(\frac{8}{3}) = 80(\frac{8}{3}) - 15(\frac{8}{3})^2$ ft
 $= 106.667$ ft

c) $s(t) = 96$

$$80t - 15t^2 = 96$$

$$15t^2 - 80t + 96 = 0$$

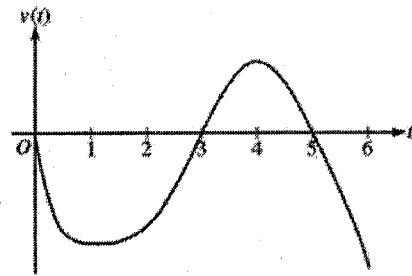
which occurs at $t = 1.8233926$ sec
 (on the way up)

and $t = 3.5099407$ sec
 (on the way down)

$$v(1.8233926) = 25.298 \text{ ft/sec (upward)}$$

$$v(3.5099407) = -25.298 \text{ ft/sec (downward)}$$

- #5. A particle moves along the x-axis so that its velocity at time t , $0 \leq t \leq 6$, is given by a differentiable function v whose graph is shown.



Graph of v

During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

$a(t) = v'(t)$ so $a(t) < 0$ when $v'(t) < 0$,
this occurs when $v(t)$ is decreasing
which is true for $0 < t < 1$ and $4 < t < 6$.

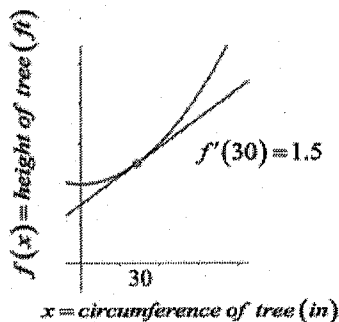
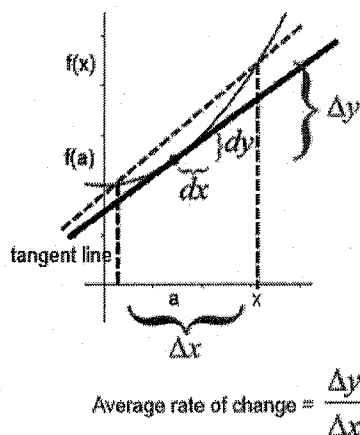
Linearization of a function in the region of an x-value and the meaning of 'differential'

For a given function curve, as long as we stay in a small region around an x-value, then the tangent line is a good approximation to the curve $f(x)$:

dx and dy are called 'differentials'

A differential is an infinitely small change in a variable's value.

You can think of $f'(x)$, the derivative, as the ratio of how much change in the y variable is produced by a small change in the x variable.



For example, if the y variable is the height of a tree and the x variable is the circumference of the tree, a derivative of 1.5 at $x = 30$ means that for every 1 additional inch of circumference a tree's height increases by 1.5 feet.

More precisely, the ratio of changes in height to changes in circumference is 1.5 in the region of circumference of 30 inches. If you made an infinitely small change in circumference the infinitely small change in height would be 1.5 times the change in circumference.

If we have one known point on a function curve, and information about the derivative of a function, we can use a tangent line to find an approximate value for the function near the starting point:

- #1. Given $f(x) = x^2 + 2$ write the equation of a tangent line to the curve at $x = 1$, and then use this tangent line to approximate the value of the function at $x = 2.2$

$$f(1) = 1^2 + 2 = 3$$

$$f'(x) = 2x$$

$$f'(1) = 2(1) = 2 \quad (1, 3) \quad m = 2$$

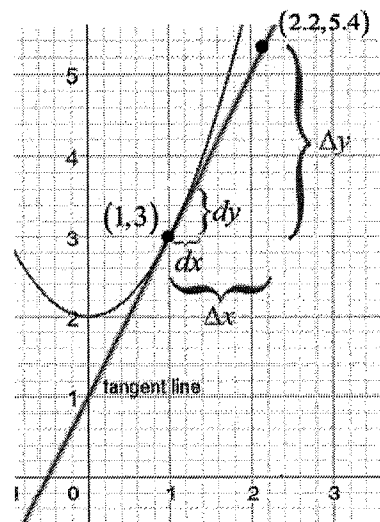
$$(y - 3) = 2(x - 1)$$

$$y = 2x - 2 + 3$$

$$y = 2x + 1$$

$$y(2.2) = 2(2.2) + 1 = 5.4$$

$$\therefore f(2.2) \approx 5.4$$

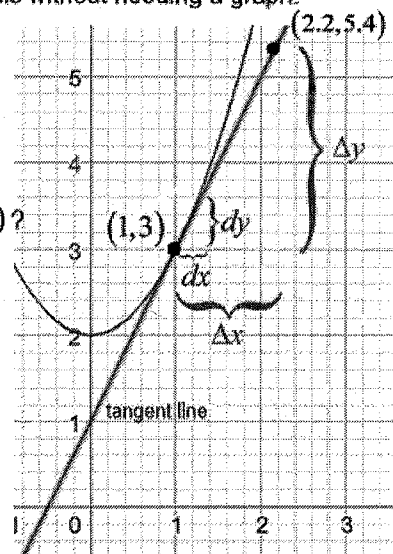


Use the 2nd derivative to determine whether an approximation is an over- or under-estimate

From the graph, we can see that this estimation of $f(2.2)$ is an underestimate of the actual value of $f(2.2)$

But we can use the 2nd derivative to find the concavity of $f(x)$ to determine this without needing a graph:

- #2. Given $f(x) = x^2 + 2$ write the equation of a tangent line to the curve at $x = 1$, and then use this tangent line to approximate the value of the function at $x = 2.2$



Does this approximation over- or under-estimate the true value of $f(2.2)$?

$$f(1) = (1)^2 + 2 = 3$$

$$f'(x) = 2x, f'(1) = 2 \quad (1, 3) \text{ } m = 2$$

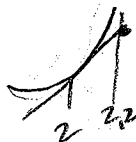
$$(y - 3) = 2(x - 1) \rightarrow y = 2x + 1$$

$$y(2.2) \approx f(2.2) \approx 2(2.2) + 1 = 5.4$$

$$f''(x) = 2 > 0$$

$\therefore f$ is concave up in this region

$\therefore 5.4$ is an underestimation of $f(2.2)$



- #3. Given $f(x) = 3x - x^2 + 4$

a) Write the equation of a tangent line to the curve at $x = 2$.

b) Use this tangent line to approximate the value of the function at $x = 3$.

c) Does this approximation over- or under-estimate the true value of $f(3)$?

$$(a) f(2) = 3(2) - (2)^2 + 4 = 6$$

$$f'(x) = 3 - 2x$$

$$f'(2) = 3 - 2(2) = -1 \quad (2, 6) \text{ } m = -1$$

$$(y - 6) = -(x - 2)$$

$$y = -x + 2 + 6$$

$$y = -x + 8$$

$$(b) y(3) = -(3) + 8 = 5 \approx f(3)$$

$$(c) f''(x) = -2 < 0$$

$\therefore f$ is concave down in this region

$\therefore 5$ is an overestimation of $f(3)$



Examples

#4. a) Use the tangent line at $x = 0$ to approximate the function $f(x) = \sin(x)$

b) Use this tangent line to approximate $\sin(0)$, $\sin(0.1)$, and $\sin(0.5)$.

c) For each of these approximately, what is the 'error' (difference between the actual function value and the approximated value)?

(a) $f(0) = \sin 0 = 0$

$f'(x) = \cos x$

$f'(0) = \cos 0 = 1$ $(0, 0)$ $m = 1$

$(y - 0) = 1(x - 0)$

$y = x$

(b) $f(0) \approx 0$

$f(0.1) \approx 0.1$

$f(0.5) \approx 0.5$

(c) actual

$\sin(0) = 0$

$\sin(0.1) = 0.09983341666$

$\sin(0.5) = 0.4794255386$

approx

0

0.1

0.5

error

0

$1.666 \cdot 10^{-4}$
 $= 0.0001666$

0.02057

these are overestimates

#5. Use a linear approximation to estimate the given number: $\sqrt{36.1}$

$f(x) = \sqrt{x} = x^{1/2}$

$f(36) = 6$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$f'(36) = \frac{1}{2\sqrt{36}} = \frac{1}{12}$

$(36, 6)$ $m = \frac{1}{12}$

$(y - 6) = \frac{1}{12}(x - 36)$

$y = \frac{1}{12}x - 3 + 6$

$y = \frac{1}{12}x + 3$

$y(36.1) = \frac{1}{12}(36.1) + 3 = 6.00833$

Unit 3-8: Related Rates

In Related Rates problems, two different variables are related and both vary with time

In natural, social, and behavioral sciences, two different quantities represented by variables are related to each other, but also both vary over time. If one quantity is changing over time at a known rate, we can determine the rate of change of the other quantity (at a specific point in time). These problems are referred to as **related rate problems**.

Solving Related Rates Problems

Step 1: Draw a picture if possible.

Step 2: Identify what rate you are trying to find...this will be a derivative, so define a variable for this quantity.

Step 3: Identify the rate you are given, and define a variable for this quantity.

Step 4: Use the picture to write an algebra equation linking these variables.

Step 5: Differentiate implicitly with respect to time.

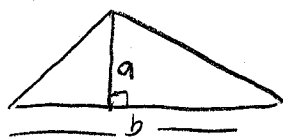
Step 6: Substitute numerical values for the variables and the derivatives at the specified point in time, and solve for the unknown rate.

What can make these problems tricky is each one is a little different. But there are some frequently occurring types of equations that are produced, so we'll look at some common ones...

Solving Related Rates Problems

Geometry formulas - the equation relating the two variables is often a geometry formula.

- #1. The altitude of a triangle is increasing at a rate of 1 cm/minute while the area of the triangle is increasing at a rate of 2 cm²/minute. How fast is the length of the base of the triangle changing when the altitude is 10 cm?



Find $\frac{db}{dt}$

$$\frac{da}{dt} = 1 \text{ cm/min}$$

$$\frac{dA}{dt} = 2 \text{ cm}^2/\text{min}$$

$$A = \frac{1}{2} b a \quad (\text{product rule})$$

$$\frac{d}{dt}(A) = \frac{1}{2} b \frac{d}{dt}(a) + a \frac{d}{dt}(\frac{1}{2} b)$$

$$1 \frac{dA}{dt} = \frac{1}{2} b (1 \frac{da}{dt}) + a (\frac{1}{2} \frac{db}{dt})$$

$$(2) = \frac{1}{2} () (1) + () \frac{1}{2} \frac{db}{dt}$$

Unit 3-8: Related Rates

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Solving Related Rates Problems

Geometry formulas - the equation relating the two variables is often a geometry formula.

- #1. The altitude of a triangle is increasing at a rate of 1 cm/minute while the area of the triangle is increasing at a rate of 2 cm²/minute. How fast is the length of the base of the triangle changing when the altitude is 10 cm and the area is 100 cm²?

$$A = \frac{1}{2}bh \quad (\text{product rule})$$

$$\frac{d}{dt}[A] = \frac{1}{2}b \frac{dh}{dt} + h \frac{d}{dt}\left[\frac{1}{2}b\right]$$

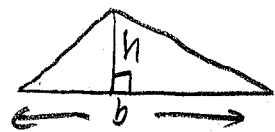
$$1 \frac{dA}{dt} = \frac{1}{2}b \left(1 \frac{dh}{dt}\right) + h \left(\frac{1}{2} \frac{db}{dt}\right)$$

$$(2) = \frac{1}{2}(20)(1) + (10) \frac{1}{2} \frac{db}{dt}$$

$$2 = 10 + 5 \frac{db}{dt}$$

$$-8 = 5 \frac{db}{dt}$$

$$\frac{db}{dt} = \frac{-8}{5} \text{ cm/minute}$$



Find $\frac{db}{dt}$

$$\frac{dh}{dt} = +1 \text{ cm/min}$$

$$\frac{dA}{dt} = +2 \text{ cm}^2/\text{min}$$

(instant)

$$A = \frac{1}{2}bh$$

$$100 = \frac{1}{2}b(10)$$

$$\frac{200}{10} = b$$

$$20 = b$$

Solving Related Rates Problems

Pythagorean Theorem - when the two variables are lengths or distances moving perpendicularly.

- #2. A plane flying horizontally at an altitude of 1 mile and a speed of 500 miles/hr passes directly over a radar station. Find the rate at which the distance from the plane to the station is changing when it is 2 miles away from the station.

$$x^2 + 1^2 = s^2$$

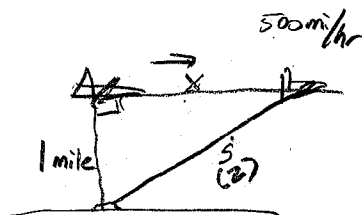
$$\frac{d}{dt}[x^2] + \frac{d}{dt}[1] = \frac{d}{dt}[s^2]$$

$$2x \frac{dx}{dt} + 0 = 2s \frac{ds}{dt}$$

$$2(\sqrt{3})(500) = 2(2) \frac{ds}{dt}$$

$$1000\sqrt{3} = 4 \frac{ds}{dt}$$

$$\frac{ds}{dt} = \frac{1000\sqrt{3}}{4} = 250\sqrt{3} = 433.013 \frac{\text{mi}}{\text{hr}}$$



Find $\frac{ds}{dt}$

$$\frac{dx}{dt} = 500 \text{ mi/hr}$$

$$x^2 + 1 = s^2$$

$$x^2 + 1 = (2)^2 = 4$$

$$x^2 = 3, x = \sqrt{3}$$

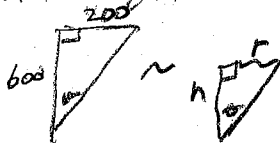
Quantity accumulation - when the problem is about a quantity (often a liquid) accumulating/depleting.

- #3. Water is leaking out of an inverted conical tank at a rate of 10,000 cm³/minute at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/minute, find the rate at which water is being pumped into the tank when the height of the water is 2 m.

$$V = \frac{1}{3}\pi r^2 h$$

don't have a derivative for r , so need to substitute it... need another equation!

$$V = \frac{1}{3}\pi \left(\frac{1}{3}h\right)^2 h$$



$$\frac{r}{h} = \frac{200}{600} = \frac{1}{3}$$

$$r = \frac{1}{3}h$$

$$V = \frac{\pi}{27} h^3$$

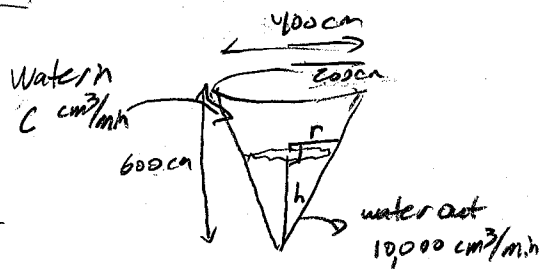
$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{27} h^3\right]$$

$$\frac{dV}{dt} = \frac{\pi}{27} 3h^2 \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{9} (200)^2 (20) = 279252.6803 \text{ cm}^3/\text{min}$$

$$\frac{dV}{dt} = C - 10000$$

$$\therefore C = 279252.6803 + 10000 = 289252.680 \text{ cm}^3/\text{min}$$



Find C

$$\frac{dh}{dt} = +20 \text{ cm/min}$$

Volume of water:

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = C - 10000$$

$$\text{when } h=2\text{m, } h=200\text{cm}$$

Solving Related Rates Problems

Geometry formulas - the equation relating the two variables is often a geometry formula.

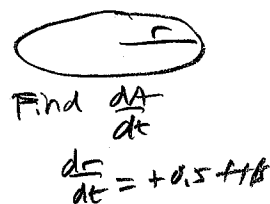
- #4. A child throws a stone into a still pond, causing a circular ripple to spread. If the radius of the circle increases at a constant rate of 0.5 feet per second, how fast is the area of the ripple increasing when the radius of the ripple is 30 feet?

$$A = \pi r^2$$

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi (30)(0.5) = 94.248 \text{ ft}^2/\text{s}$$



Pythagorean Theorem - when the two variables are lengths or distances moving perpendicularly.

- #5. A 20 foot long ladder is leaning against a wall. If the foot of the ladder is slipping away from the wall at a rate of 7 ft/s, how fast is the top of the ladder (the point where the ladder touches the wall) moving downward when the foot of the ladder is 4 ft from the wall?

$$x^2 + y^2 = 20^2$$

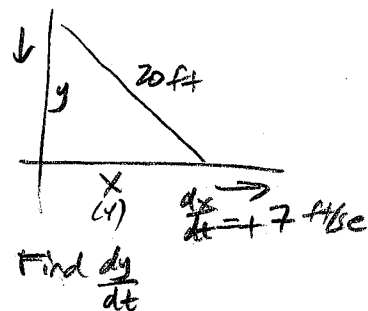
$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(20^2)$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$2(4)(7) + 2(\sqrt{384}) \frac{dy}{dt} = 0$$

$$2\sqrt{384} \frac{dy}{dt} = -2(4)(7)$$

$$\frac{dy}{dt} = -\frac{28}{\sqrt{384}} = -1.429 \text{ ft/sec}$$



$$(4)^2 + y^2 = 20^2$$

$$y^2 = 384$$

$$y = \sqrt{384}$$

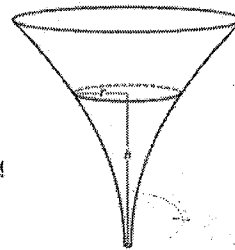
Solving Related Rates Problems

Quantity accumulation - when the problem is about a quantity (often a liquid) accumulating/depleting.

#6. The radius of a funnel depends upon height according to the function

$$r = \frac{1}{20}(3 + h^2) \text{ where } 0 \leq h \leq 10 \text{ and both } r \text{ and } h \text{ are in inches.}$$

The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, how fast is the height of the liquid changing?



$$r = \frac{1}{20}(3 + h^2)$$

$$\frac{d}{dt}[r] = \frac{d}{dt}\left[\frac{1}{20}(3 + h^2)\right]$$

$$\frac{dr}{dt} = \frac{2}{10}h \frac{dh}{dt}$$

$$\left(-\frac{1}{5}\right) = \frac{1}{10}(3) \frac{dh}{dt}$$

$$-2 = 3 \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{2}{3} \frac{\text{in}}{\text{sec}}$$

$$\text{Find } \frac{dh}{dt}$$

$$\frac{dr}{dt} = -\frac{1}{5} \text{ in/sec}$$