

AP Calc BC – Lesson Notes – Unit 3: Finding Derivatives

Unit 3-1: Derivative 'Shortcuts'

Larsen: 2.2 (Stewart: 3.1)

We can evaluate a derivative of a given function at a generic x to get a 'shortcut'

The limit definition of the derivative is the official way to determine a derivative equation for a given function, but it can be very time-consuming to compute. To accelerate finding derivatives so that we can focus later on how derivatives are used, we employ 'derivative shortcuts' (not an official term).

We will now allow the use of derivative shortcuts to compute derivatives, *however, you must be prepared to derive (prove) the derivative short for any simple function.*

Proofs (derivations) of selected derivative shortcuts

Derive the derivative shortcut for $f(x) = c$ where c is any constant

proof

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[c] - [c]}{h} \\ &= \lim_{h \rightarrow 0} 0 \\ &= 0 \end{aligned}$$

shortcut

$$\frac{d}{dx}[c] = 0$$

Derive the derivative shortcut for $f(x) = x^n$ where n is any real number

proof

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^n] - [(x)^n]}{h} \end{aligned}$$

shortcut

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

(binomial theorem)

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{\left[x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 \dots + \binom{n}{n-1}x^1h^{n-1} + h^n \right] - [(x)^n]}{h} \\ &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 \dots + \binom{n}{n-1}x^1h^{n-1} + h^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{h \left[nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 \dots + \binom{n}{n-1}x^1h^{n-2} + h^{n-1} \right]}{h} \\ &= \lim_{h \rightarrow 0} \left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 \dots + \binom{n}{n-1}x^1h^{n-2} + h^{n-1} \right) \\ &= nx^{n-1} \end{aligned}$$

Proofs (derivations) of selected derivative shortcuts

Derive the derivative shortcut for $f(x) = e^x$

proof

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[e^{x+h}] - [e^x]}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x (e^h - 1)}{h} \end{aligned}$$

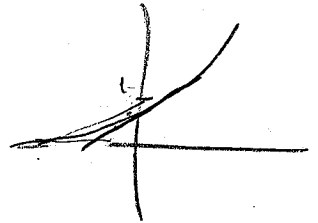
definition of: $e = \lim_{h \rightarrow 0} (1+h)^{\frac{1}{h}}$ means that for small h , $e \approx (1+h)^{\frac{1}{h}}$

therefore, $e^h \approx 1+h$ Substituting this...

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{e^x ([1+h] - 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x h}{h} \\ &= \lim_{h \rightarrow 0} e^x \\ &= e^x \end{aligned}$$

shortcut

$$\frac{d}{dx} [e^x] = e^x$$



Derive the derivative shortcut for $f(x) = \sin(x)$

proof

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[\sin(x+h)] - [\sin(x)]}{h} \end{aligned}$$

shortcut

$$\boxed{\frac{d}{dx} [\sin(x)] = \cos(x)}$$

trig identity: $\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$

Substituting this...

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{[\sin(x)\cos(h) + \cos(x)\sin(h)] - [\sin(x)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) - \sin(x) + \cos(x)\sin(h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h)-1) + \cos(x)\sin(h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x)\sin(h)}{h} \end{aligned}$$

The functions not containing h can be moved outside the limits...

$$= \sin(x) \lim_{h \rightarrow 0} \frac{(\cos(h)-1)}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h}$$

Special trig limits: $\lim_{h \rightarrow 0} \frac{(\cos(h)-1)}{h} = 0$, $\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$...

$$\begin{aligned} &= \sin(x)[0] + \cos(x)[1] \\ &= \cos(x) \end{aligned}$$

List of derivative shortcuts (so far) that you must memorize

$$\frac{d}{dx} [c] = 0$$

$$\frac{d}{dx} [x^n] = nx^{n-1} \quad \text{"power rule"}$$

$$\frac{d}{dx} [e^x] = e^x$$

$$\frac{d}{dx} [a^x] = (\ln a)a^x$$

$$\frac{d}{dx} [\sin(x)] = \cos(x)$$

$$\frac{d}{dx} [\cos(x)] = -\sin(x)$$

(← derived later, we've used $\frac{d}{dx} \ln(x)$)

Proofs (derivations) of selected derivative properties (rules)

What is the derivative of a constant times a function? $\frac{d}{dx}[cf(x)]$

proof

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[cf(x+h)] - [cf(x)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{c(f(x+h) - f(x))}{h} \\ &= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= cf'(x)\end{aligned}$$

property / rule

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}[f(x)]$$

"constant multiple rule"

What is the derivative of the sum of two different functions? $\frac{d}{dx}[f(x) + g(x)]$

proof

$$\begin{aligned}\frac{d}{dx}[f(x) + g(x)] &= \lim_{h \rightarrow 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]\end{aligned}$$

property / rule

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

"sum rule"

List of derivative properties / rules (so far) that you must memorize

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}[f(x)] \quad \text{"constant multiple rule"}$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \quad \text{"sum rule"}$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] \quad \text{"difference rule"}$$

(The sum and difference rules are what allow us to take derivatives of each term separately in a function with multiple terms...each term is being treated as a separate function.)

Examples

Find the derivatives of the given functions.

$$f(x) = x^5$$

$$f'(x) = 5x^4$$

$$f(x) = x^{\frac{2}{3}}$$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

$$= \frac{2}{3\sqrt[3]{x}}$$

$$g(t) = \frac{2}{t^5} = 2t^{-5}$$

$$g'(t) = -10t^{-6}$$

$$= \frac{-10}{t^6}$$

$$f(x) = 3x^5 + 4e^x + 7$$

$$f'(x) = 15x^4 + 4e^x$$

$$g(x) = \frac{x^4 - 3x^2}{2x} = \frac{x^3}{2} - \frac{3x}{2}$$

$$g(x) = \frac{1}{2}x^3 - 3x$$

$$\text{so } g'(x) = \frac{3}{2}x^2 - 3$$

For what values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?

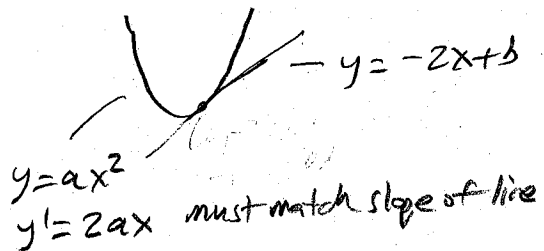
now, point of tangency must lie on both curve & line at their intersection (ys are equal)

$$-2x + b = ax^2 \quad a = \frac{1}{2}$$

$$-2(2) + b = \left(\frac{1}{2}\right)(2)^2 \quad x = 2$$

$$-4 + b = -2$$

$$b = 2$$



$$\text{so } 2ax = -2$$

$$\text{at } x = 2!$$

$$2a(2) = -2$$

$$4a = -2$$

$$a = -\frac{2}{4}$$

$$a = -\frac{1}{2}$$

Examples

Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

$$y' = 3ax^2 + 2bx + c = 0 \text{ at horiz tangents}$$

at $(-2, 6)$

at $(2, 0)$

$$3a(-2)^2 + 2b(-2) + c = 0$$

$$3a(2)^2 + 2b(2) + c = 0$$

$$12a - 4b + c = 0$$

$$12a + 4b + c = 0$$

$$a(-2)^3 + b(-2)^2 + c(-2) + d = 6$$

$$a(2)^3 + b(2)^2 + c(2) + d = 0$$

$$-8a + 4b - 2c + d = 6$$

$$8a + 4b + 2c + d = 0$$

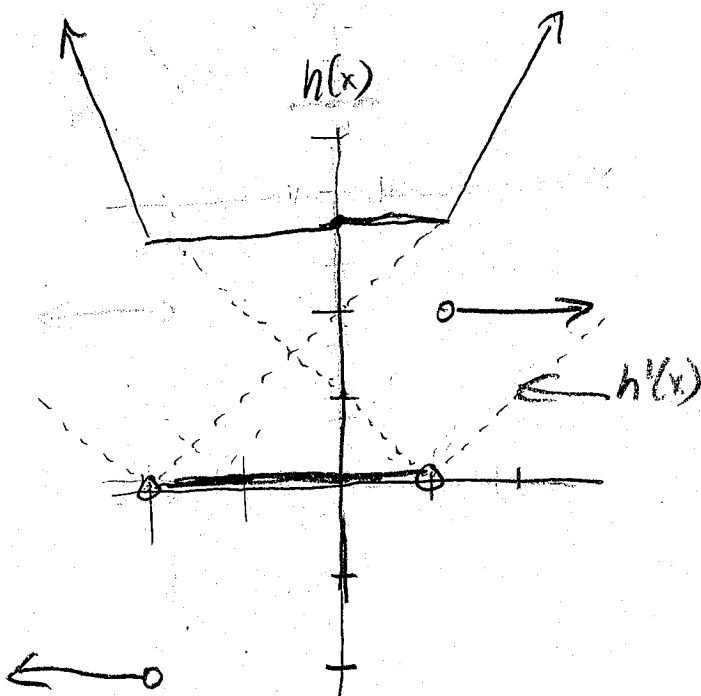
System:
$$\begin{cases} 12a - 4b + c = 0 \\ 12a + 4b + c = 0 \\ -8a + 4b - 2c + d = 6 \\ 8a + 4b + 2c + d = 0 \end{cases}$$

$$\rightarrow \left[\begin{array}{cccc|c} 12 & -4 & 1 & 0 & 0 \\ 12 & 4 & 1 & 0 & 0 \\ -8 & 4 & -2 & 1 & 6 \\ 8 & 4 & 2 & 1 & 0 \end{array} \right]$$

$$\text{rref} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/16 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -9/4 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right]$$

$$y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$$

Where is the function $h(x) = |x-1| + |x+2|$ differentiable?
Find a formula for h' and sketch the graphs of h and h' .



$$h'(x) = \begin{cases} -2, & x < -2 \\ 0, & -2 < x < 1 \\ 2, & x > 1 \end{cases}$$

Unit 3-2: Derivative Product and Quotient Rules
Larsen: 2.3 (Stewart: 3.3)

Proofs (derivations) of selected derivative properties (rules)

What is the derivative of the product of two functions? $\frac{d}{dx}[f(x)g(x)]$

proof

property / rule

$$\frac{d}{dx}[f(x)g(x)] =$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\lim_{h \rightarrow 0} \frac{[f(x+h)g(x+h)] - [f(x)g(x)]}{h}$$

"product rule"
(first times derivative of the second, plus
second times derivative of the first)

add and subtract a specific term in the middle...

$$\lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

factor each pair...

$$\lim_{h \rightarrow 0} \frac{f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x))}{h}$$

$$\lim_{h \rightarrow 0} \frac{f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x))}{h}$$

$$\lim_{h \rightarrow 0} f(x+h) \frac{(g(x+h) - g(x))}{h} + \lim_{h \rightarrow 0} g(x) \frac{(f(x+h) - f(x))}{h}$$

$$\lim_{h \rightarrow 0} f(x+h) \lim_{h \rightarrow 0} \frac{(g(x+h) - g(x))}{h} + \lim_{h \rightarrow 0} g(x) \lim_{h \rightarrow 0} \frac{(f(x+h) - f(x))}{h}$$

$$f(x)g'(x) + g(x)f'(x)$$

A product rule example

Differentiate each function:

$$f(x) = x^3 e^x$$

$$f'(x) = x^3 e^x + e^x (3x^2)$$

$$= \boxed{e^x x^2 (x+3)}$$

$$f(x) = 2x^3$$

could do product rule...

$$f'(x) = 2(3x^2) + x^3(0)$$

$$= \boxed{6x^2}$$

but not needed...

$$f'(x) = \frac{d}{dx}[2x^3] = 2 \frac{d}{dx}[x^3]$$

$$= 2(3x^2)$$

$$= \boxed{6x^2}$$

(just take out constant
and multiply back in
at the end.)

$$f(x) = 2x^3 e^x$$

2 ways...

$$f'(x) = 2 \frac{d}{dx}[x^3 e^x]$$

$$= 2[x^3 e^x + e^x (3x^2)]$$

$$= 2x^3 e^x + 6x^2 e^x$$

$$= \boxed{2x^2 e^x (x+3)}$$

... or just group the constant
with one of the terms

$$f(x) = (2x^3)(e^x)$$

$$f'(x) = (2x^3)e^x + e^x(6x^2)$$

$$= 2x^3 e^x + 6e^x x^2$$

$$= \boxed{2x^2 e^x (x+3)}$$

Proofs (derivations) of selected derivative properties (rules)

What is the derivative of the quotient of two functions?

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right]$$

proof

property / rule

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] =$$

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

"quotient rule"
(low d-high minus high d-low divided by low squared)

$$\lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}$$

add and subtract a specific term in the middle...

$$\lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h}$$

factor each pair...

$$\lim_{h \rightarrow 0} \frac{g(x)(f(x+h) - f(x)) + f(x)(g(x) - g(x+h))}{hg(x)g(x+h)}$$

$$\lim_{h \rightarrow 0} \frac{g(x)(f(x+h) - f(x)) - f(x)(g(x+h) - g(x))}{hg(x)g(x+h)}$$

$$\lim_{h \rightarrow 0} \frac{g(x) \frac{(f(x+h) - f(x))}{h} - f(x) \frac{(g(x+h) - g(x))}{h}}{g(x)g(x+h)}$$

$$\frac{g(x) \lim_{h \rightarrow 0} \frac{(f(x+h) - f(x))}{h} - f(x) \lim_{h \rightarrow 0} \frac{(g(x+h) - g(x))}{h}}{\lim_{h \rightarrow 0} g(x)g(x+h)}$$

$$\frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$f'(x) = \frac{(x^2+3x)(10x^4-3x^2) - (2x^5-x^3)(2x+3)}{(x^2+3x)^2}$$

A quotient rule example

$$f(x) = \frac{2x^5 - x^3}{x^2 + 3x}$$

$$= \frac{10x^6 + 30x^5 - 3x^4 - 9x^3 - 4x^6 + 2x^4 - 6x^5 + 3x^3}{(x^2+3x)^2}$$

$$= \frac{6x^6 + 24x^5 - x^4 - 6x^3}{(x^2+3x)^2}$$

Shortcuts

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[a^x] = (\ln a)a^x$$

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

Properties

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}[f(x)]$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Examples

Differentiate: $y = \sqrt{x}(x^2 - 2x)$

$$y' = \sqrt{x}(2x-2) + (x^2-2x)\left(\frac{1}{2}x^{-1/2}\right)$$

$$= \frac{2x\sqrt{x} - 2\sqrt{x}}{1} + \frac{x(x-2)}{2\sqrt{x}}$$

$$= \frac{4x^2 - 2x + x^2 - 2x}{2\sqrt{x}}$$

$$= \frac{5x^2 - 4x}{2\sqrt{x}}$$

$$= \boxed{\frac{x(5x-4)}{2\sqrt{x}}}$$

$$y = \frac{x^2 + 4x + 3}{\sqrt{x}} = \frac{x^2 + 4x + 3}{x^{1/2}}$$

$$y = x^{3/2} + 4x^{1/2} + 3x^{-1/2}$$

$$y' = \left[\frac{3}{2}x^{1/2} + 2x^{-1/2} - \frac{3}{2}x^{-3/2} \right]$$

factor out $\frac{1}{2}x^{-3/2}$

$$y' = \frac{1}{2}x^{-3/2}(3x^2 + 4x - 3)$$

try simplifying 1st

$$y = \frac{u^2 - u - 2}{u+1} = \frac{(u-2)(u+1)}{(u+1)} = u-2$$

$y' = 1$

otherwise...

$$y' = \frac{(u+1)(2u-1) - (u^2-u-2)(1)}{(u+1)^2}$$

$$= \frac{2u^2 + 2u - u - 1 - u^2 + u + 2}{(u+1)^2}$$

$$= \frac{u^2 + 2u + 1}{(u+1)^2}$$

$$= \frac{(u+1)(u+1)}{(u+1)^2}$$

$$= \boxed{1}$$

$$y = \frac{\sqrt{x}-1}{\sqrt{x}+1} = \frac{x^{1/2}-1}{x^{1/2}+1}$$

$$y' = \frac{(x^{1/2}+1)\left(\frac{1}{2}x^{-1/2}\right) - (x^{1/2}-1)\left(\frac{1}{2}x^{-1/2}\right)}{(\sqrt{x}+1)^2}$$

$$= \frac{\frac{1}{2}x^{-1/2}(2)}{(\sqrt{x}+1)^2}$$

$$= \boxed{\frac{1}{\sqrt{x}(\sqrt{x}+1)^2}}$$

Examples

Find the equation of the tangent line to the curve at the given point.

$y = 2xe^x, (0,0)$

$m = y' = (2x)e^x + e^x(2) \Big|_{x=0} = 0 + e^2(2) = 2$

$(y-0) = 2(x-0)$

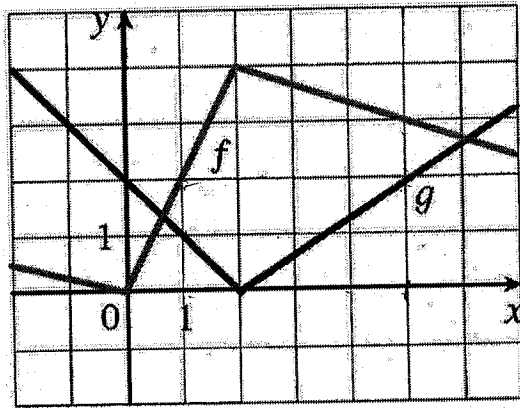
$y = 2x$

If f and g are the functions whose graphs are show, let

$u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.

a) Find $u'(1)$.

b) Find $v'(5)$.



(a) $u'(x) = f(x)g'(x) + g(x)f'(x)$

$u'(1) = f(1)g'(1) + g(1)f'(1)$

$= (2)(-1) + (1)(2)$

$= -2 + 2$

$= 0$

(b) $v'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$v'(5) = \frac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2}$

$= \frac{(2)(-\frac{1}{3}) - (3)(\frac{2}{3})}{(2)^2}$

$= \frac{-\frac{2}{3} - 2}{4} = \boxed{-\frac{2}{3}}$

If f is a differentiable function, find an expression for the derivative of each of the following functions.

$y = \frac{x^2}{f(x)}$

$y' = \frac{f(x)(2x) - x^2(f'(x))}{[f(x)]^2}$

$y = \frac{1+xf(x)}{\sqrt{x}} = \frac{1+xf(x)}{x^{1/2}}$

$y' = \frac{\sqrt{x}(0 + xf'(x) + f(x)(1)) - (1+xf(x))(\frac{1}{2}x^{-1/2})}{(\sqrt{x})^2}$

$= \frac{\sqrt{x}(xf'(x) + f(x)) - \frac{1}{2\sqrt{x}} - \frac{xf(x)}{2\sqrt{x}}}{(\sqrt{x})^2}$

$= \frac{2x(xf'(x) + f(x)) + 1 + xf(x)}{2x\sqrt{x}}$

Unit 3-3: More Trig Derivatives, Higher-order Derivatives
Larsen: 2.3 (Stewart: 3.3)

Proofs (derivations) of selected derivative shortcuts (rules)

What is the derivative of tangent? $f(x) = \tan(x)$

proof

$$\begin{aligned} & \frac{d}{dx}[\tan(x)] \\ &= \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] \\ &= \frac{\cos(x)\frac{d}{dx}[\sin(x)] - \sin(x)\frac{d}{dx}[\cos(x)]}{[\cos(x)]^2} \\ &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{[\cos(x)]^2} \\ &= \frac{\cos^2(x) + \sin^2(x)}{[\cos(x)]^2} \\ &= \frac{1}{[\cos(x)]^2} \\ &= \sec^2(x) \end{aligned}$$

shortcut

$$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

What is the derivative of cotangent? $f(x) = \cot(x)$

proof

$$\begin{aligned} & \frac{d}{dx}[\cot(x)] \\ &= \frac{d}{dx}\left[\frac{\cos(x)}{\sin(x)}\right] \\ &= \frac{\sin(x)\frac{d}{dx}[\cos(x)] - \cos(x)\frac{d}{dx}[\sin(x)]}{[\sin(x)]^2} \\ &= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{[\sin(x)]^2} \\ &= \frac{-(\sin^2(x) + \cos^2(x))}{[\sin(x)]^2} \\ &= \frac{-1}{[\sin(x)]^2} \\ &= -\csc^2(x) \end{aligned}$$

shortcut

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$$

Proofs (derivations) of selected derivative shortcuts (rules)

What is the derivative of secant? $f(x) = \sec(x)$

proof

$$\begin{aligned} & \frac{d}{dx} [\sec(x)] \\ &= \frac{d}{dx} \left[\frac{1}{\cos(x)} \right] \\ &= \frac{\cos(x) \frac{d}{dx} [1] - 1 \frac{d}{dx} [\cos(x)]}{[\cos(x)]^2} \\ &= \frac{\cos(x)(0) - 1(-\sin(x))}{[\cos(x)]^2} \\ &= \frac{\sin(x)}{[\cos(x)]^2} \\ &= \frac{1}{\cos(x)} \frac{\sin(x)}{\cos(x)} \\ &= \sec(x) \tan(x) \end{aligned}$$

shortcut

$$\frac{d}{dx} [\sec(x)] = \sec(x) \tan(x)$$

What is the derivative of cosecant? $f(x) = \csc(x)$

proof

$$\begin{aligned} & \frac{d}{dx} [\csc(x)] \\ &= \frac{d}{dx} \left[\frac{1}{\sin(x)} \right] \\ &= \frac{\sin(x) \frac{d}{dx} [1] - 1 \frac{d}{dx} [\sin(x)]}{[\sin(x)]^2} \\ &= \frac{\sin(x)(0) - 1(\cos(x))}{[\sin(x)]^2} \\ &= \frac{-\cos(x)}{[\sin(x)]^2} \\ &= -\frac{1}{\csc(x)} \frac{\cos(x)}{\sin(x)} \\ &= -\csc(x) \cot(x) \end{aligned}$$

shortcut

$$\frac{d}{dx} [\csc(x)] = -\csc(x) \cot(x)$$

Shortcuts

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[a^x] = (\ln a)a^x$$

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

$$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

Properties

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}[f(x)]$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

should also know...

$$\left(\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{and} \quad \sin^2(x) + \cos^2(x) = 1\right)$$

Higher-order Derivatives

Because the derivative of a function produces another function, you can then take the derivative of that derivative function. This is called a **second-derivative**, and this process can repeat to create third or other **higher-order derivatives**.

Notations

original function $y = f(x)$

first derivative $y', f'(x), \frac{dy}{dx}, \frac{d}{dx}[f(x)]$

second derivative $y'', f''(x), \frac{d^2y}{dx^2}, \frac{d^2}{dx^2}[f(x)]$

third derivative $y''', f'''(x), \frac{d^3y}{dx^3}, \frac{d^3}{dx^3}[f(x)]$

fourth derivative $y^{(4)}, f^{(4)}(x), \frac{d^4y}{dx^4}, \frac{d^4}{dx^4}[f(x)]$

n-th derivative $y^{(n)}, f^{(n)}(x), \frac{d^ny}{dx^n}, \frac{d^n}{dx^n}[f(x)]$

Higher-order Derivatives if original function is position vs. time

If the original function is a position function (position as a function of time), then the first and higher-order derivatives have specific meanings and names...

original function $s(t)$ position at a specific time "position"

first derivative $v(t) = s'(t)$ how fast position is changing at a specific time "velocity"

second derivative $a(t) = v'(t) = s''(t)$ how fast velocity is changing at a specific time "acceleration"

third derivative although we don't see it often, the derivative of acceleration, how fast acceleration changes at a specific time (the 3rd derivative of position) is called "jerk".

Examples

Differentiate: $y = \frac{\sin x}{1 + \cos x}$

$$y' = \frac{(1 + \cos x)(\cos x) - \sin x(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$$

$$= \frac{\cos x + 1}{(1 + \cos x)^2}$$

$$= \boxed{\frac{1}{1 + \cos x}}$$

$$\begin{aligned} \text{find } \frac{d^2y}{dx^2} &= \frac{(1 + \cos x)(0) - (1)(-\sin x)}{(1 + \cos x)^2} \\ &= \frac{\sin x}{(1 + \cos x)^2} \end{aligned}$$

Find the equation of the tangent line to the curve

$$y = x + \cos x \text{ at } (0, 1)$$

$$m = y' = 1 - \sin x \Big|_{x=0} = 1 - \sin(0) = 1 - 0 = 1$$

$$\boxed{y - 1 = 1(x - 0)}$$

$$y - 1 = x$$

$$\boxed{y = x + 1}$$

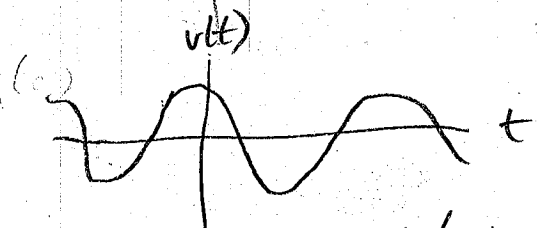
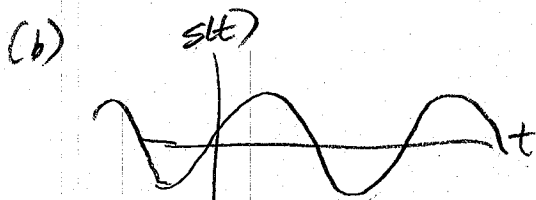
Examples

An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2\cos t + 3\sin t$, $t \geq 0$, where s is measured in centimeters and t in seconds. (We take the positive direction to be downward.)

- Find the velocity at time t .
- Graph the velocity and position functions.
- When does the mass pass through the equilibrium position for the first time?
- How far from its equilibrium position does the mass travel?
- When is the speed the greatest?

$$s(t) = 2\cos t + 3\sin t$$

(a) $v(t) = s'(t) = -2\sin t + 3\cos t$



(interestingly, $2\cos t + 3\sin t$ looks like a single sinusoid (as long as frequencies are the same) (with a phase shift) \Rightarrow see if you can show this w/ trig identities \Leftarrow (see page 100)

(c) must use graph/calculator - 1st x-intercept: $t = 2.553 \text{ sec}$

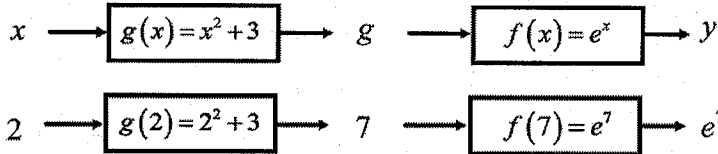
(d) max position is $s(1.9828) = 3.606 \text{ cm}$

(e) v_{\max} @ $t = 2.554 \text{ sec}$ & $t = 5.695 \text{ sec}$
(largest -) (largest +)

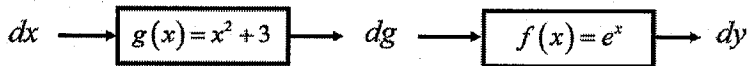
The Chain Rule - Derivatives of a Composition of Functions

When a function can be defined as a composition of two other functions (one 'inside' the other), we have to use the **Chain Rule** when computing the derivative.

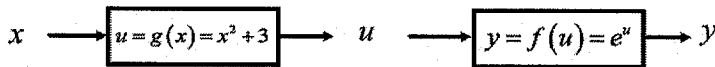
If we have a composite function... $y = f(g(x)) = e^{x^2+3}$ $f(x) = e^x$
 $g(x) = x^2 + 3$
 and view the functions as 'machines'...



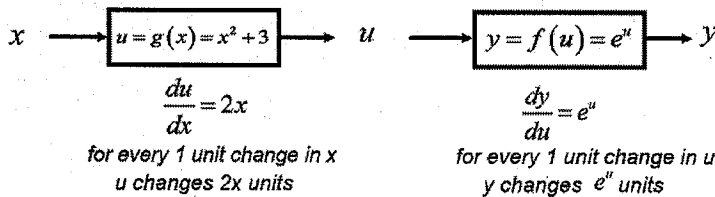
If we imagine making a small change in x, this change must propagate through the functions...



You could say that the variable in the middle is some new variable, and rewrite like this:



Then the two derivatives of the functions have the following meaning...



Which means if we want to know how much y changes for each unit change in x we need to propagate this change by multiplying the two derivatives:

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

The Chain Rule is almost like unit analysis / unit conversion.

Derivatives are 'slopes' which are like ratios to show how change in x converts to change in y. Imagine if you start with 1 mile and want to know how many inches that is equivalent to. Each unit conversion is like a derivative:

There are 5280 feet per mile, so $\frac{df}{dm} = \frac{5280 \text{ ft}}{1 \text{ mi}}$ There are 12 inches per foot, so $\frac{di}{df} = \frac{12 \text{ in}}{1 \text{ ft}}$

To find how many inches there are per mile...

$$\frac{di}{dm} = \frac{12 \text{ in}}{1 \text{ ft}} \frac{5280 \text{ ft}}{1 \text{ mi}}$$

$$\frac{di}{dm} = \frac{di}{df} \frac{df}{dm}$$

The Chain Rule - Derivatives of a Composition of Functions

So the **Chain Rule** says that if you have a composition of functions, you first take the derivative of the 'outside' function, but then you multiply by the derivative of the 'inside' function:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

derivative of the outside function (with inside just copied as is)

derivative of the inside function

Chain Rule examples

Find y' : $y = (x^3 + 3)^{14}$

Identify the outside and inside functions...

$$y = f(g(x))$$

$$f(x) = x^{14}$$

$$g(x) = x^3 + 3$$

Take the derivative of the outside function (with the inside function just copied)...

$$\frac{dy}{du} = 14(x^3 + 3)^{13}$$

...but then multiply by the derivative of the inside function:

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 14(x^3 + 3)^{13} \cdot (3x^2)$$

This applies to any number of nested functions

The reason this is called the chain rule is that if you have many nested functions, you always start with the outer most function, take its derivative, then multiply by ('chain on') the derivative of the inside, repeating until you finally stop when the inside function is an x .

$$y = \ln(\sin(e^{x^5}))$$

$$y' = \frac{d}{dx}[\ln(\sin(e^{x^5}))]$$

$$= \frac{1}{\sin(e^{x^5})} \frac{d}{dx}[\sin(e^{x^5})]$$

$$= \frac{1}{\sin(e^{x^5})} \cos(e^{x^5}) \frac{d}{dx}[e^{x^5}]$$

$$= \frac{1}{\sin(e^{x^5})} \cos(e^{x^5}) e^{x^5} \frac{d}{dx}[x^5]$$

$$= \frac{1}{\sin(e^{x^5})} \cos(e^{x^5}) e^{x^5} (5x^4) \frac{d}{dx}[x]$$

$$= \cot(e^{x^5}) e^{x^5} 5x^4$$

= 1 (step)

Shortcut for derivative of logarithmic function

We can use the chain rule to develop a shortcut for derivative of a log function.

$$y = \ln(x)$$

$$e^y = x$$

taking derivative of each side

(using chain rule on left)

$$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$$

$$e^y \frac{d}{dx}[y] = 1$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{x \ln(b)}$$

The extended (general) derivative shortcuts

With the chain rule we can make more general (extended) forms of each of our derivative shortcuts:

$$\frac{d}{dx}[x^n] = nx^{n-1} \Rightarrow \frac{d}{dx}[(\text{something})^n] = n(\text{something})^{n-1} \cdot \frac{d(\text{something})}{dx}$$

$$\text{ex) } y = (x^3 + 4x)^5 \quad y' =$$

$$\frac{d}{dx}[\sin(x)] = \cos(x) \Rightarrow \frac{d}{dx}[\sin(\text{something})] = \cos(\text{something}) \cdot \frac{d(\text{something})}{dx}$$

$$\text{ex) } y = \sin(3x^2 + x) \quad y' =$$

Chain Rule examples

$$y = \sin^3(x^3 + 7x) = (\sin(x^3 + 7x))^3$$

$$y' = 3(\sin(x^3 + 7x))^2 \cos(x^3 + 7x) (3x^2 + 7)$$

$$f(t) = \sqrt[3]{1 + \tan t} = (1 + \tan t)^{1/3}$$

$$f'(t) = \frac{1}{3}(1 + \tan t)^{-2/3} (\sec^2 t)$$

$$= \frac{\sec^2 t}{3 \sqrt[3]{(1 + \tan t)^2}}$$

Chain Rule examples

$$F(y) = \left(\frac{y-6}{y+7} \right)^3$$

$$F'(y) = 3 \left(\frac{y-6}{y+7} \right)^2 \left[\frac{(y+7)(1) - (y-6)(1)}{(y+7)^2} \right]$$

$$= 3 \frac{(y-6)^2}{(y+7)^2} \frac{y+7-y+6}{(y+7)^2}$$

$$= \frac{3(y-6)^2 \cdot 13}{(y+7)^4}$$

$$= \boxed{\frac{39(y-6)^2}{(y+7)^4}}$$

$$y = 5^{-1/x} \quad -x^{-1}$$

$$y' = (\ln 5) 5^{-1/x} \cdot \frac{d}{dx} \left[-\frac{1}{x} \right]$$

$$= (\ln 5) 5^{-1/x} (-x^{-2})$$

$$= \boxed{-\frac{(\ln 5) 5^{-1/x}}{x^2}}$$

$y = \sin(\sin(\sin x))$ (careful!)

$$y' = \cos(\sin(\sin x)) \cdot \frac{d}{dx} [\sin(\sin x)]$$

$$\cos(\sin x) \cdot \frac{d}{dx} [\sin x]$$

cos x

$$\boxed{y' = \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x}$$

$$y = 2^{3^{x^2}}$$

$$y' = (\ln 2) 2^{3^{x^2}} \cdot \frac{d}{dx} [3^{x^2}]$$

$$(\ln 3) 3^{x^2} \cdot \frac{d}{dx} [x^2]$$

2x

$$\boxed{y' = (\ln 2) 2^{3^{x^2}} (\ln 3) 3^{x^2} 2x}$$

Chain Rule examples

Find an equation of the tangent line to

$y = \sin x + \cos 2x$ at the point $\left(\frac{\pi}{6}, 1\right)$



$$y' = \cos x + (-\sin(2x))(2)$$

$$y' \Big|_{x=\frac{\pi}{6}} = \cos \frac{\pi}{6} - 2 \sin \left(\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2} - 2 \cdot \frac{\sqrt{3}}{2}$$

$$m = -\frac{\sqrt{3}}{2}$$

$$(y-1) = -\frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6}\right)$$

57. If f and g are the functions whose graphs are shown, let

$u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$.

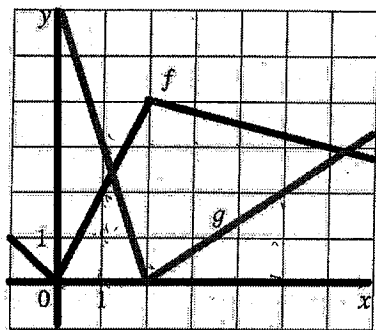
Find each derivative, if it exists.

If it does not exist, explain why.

a) $u'(1)$

b) $v'(1)$

c) $w'(1)$



(a) $u'(x) = f'(g(x)) \cdot g'(x)$

$$u'(1) = f'(g(1)) \cdot g'(1)$$

$$= f'(3) \cdot (-3)$$

$$= \left(-\frac{1}{4}\right) \cdot (-3)$$

$$= \boxed{\frac{3}{4}}$$

(b) $v'(x) = g'(f(x)) \cdot f'(x)$

$$v'(1) = g'(f(1)) \cdot f'(1)$$

$$= g'(2) \cdot (2)$$

$$= \text{DNE} \cdot (2)$$

$$= \boxed{\text{DNE}}$$

(c) $w'(x) = g'(g(x)) \cdot g'(x)$

$$w'(1) = g'(g(1)) \cdot g'(1)$$

$$= g'(3) \cdot (-3)$$

$$= \left(\frac{2}{3}\right) \cdot (-3)$$

$$= \boxed{-2}$$

Shortcuts

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[a^x] = (\ln a)a^x$$

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{x \ln(b)}$$

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

$$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

Properties

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}[f(x)]$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

should also know...

$$\left(\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{and} \quad \sin^2(x) + \cos^2(x) = 1 \right)$$

Unit 3-5: Implicit and Logarithmic Differentiation
Larsen: 2.5 (Stewart: 3.5)

Differentiation Procedures: Implicit Differentiation and Logarithmic Differentiation

In some cases, we can employ a procedure which will help us to more easily compute the derivative of a function in the normal way (by solving for y , then employing shortcuts and properties). There are two such procedures...

Implicit Differentiation: Is useful when we have an equation which is not easy to solve for y in terms of x . We use this procedure to take the derivative of each term where it is in the original equation, then solve for the derivative at the end of the procedure.

Implicit Differentiation is useful for something like: $x^3 + xy + y^3 = 5$

Logarithmic Differentiation: Is useful when we have an equation which employs exponential expressions or many factors multiplied/divided. By taking the logarithm of both sides of the equation, we can use log properties to simplify the expression before taking the derivative.

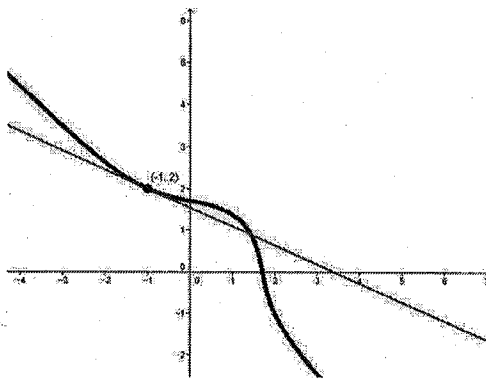
Logarithmic Differentiation is useful for something like: $y = \frac{(x^2 + 3x)^4 (5x^3)}{(2x^4 - 4x)^2}$

...or: $y = x^{5x^3+2x}$

Implicit Differentiation

$$x^3 + xy + y^3 = 5$$

How could you find the slope of the line tangent to this curve at the point $(-1, 2)$?



Some equations can be solved for y . These are called 'explicit equations'.

But some equations are either difficult or impossible to solve for y . These are called 'implicit equations'.

We can still differentiate (find the derivative) of an implicit equation, but we need a procedure which doesn't require solving for y first.

This procedure is called **implicit differentiation**.

Implicit Differentiation Procedure

1) Take the derivative of each term in place in the original equation. For each term...

- If the variable for the term is x then take the derivative as normal.

$$3x^4 \Rightarrow 12x^3$$

- If the variable for the term is y then use the chain rule by multiplying by $\frac{dy}{dx}$

$$3y^4 \Rightarrow 3 \frac{d}{dx}[y^4] \Rightarrow 3(4y^3) \frac{d}{dx}[y] \Rightarrow 3(4y^3) \frac{dy}{dx}$$

- If the term contains both x and y , use the product rule to split the term first.

2) Solve algebraically for $\frac{dy}{dx}$

Implicit Differentiation example

DL

$$x^3 + xy + y^3 = 5$$

Find the slope of the line tangent to this curve at the point (-1, 2)?

$$\frac{d}{dx}[x^3] + \frac{d}{dx}[xy] + \frac{d}{dx}[y^3] = \frac{d}{dx}[5]$$

$$\frac{d}{dx}[x^3] + x \frac{d}{dx}[y] + y \frac{d}{dx}[x] + \frac{d}{dx}[y^3] = \frac{d}{dx}[5]$$

$$3x^2 + x(1 \frac{dy}{dx}) + y(1) + (3y^2 \frac{dy}{dx}) = 0$$

$$\frac{dy}{dx}(x + 3y^2) + 3x^2 + y = 0$$

$$\frac{dy}{dx}(x + 3y^2) = -3x^2 - y$$

$$\frac{dy}{dx} = \frac{-3x^2 - y}{x + 3y^2} \quad \text{so } m = \frac{-3(-1)^2 - (2)}{(-1) + 3(2)^2} = \boxed{\frac{-5}{11}}$$

Key idea:
don't solve for y 1st,
just take derivatives
term by term and
solve for $\frac{dy}{dx}$ after

Logarithmic Differentiation Procedure

- 1) Take the natural logarithm of both sides of the equation.
- 2) Use log properties to simplify the equation.
- 3) Take the derivative of both sides of the equation.
- 4) Solve algebraically for $\frac{dy}{dx}$

Logarithmic Differentiation Examples

$$y = \frac{(x^2 + 3x)^4 (5x^3)}{(2x^4 - 4x)^2}$$

(3) ✓

$$\ln y = \ln \frac{(x^2 + 3x)^4 (5x^3)}{(2x^4 - 4x)^2}$$

$$\ln y = 4 \ln(x^2 + 3x) + \ln(5x^3) - 2 \ln(2x^4 - 4x)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[4 \ln(x^2 + 3x)] + \frac{d}{dx}[\ln(5x^3)] - \frac{d}{dx}[2 \ln(2x^4 - 4x)]$$

$$\frac{1}{y} \frac{dy}{dx} = 4 \frac{1}{x^2 + 3x} (2x + 3) + \frac{1}{5x^3} (15x^2) - 2 \frac{1}{2x^4 - 4x} (8x^3 - 4)$$

$$\frac{1}{y} \frac{dy}{dx} = \left[\frac{4(2x+3)}{x^2+3x} + \frac{3}{x} - \frac{2(8x^3-4)}{2x^4-4x} \right]$$

$$\frac{dy}{dx} = \left[\frac{4(2x+3)}{x^2+3x} + \frac{3}{x} - \frac{2(8x^3-4)}{2x^4-4x} \right] y$$

$$\boxed{\frac{dy}{dx} = \left[\frac{4(2x+3)}{x^2+3x} + \frac{3}{x} - \frac{2(8x^3-4)}{2x^4-4x} \right] \frac{(x^2+3x)^4 (5x^3)}{(2x^4-4x)^2}}$$

(2) b
 $y = x^{5x^3+2x}$

$$\ln y = \ln x^{5x^3+2x}$$

$$\ln y = (5x^3+2x) \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = (5x^3+2x) \frac{1}{x} + \ln x (15x^2+2)$$

$$\boxed{\frac{dy}{dx} = [5x^2+2 + \ln x (15x^2+2)] x^{5x^3+2x}}$$

More Examples

Find $\frac{d^2y}{dx^2}$ $x^2 + y^2 = 4$

$$2x + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y} \quad \text{--- substitute in}$$

quotient rule...

$$\frac{d^2y}{dx^2} = \frac{(y)(-1) - (-x)(1 \frac{dy}{dx})}{y^2}$$

$$= \frac{-y + x(\frac{-x}{y})}{y^2} \quad \frac{(y)}{(y)}$$

$$= \frac{-y^2 - x^2}{y^3} = \frac{-(x^2 + y^2)}{y^3}$$

$$= \boxed{\frac{-4}{y^3}}$$

$$g(x) = \ln \frac{a-x}{a+x}$$

$$g(x) = \ln(a-x) - \ln(a+x)$$

$$g'(x) = \frac{1}{a-x}(-1) - \frac{1}{a+x}(1)$$

$$g'(x) = -1 \left[\frac{1}{a-x} + \frac{1}{a+x} \right]$$

$$= -1 \left[\frac{a+x}{(a-x)(a+x)} + \frac{a-x}{(a-x)(a+x)} \right]$$

$$= -1 \left[\frac{a+x+a-x}{(a-x)(a+x)} \right]$$

$$= \boxed{\frac{-2a}{(a-x)(a+x)}}$$

$$(1+x^2y^2)^{1/2}$$

$$\sqrt{1+x^2y^2} = 2xy \quad \text{square both sides } |^{2x}$$

$$\frac{1}{2} (1+x^2y^2)^{-1/2} = 2xy^2$$

$$0^{1/2} + x^2(2y \frac{dy}{dx}) + y^2(2x) = 4x^2(2y \frac{dy}{dx}) + y^2(8x)$$

$$\frac{dy}{dx} (2x^2y - 8x^2y) = 8xy^2 - 2xy^2$$

$$\frac{dy}{dx} (-6x^2y) = 6xy^2$$

$$\frac{dy}{dx} = \frac{6xy^2}{-6x^2y} = \boxed{\frac{-y}{x}}$$

If $[g(x)]^2 + 12x = x^2g(x)$

and $g(4) = 12$, find $g'(4)$.

$$[g(x)]^2 + 12x = x^2g(x)$$

$$2g(x)g'(x) + 12 = x^2g'(x) + g(x)(2x)$$

$$g'(x)(2g(x) - x^2) = 2xg(x) - 12$$

$$g'(x) = \frac{2xg(x) - 12}{2g(x) - x^2} \quad g(4) = 12$$

$$\text{so } g'(4) = \frac{2(4)(12) - 12}{2(12) - (4)^2} = \frac{84}{8} = \boxed{\frac{21}{2}}$$

$$y = \sqrt{x} e^{x^2} (x^2 + 1)^{10}$$

$$\ln y = \ln(x^{1/2} e^{x^2} (x^2 + 1)^{10})$$

$$\ln y = \frac{1}{2} \ln x + \ln(e^{x^2}) + 10 \ln(x^2 + 1)$$

$$\ln y = \frac{1}{2} \ln x + x^2 + 10 \ln(x^2 + 1)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \frac{1}{x} + 2x + 10 \frac{1}{x^2 + 1} (2x)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2x} + 2x + \frac{20x}{x^2 + 1}$$

$$\frac{dy}{dx} = \left[\frac{1}{2x} + 2x + \frac{20x}{x^2 + 1} \right] \sqrt{x} e^{x^2} (x^2 + 1)^{10}$$

$$y = x^{1/x} \quad \ln y = \frac{1}{x} \ln x = x^{-1} \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \frac{1}{x} + \ln x (-x^{-2})$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x^2} - \frac{\ln x}{x^2}$$

$$\frac{dy}{dx} = \left[\frac{1 - \ln x}{x^2} \right] x^{1/x}$$

$$y = \ln(e^{-x} + x e^{-x})$$

$$y = \ln(e^{-x}(1+x))$$

$$y = \ln e^{-x} + \ln(1+x)$$

$$y = -x + \ln(1+x)$$

$$\frac{dy}{dx} = -1 + \frac{1}{x+1} (1)$$

$$\frac{dy}{dx} = -1 + \frac{1}{x+1}$$

$$= \frac{-(x+1) + 1}{(x+1)(x+1)}$$

$$= \frac{-x-1+1}{x+1}$$

$$= \frac{-x}{x+1}$$

Find dy/dx : $x = 3y^2 + 2y$ find $\frac{dy}{dy}$ 1st...

$$\frac{dx}{dy} = 6y + 2$$

$$\frac{dy}{dx} = \frac{1}{6y+2}$$

$$y = \arcsin x$$

$$\sin y = x = \frac{\text{opp}}{\text{hyp}}$$

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$

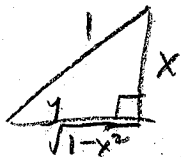
$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

$$(\text{since } y = \sin^{-1}(x))$$

$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$$



$$\sin y = \frac{x}{1} = \frac{\text{opp}}{\text{hyp}}$$

$$\cos y = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{1-x^2}}{1}$$

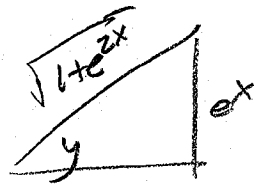
$$y = \tan^{-1}(e^x) \quad \tan y = e^x = \frac{\text{opp}}{\text{adj}}$$

$$\frac{d}{dx}[\tan y] = \frac{d}{dx}[e^x]$$

$$\sec^2 y \frac{dy}{dx} = e^x$$

$$\frac{dy}{dx} = \frac{e^x}{\sec^2 y}$$

$$\frac{dy}{dx} = e^x \cos^2 y$$



$$\cos y = \frac{1}{\sqrt{1+e^{2x}}}$$

$$\frac{dy}{dx} = e^x \frac{1}{1+e^{2x}}$$

$$\boxed{\frac{dy}{dx} = \frac{e^x}{1+e^{2x}}}$$

Unit 3-6: Derivatives of Inverse Functions

Larsen: 2.6 (Stewart: 3.6)

Extra topic: hyperbolic functions (Stewart 3.11)

Two Theorems about Functions and their Inverses

There are two theorems which are sometimes useful about functions and their inverse functions:

Let f be a function whose domain is an interval I .

If f has an inverse function, then the following statements are true:

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is differentiable on an interval containing c and $f'(c) \neq 0$ then f^{-1} is differentiable at $f(c)$.

Let f be a function that is differentiable on an interval I .

If f has an inverse function g , then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, f'(g(x)) \neq 0$$

Deriving this if you need it on the AP Exam...

This theorem doesn't seem intuitive, but is easy to derive if you need it using the Chain Rule:

If $f(x)$ and $g(x)$ are inverses of each other, then: $f(g(x)) = x$

Differentiate on both sides (Chain Rule on left): $\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[x]$

$$f'(g(x))g'(x) = 1$$

Solve for $g'(x)$: $g'(x) = \frac{1}{f'(g(x))}$

An example of this theorem's use from the 2003 AP Calculus AB exam...

If $f(x) = x^3 + x$ and $g(x) = f^{-1}(x)$ and $g(2) = 1$

what is the value of $g'(2)$?

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ g'(2) &= \frac{1}{f'(g(2))} & f'(x) &= 3x^2 + 1 \\ &= \frac{1}{3(g(2))^2 + 1} \\ &= \frac{1}{3(1)^2 + 1} \\ &= \boxed{\frac{1}{4}} \end{aligned}$$

For inverse trig functions, rather than use the theorem, it is easier to derive with implicit differentiation:

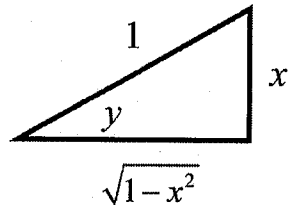
Find y' , if $y = \arcsin(x)$.

$$\sin(y) = x$$

$$\frac{d}{dx}[\sin(y)] = \frac{d}{dx}[x]$$

$$\cos(y) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\left(\frac{\sqrt{1-x^2}}{1}\right)} = \boxed{\frac{1}{\sqrt{1-x^2}}}$$



The derivative of arcsin with an arbitrary argument

If the argument of an arcsin function is not just x , how would that affect the derivative?

Find y' , if $y = \arcsin(3x^2)$.

If we start with our previous result... $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$

...and modify to allow the argument of the arcsin to be a function of x : $u = h(x)$

Then we can use the Chain Rule to find: $\frac{d}{dx}[\arcsin u]$

$$\frac{d}{dx}[\arcsin u] = \frac{1}{\sqrt{1-u^2}} \frac{d}{dx}[u(x)] = \frac{1}{\sqrt{1-u^2}} u'$$

$$\boxed{\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}}$$

Find y' , if $y = \arcsin(3x^2)$. So, in this case, $u = 3x^2$

$$\frac{d}{dx}[\arcsin(3x^2)] = \frac{6x}{\sqrt{1-(3x^2)^2}} = \frac{6x}{\sqrt{1-9x^4}}$$

The Derivatives of the Inverse Trigonometric Functions

$$\begin{aligned}\frac{d}{dx}[\arcsin u] &= \frac{u'}{\sqrt{1-u^2}} & \frac{d}{dx}[\arccos u] &= \frac{-u'}{\sqrt{1-u^2}} \\ \frac{d}{dx}[\arctan u] &= \frac{u'}{1+u^2} & \frac{d}{dx}[\operatorname{arccot} u] &= \frac{-u'}{1+u^2} \\ \frac{d}{dx}[\operatorname{arcsec} u] &= \frac{u'}{|u|\sqrt{u^2-1}} & \frac{d}{dx}[\operatorname{arccsc} u] &= \frac{-u'}{|u|\sqrt{u^2-1}}\end{aligned}$$

Another example

Find y' for $y = \ln(t^2 + 4) - \frac{1}{2} \arctan\left(\frac{t}{2}\right)$

$$y' = \frac{1}{t^2+4} (2t) - \frac{1}{2} \frac{1}{1+(\frac{t}{2})^2} \left(\frac{1}{2}\right)$$

$$y' = \frac{2t}{t^2+4} - \frac{1}{4} \frac{1}{1+\frac{t^2}{4}}$$

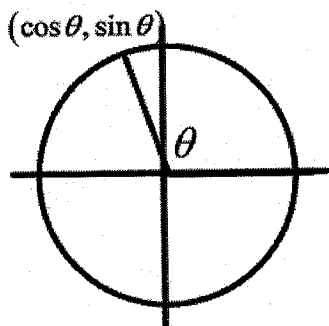
$$y' = \frac{2t}{t^2+4} - \frac{1}{t^2+4}$$

$$y' = \frac{2t-1}{t^2+4}$$

Extra topic: Hyperbolic Functions

This topic is not on the AP Calc BC exam, but is sometimes included in Calc1/Calc2 courses, so I think you should see it. You may encounter this later if you major in physics or engineering in college.

The standard trigonometric functions sine and cosine use a unit circle to convert an input interpreted as an angle into x and y coordinates:

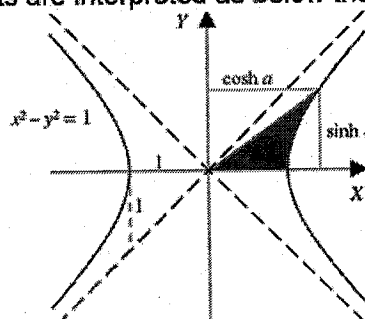


$$\cos(\theta) = x - \text{coordinate}$$

$$\sin(\theta) = y - \text{coordinate}$$

Then tan, cot, sec, and csc are defined from the sin and cos.

The hyperbolic trigonometric functions hyperbolic sine and hyperbolic cosine similarly use a unit hyperbola but the input is interpreted as the 'hyperbolic angle' which is twice the 'hyperbolic sector area' (shown in red here). Negative arguments are interpreted as below the x-axis.



$$\cosh(\theta) = x - \text{coordinate}$$

$$\sinh(\theta) = y - \text{coordinate}$$

Then tanh, coth, sech, and csch are defined from the sinh and cosh.

Interestingly, it turns out that you can also write the hyperbolic trigonometric functions as combinations of exponential functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

(You won't encounter these further in this course, but I wanted you to at least be exposed to what they are in case you encounter them in the future :)

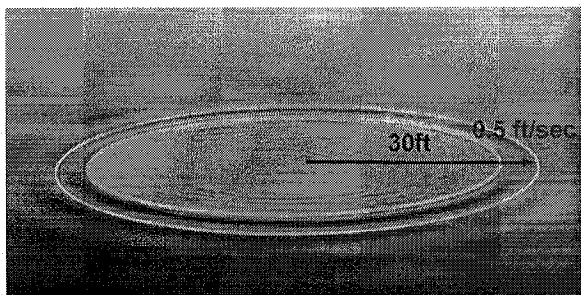
Unit 3-7: Related Rates

Larsen: 2.7 (Stewart: 3.7)

In Related Rates problems, two different variables are related and both vary with time

In natural, social, and behavioral sciences, two different quantities represented by variables are related to each other, but also both vary over time. If one quantity is changing over time at a known rate, we can determine the rate of change of the other quantity (at a specific point in time). These problems are referred to as **related rate problems**.

Ex) A child throws a stone into a still pond, causing a circular ripple to spread. If the radius of the circle increases at a constant rate of 0.5 feet per second, how fast is the area of the ripple increasing when the radius of the ripple is 30 feet?



Solving Related Rates Problems

Step 1: Draw a picture if possible.

Step 2: Identify what is varying and assign variables.

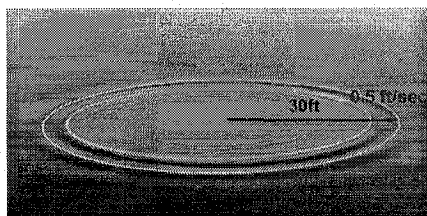
Step 3: Identify and interpret rates of change as derivatives.

Step 4: Write an equation which relates the variable to each other.

Step 5: Take the derivative with respect to time (implicitly).

Step 6: Substitute numerical values for the variables and the derivatives at the specified point in time, and solve for the unknown rate.

A child throws a stone into a still pond, causing a circular ripple to spread. If the radius of the circle increases at a constant rate of 0.5 feet per second, how fast is the area of the ripple increasing when the radius of the ripple is 30 feet?



$$\begin{array}{l} \text{A} \rightarrow r \\ (30) \quad 15 \text{ ft/s} \\ \frac{dA}{dt} = ? \quad \frac{dr}{dt} = 1.5 \text{ ft/s} \end{array}$$

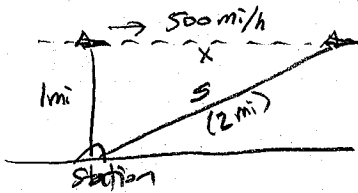
$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(30)(1.5) = \boxed{300\pi \text{ ft}^2/\text{s}}$$

'Snapshot'
 $r = 30$

A plane flying horizontally at an altitude of 1 mile and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station.



$$\frac{dx}{dt} = 500$$

'Snapshot'

$$s = 2 \text{ mi}$$

$$\sqrt{2^2 - 1^2} = \sqrt{3} = x$$

$$x^2 + 1^2 = s^2$$

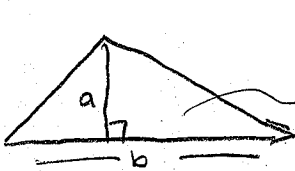
$$x^2 + 1 = s^2$$

$$2x \frac{dx}{dt} = 2s \frac{ds}{dt}$$

$$\frac{ds}{dt} = \frac{x}{s} \frac{dx}{dt}$$

$$\frac{ds}{dt} = \frac{(\sqrt{3})}{(2)} (500) = \boxed{250\sqrt{3} \text{ mi/h}}$$

The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of $2 \text{ cm}^2/\text{min}$. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2 ?



$$\frac{da}{dt} = 1 \text{ cm/min}$$

$$\frac{dA}{dt} = 2 \text{ cm}^2/\text{min}$$

$$\frac{db}{dt} = ?$$

$$A = \frac{1}{2}ba \quad (\text{product rule})$$

$$\frac{dA}{dt} = \frac{1}{2}b \frac{da}{dt} + a \left(\frac{1}{2} \frac{db}{dt} \right)$$

'Snapshot': $a = 10$
 $A = 100$ \Rightarrow $A = \frac{1}{2}ba$
 $100 = \frac{1}{2}b(10)$
 $20 = b$

$$2 = \frac{1}{2}(20)(1) + (10) \frac{1}{2} \frac{db}{dt}$$

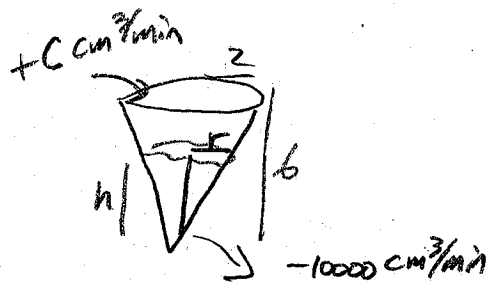
$$2 = 10 + 5 \frac{db}{dt}$$

$$5 \frac{db}{dt} = -8$$

$$\boxed{\frac{db}{dt} = -\frac{8}{5} \text{ cm/min}}$$

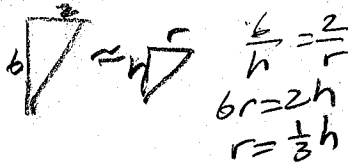
(base is decreasing)

Water is leaking out of an inverted conical tank at a rate of $10,000 \text{ cm}^3/\text{min}$ at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.



$$V = \frac{1}{3}\pi r^2 h \quad \text{need another relationship}$$

$$V = \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 h$$



$$V = \frac{\pi}{27} h^3$$

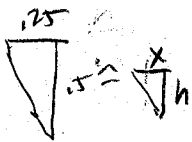
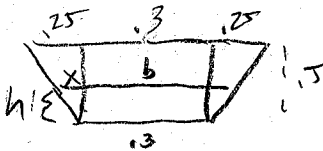
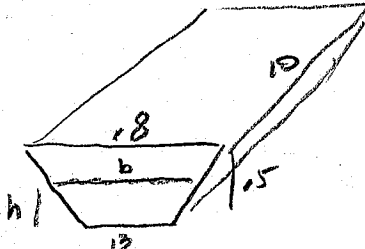
$$\frac{dV}{dt} = \frac{3\pi}{27} h^2 \frac{dh}{dt} = \frac{\pi}{9} (20)^2 (2)$$

$$\frac{dV}{dt} - 10000 = C$$

$$\frac{dV}{dt} = \frac{2400000\pi}{27} + 10000 \text{ cm}^3/\text{min}$$

$$C = 289252.680 \text{ cm}^3/\text{min}$$

A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top and has height of 50 cm. If the trough is being filled with water at the rate of 0.2 cubic meters/min., how fast is the water level rising when the water is 30 cm deep?



$$\frac{x}{h} = \frac{1.25}{25}$$

$$15x = 1.25h$$

$$x = \frac{1}{20}h$$

$$b = 2x + 30$$

$$b = 2\left(\frac{1}{20}h\right) + 30$$

$$b = h + 30$$

$$V = A_{\text{trap}} L$$

$$= \frac{1}{2}(b_1 + b_2)hL$$

$$= \frac{1}{2}(b + 30)h(10) \quad \text{but } b = h + 30$$

$$V = 5h(h + 30)$$

$$V = 5h(h + 30)$$

$$\frac{dV}{dt} = [5h(1) + (h + 30)(5)] \frac{dh}{dt}$$

$$\text{at this time: } \frac{dV}{dt} = 0.2 \text{ m}^3/\text{min} \quad h = 0.3 \text{ m}$$

$$0.2 = [5(0.3) + (0.3 + 30)(5)] \frac{dh}{dt}$$

$$0.2 = 6 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{30} \text{ m/min}$$

A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 min after the woman starts walking?

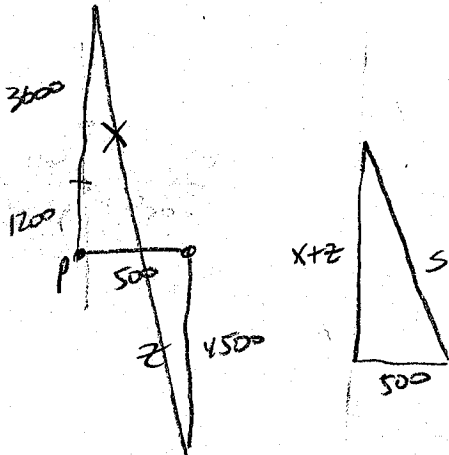
$$5 \text{ min} = 300 \text{ sec}$$

$$15 \text{ min} = 900 \text{ sec}$$

$$\text{man north: } \frac{4 \text{ ft}}{s} \cdot 300 = 1200 \text{ ft}$$

$$\text{man continues north: } \frac{4 \text{ ft}}{s} \cdot 900 = 3600 \text{ ft}$$

$$\text{woman south: } \frac{5 \text{ ft}}{s} \cdot 900 = 4500 \text{ ft}$$



$$s^2 = (x+z)^2 + 500^2$$

$$s^2 = x^2 + 2xz + z^2 + 250000$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2z \frac{dz}{dt} + z \left(2 \frac{dx}{dt} \right) + 2z \frac{dz}{dt}$$

at this time: $\frac{dx}{dt} = 4$ $\frac{dz}{dt} = 5$

$$s^2 = (4800 + 4500)^2 + 250000$$

$$s = \sqrt{86740000} = 9313.431162$$

$$2\sqrt{86740000} \frac{ds}{dt} = 2(4800)(4) + 2(4800)(5) + (4500)2(4) + 2(4500)5$$

$$2\sqrt{86740000} \frac{ds}{dt} = 167400$$

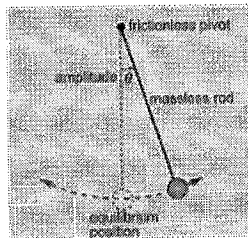
$$\frac{ds}{dt} = \frac{83700}{\sqrt{86740000}} = 8.987 \text{ ft/sec}$$

Unit 3-8: Linearization and Differentials, Newton's Method
 Larsen: 2.8 (Stewart: 3.10)

Procedures which use the tangent line as an approximation to a curve in a region

There are times when it is difficult to work directly with a function, but much easier if we can find a linear equation which approximates the behavior of the function, at least around a specific x -value.

An example from physics: If you analyze the motion of an idealized pendulum acted upon by gravity:



...the tangential acceleration turns out to be given by the equation: $a_t = -g \sin \theta$

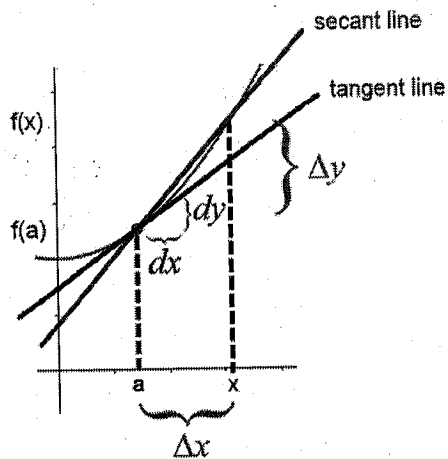
But by using one of the procedures we are learning today, you can show that as long as the perturbation of the pendulum is small (for θ near zero), the sine function is approximately equal to a linear function:

$$\sin \theta \approx \theta$$

The idea of oscillation modeled by a pendulum can be used to model other periodic phenomena where you need to combine the effects of many oscillations. Being able to simplify the function makes the resulting computations much more manageable.

Linearization of a function in the region of an x -value

For a given function curve, as long as we stay in a small region around an x -value, then the tangent line is a good approximation to the curve $f(x)$:



For the secant line (on the actual curve):

$$(y - y_a) = m_{\text{secant}} (x - x_a)$$

$$(f(x) - f(a)) = \frac{\Delta y}{\Delta x} (x - a)$$

$$f(x) = f(a) + \frac{\Delta y}{\Delta x} (x - a)$$

For the tangent line (with slope which is the limit as change in x approaches zero = the derivative):

$$(y - y_a) = m_{\text{tangent}} (x - x_a)$$

$$(f(x) - f(a)) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} (x - a)$$

$$(f(x) - f(a)) = f'(x)(x - a)$$

$$f(x) = f(a) + f'(x)(x - a)$$

Then the $f(x)$ using the tangent line is an approximation of the actual $f(x)$ on the curve:

$$f(x)_{\text{tangent}} \approx f(x)_{\text{secant}}$$

$$f(x) \approx f(a) + f'(x)(x - a)$$

The resulting linear function is called the **linearization of f at a** .

What is the linearization of the function $f(x) = \sin x$ at $x = 0$?

$$f(x) = \sin x \quad f'(x) = \cos x$$

$$f(0) = \sin 0 = 0 \quad f'(0) = \cos 0 = 1$$

$$f(x) \approx f(0) + f'(0)(x - 0)$$

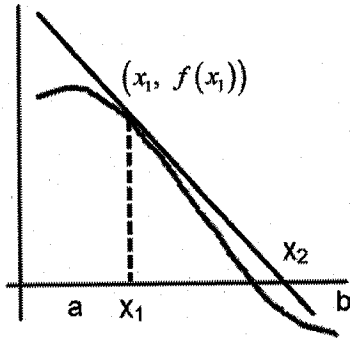
$$f(x) \approx 0 + (1)(x - 0)$$

$$f(x) \approx x$$

$$\boxed{\sin x \approx x} \quad (\text{for small } x)$$

Newton's Method

A 2nd procedure which uses a linearization to approximate a function allows us to find the approximate location of one of a function's zeros to any accuracy we choose. This procedure is called **Newton's Method**.



If the Intermediate Value Theorem has shown that there exists a zero for the function f in the interval $[a, b]$, we can select an x_1 and find the linearization of the function at this x -value:

$$f(x) \approx f(x_1) + f'(x_1)(x - x_1)$$

Solving for the x -value when this function equals zero (which we will call x_2)...

$$f(x) \approx f(x_1) + f'(x_1)(x - x_1)$$

$$0 = f(x_1) + f'(x_1)(x_2 - x_1)$$

$$0 = f(x_1) + f'(x_1)x_2 - f'(x_1)x_1$$

$$f'(x_1)x_2 = f'(x_1)x_1 - f(x_1)$$

$$x_2 = \frac{f'(x_1)}{f'(x_1)}x_1 - \frac{f(x_1)}{f'(x_1)}$$

This new x -value, x_2 is not exactly at the zero of the function, but it is closer than x_1 .

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We can then use x_2 as the starting value to calculate an x_3 , even closer, and continue this process, call **iteration**, until we have a value as close as we wish to the actual function zero.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Use Newton's Method to find an approximate zero for the given function accurate to within 0.001.

$$f(x) = x - 2\sqrt{x+1} = x - 2(x+1)^{1/2}$$

$$f'(x) = 1 - 2\left(\frac{1}{2}(x+1)^{-1/2}\right) = 1 - \frac{1}{\sqrt{x+1}}$$

← put into Y1 in your calculator
← put into Y2

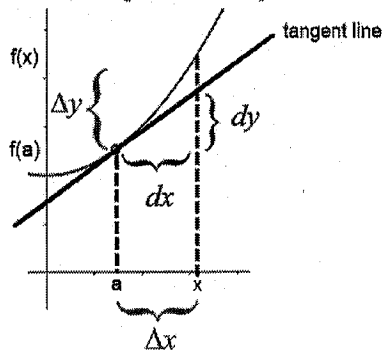
try some x values to find an interval w/ + & - :

$$f(4) = -.4721 \quad f(5) = 0.1101 \quad [4, 5] \text{ start with } x=4$$

x_n	$f(x)$	$f'(x)$	$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$
4	-.472135955	-.5527864045	4.854101966
4.854101966	0.01505166	0.586695761869	4.828447
4.828447	0.0000116426612	0.58578714387	4.828427
			4.828

Differentials

Next, we clarify the definitions of small changes in x and y for a curve.



'Average rate of change'
(algebra)

Δy is a small change in y
for a given small change in x ,
 Δx on the actual function curve.

'Instantaneous rate of change'
(calculus)

dy is a small change in y
for a given small change in x ,
 dx on the tangent line to the curve.

dx and dy are called differentials

Find the differential of the function:

$$y = \sqrt{1+t^2} = (1+t^2)^{1/2}$$

$$\frac{dy}{dt} = \frac{1}{2}(1+t^2)^{-1/2}(2t) = \frac{t}{\sqrt{1+t^2}}$$

$$\text{so } dy = \frac{t}{\sqrt{1+t^2}} dx$$

Relative Error and Percentage Error

Finally, we use an example to define something called 'relative error'.

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.03 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

$$V = \frac{4}{3}\pi r^3$$

The measurement error in the radius represents a small change in r . To determine how this translates to error in volume, we can find the differential for volume:

$$\frac{dV}{dr} = 4\pi r^2$$

$$dV = 4\pi r^2 dr$$

error = $\Delta r = 0.05$ So for $dr = 0.05$ at $r = 21$

An error in radius measurement of 0.05 cm translates to an error in volume of 277.088 cm^3 .

$$dV = 4\pi(21)^2(0.05) = 277.088 \text{ cm}^3$$

But how big is this volume error, really? We need something to compare it to. So we divide the error by the value of the quantity to see what percentage of the amount this error represents. This is called **relative error** and if expressed as a percentage, **percentage error**.

$$\text{relative error} = \frac{\Delta f(x)}{f(x)}$$

Here, $\Delta r = 0.05 \text{ cm} \rightarrow \Delta V = 277.088 \text{ cm}^3$ but the volume at $r = 21$ is...

$$V(21) = \frac{4}{3}\pi(21)^3 = 38792.386 \text{ cm}^3$$

...so the relative and percentage errors are:

$$\frac{277.088}{38792.386} = 0.007$$

0.7%

relative error percentage error

More examples

Find the linearization $L(x)$ of the function at a .

$$f(x) = e^{-2x}, a = 0$$

$$f'(x) = -2e^{-2x}$$

$$f(0) = e^{-2(0)} = 1$$

$$f'(0) = -2e^{-2(0)} = -2$$

$$L(x) \approx f(0) + f'(0)(x-0)$$

$$L(x) \approx 1 + (-2)(x-0)$$

$$\boxed{L(x) \approx 1 - 2x}$$

x	f(x)	L(x)
0.01	0.9801	0.98
0.1	0.8187	0.80
0.5	0.3678	0
1	0.1353	-1

only works over a small region:

Verify the given linear approximation at $a = 0$.

Then determine the values of x for which the linear approximation is accurate to within 0.1.

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x$$

$$f(x) = \sqrt{1+x} = (1+x)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}(1) = \frac{1}{2\sqrt{1+x}}$$

$$f(0) = \sqrt{1+0} = 1$$

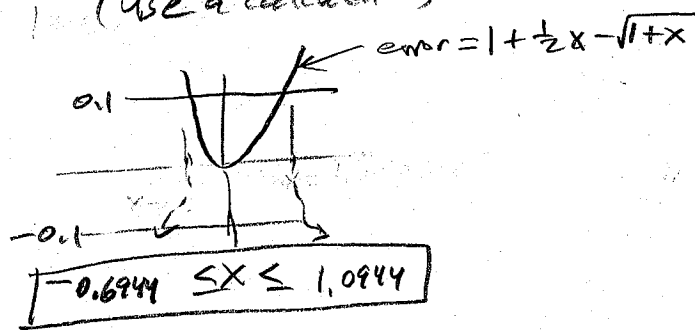
$$f'(0) = \frac{1}{2\sqrt{1+0}} = \frac{1}{2}$$

$$L(x) = f(0) + f'(0)(x-0)$$

$$L(x) = 1 + \left(\frac{1}{2}\right)(x-0)$$

$$\boxed{L(x) = 1 + \frac{1}{2}x}$$

within 0.1: $|L(x) - f(x)| < 0.1$
(use a calculator)



25. a) Find the differential dy

b) Evaluate dy for the given values of x and dx .

$$y = \cos x, x = \pi/6, dx = 0.05$$

$$\frac{dy}{dx} = -\sin x$$

$$\boxed{dy = -\sin x dx}$$

$$dy = -\sin\left(\frac{\pi}{6}\right)(0.05) = -\left(\frac{1}{2}\right)(0.05) = \boxed{-0.025}$$

(an increase of 0.05 in x causes a decrease in y of 0.025)

31. Use differentials (or equivalently, a linear approximation) to estimate the given number:

$$\sqrt{36.1} \quad f(x) = \sqrt{x} \quad a = 36 \quad f(36) = 6$$

$$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \quad f'(36) = \frac{1}{2\sqrt{36}} = \frac{1}{12}$$

$$\text{so } L(x) = f(36) + f'(36)(x-36)$$

$$L(x) = 6 + \frac{1}{12}(x-36)$$

$$\text{and } \sqrt{36.1} = L(36.1) = 6 + \frac{1}{12}(36.1-36) = \boxed{6.008\bar{3}} \quad (= 6 + \frac{1}{120})$$

34.

The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.

- (a) Use differentials to estimate the maximum error in the calculated area of the disk.

- (b) What is the relative error?

What is the percentage error?

(a) $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

$$\Delta r = dr = 0.2 \text{ cm}$$

$$\text{so } \Delta A \approx dA = 2\pi(24)(0.2)$$

$$= \boxed{30.15928 \text{ cm}^2}$$

(b) $A = \pi(24)^2 = 1809.557368 \text{ cm}^2$

$$\text{so relative error in area is } = \frac{30.15928}{1809.557368} = \boxed{0.166}$$

$$\text{percentage error is } \boxed{16.6\%}$$

what about error in radius?

$$\text{relative error in radius} = \frac{0.2}{24} = 0.00833$$

$$\text{percentage error} = 0.833\%$$

So a measurement error of only 0.8% in radius is associated with an error of 16.6% in area!

Derivative Terminology in Physics and Economics

The derivative always means the instantaneous rate of change of the function at a given input value. In physics and in economics, there are specific meaning and terminology for the use of the derivative.

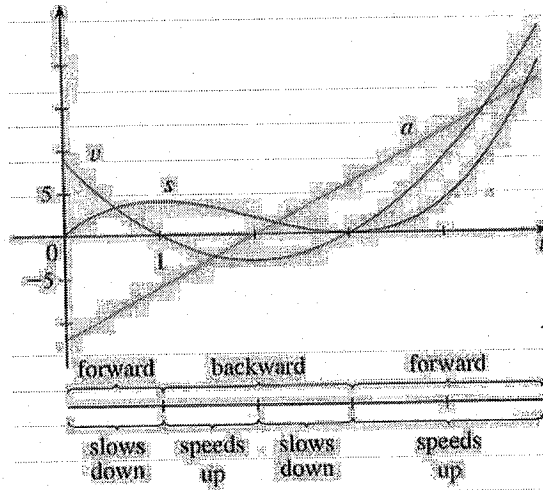
Derivative Terminology in Physics

If the original function is a position function (position as a function of time), then the first and higher-order derivatives have specific meanings and names...

original function	$s(t)$	position at a specific time	"position"
first derivative	$v(t) = s'(t)$	how fast position is changing at a specific time	"velocity"
second derivative	$a(t) = v'(t) = s''(t)$	how fast velocity is changing at a specific time	"acceleration"
third derivative		although we don't see it often, the derivative of acceleration, how fast acceleration changes at a specific time (the 3rd derivative of position) is called "jerk".	

We also say an object or particle **"speeds up"** when the velocity is positive and increasing or when the velocity is negative and decreasing (velocity and acceleration are both positive or both negative).

We also say an object or particle **"slows down"** when the velocity and acceleration have opposite signs.



Derivative Terminology in Economics

In microeconomics (economics on the scale of a single business), a business generally produces a product and then sells that product. Here are some terms to know...

x The number of units a business is making or selling.

Cost $C(x)$ How much money it takes the business to make x units.

Revenue $R(x)$ How much money the business makes in sales when it sells x units.

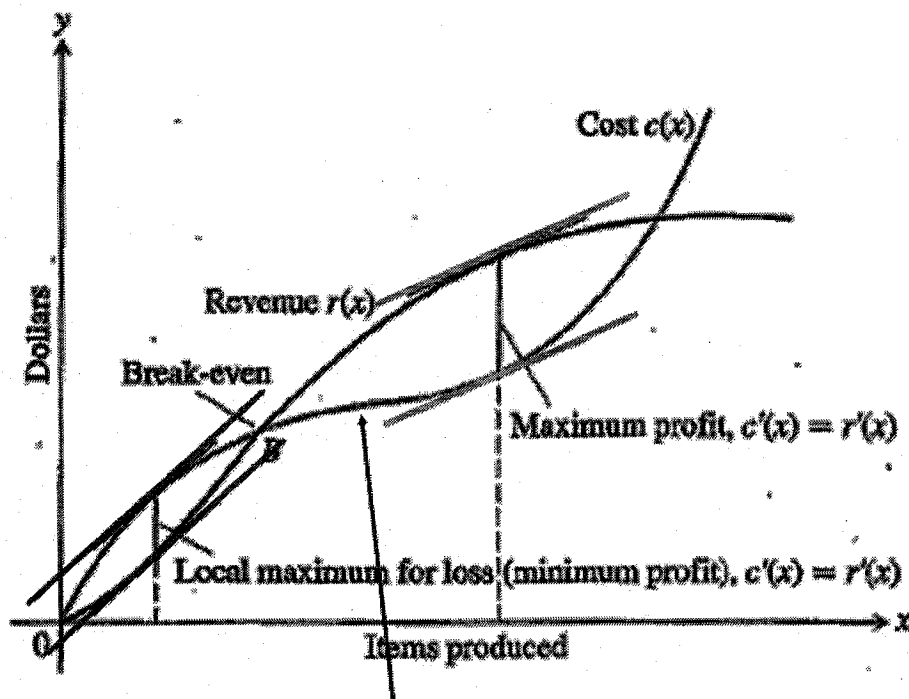
**Profit $P(x)$
(Loss)** How much money the keeps from its sales after paying the cost for making the units sold. $P(x) = R(x) - C(x)$
When profit is negative it is called 'loss'.

Marginal Cost $C'(x)$ The derivative of cost = how much cost increases to produce each additional unit at the current number of units, x .

Marginal Revenue $R'(x)$ The derivative of revenue = how much revenue increases per additional unit sold at the current number of units, x .

Breakeven
 $C(x) = R(x)$ The number of units where cost = revenue.

Maximum profit, Maximum loss When the rates of change for cost and revenue are equal, either the profit (or loss) is maximum. (Point of diminishing returns).
 $C'(x) = R'(x)$



The place where a curve changes from curving down (concave down) to curving up (concave up), or vice versa, is called an **inflection point**. It occurs when the **2nd derivative equals zero**.

Examples

p. 205 #5 A particle moves according to a law of motion where t is measured in seconds and s in feet.

$$s = f(t), t \geq 0, \quad s = \frac{t}{t^2 + 1} \quad v(t) = s'(t) = \frac{(t^2+1)(1) - (t)(2t)}{(t^2+1)^2} = \frac{t^2+1-2t^2}{(t^2+1)^2} = \frac{-t^2+1}{(t^2+1)^2}$$

a) Find the velocity at time t .

b) What is the velocity after 3 seconds?

$$v(3) = \frac{-(3^2)+1}{(3^2+1)^2} = \frac{-8}{100} \text{ ft/s}$$

c) When is the particle at rest?

$$v = \frac{-t^2+1}{(t^2+1)^2} = 0 \quad -t^2+1 = 0 \quad t^2 = 1 \quad t = 1 \text{ sec}$$

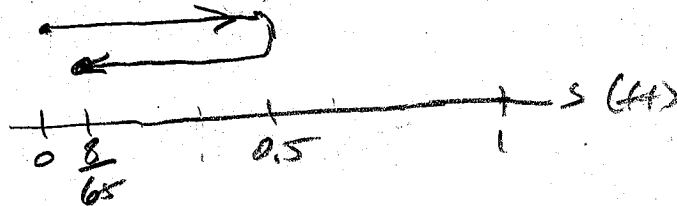
d) When is the particle moving in the positive direction?

$$\text{when } t < 1 \text{ sec} \quad 0 \leq t < 1$$

e) Find the total distance traveled during the first 8 seconds.

$$s(0) = 0 \quad s(1) = \frac{1}{2} \quad s(8) = \frac{8}{65} = .123 \text{ ft}$$

f) Draw a diagram like Figure 2 (p. 222) to illustrate the motion of the particle.



p. 205

8. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$

a) What is the maximum height reached by the ball?

b) What is the velocity of the ball when it is 96 ft. above the ground on its way up? On its way down?

$$(a) \quad v = s' = 80 - 32t = 0 \quad \text{at max height}$$

$$32t = 80$$

$$t = \frac{80}{32} = 2.5 \text{ sec.}$$

$$s(2.5) = 80(2.5) - 16(2.5)^2 = 100 \text{ ft}$$

$$(b) \quad s(t) = 80t - 16t^2 = 96$$

$$16t^2 - 80t + 96 = 0$$

$$t = \frac{80 \pm \sqrt{80^2 - 4(16)(96)}}{2(16)}$$

$$t = \frac{80 \pm 16}{32} = 2 \text{ secs on way up} \\ \text{and } 3 \text{ secs on way down}$$

$$\text{on way up} \\ t = 2$$

$$v(2) = 80 - 32(2) \\ = 16 \text{ ft/s}$$

$$\text{on way down} \\ t = 3$$

$$v(3) = 80 - 32(3) \\ = -16 \text{ ft/s}$$

Suppose that the cost, in dollars, for a company to produce x pairs of a new line of jeans is

$$C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3$$

- Find the marginal cost function.
- Find $C'(100)$ and explain its meaning. What does it predict?
- Compare $C'(100)$ with the cost of manufacturing the 101st pair.

(a) $C'(x) = 3 + 0.02x + 0.0006x^2$

(b) $C'(100) = 11$ \$/pair

if you've made 100 jeans, the cost to make the next pair of jeans is \$11.

(c) $C(100) = \$2600$
 $C(101) = \$2611.0702$

$C(101) - C(100) = \$11.0702$ for the 101st pair of jeans

The cost function for a certain commodity is

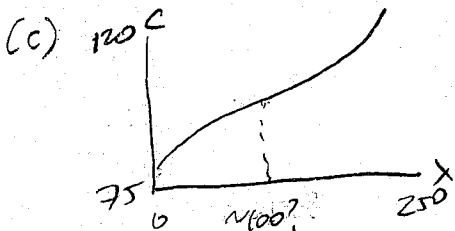
$$C(x) = 84 + 0.16x - 0.0006x^2 + 0.000003x^3$$

- Find and interpret $C'(100)$.
- Compare with the cost of producing the 101st item.
- Graph the cost function and estimate the inflection point.
- Calculate the value of x for which C has an inflection point. What is the significance of this value of x ?

(a) $C'(x) = 0.16 - 0.0012x + 0.000009x^2$

$C'(100) = \$0.13$ if you've made 100, 101st item costs \$0.13

(b) $C(100) = \$97$ $C(101) = \$97.130303$ $C(101) - C(100) = \$0.130303$ for 101st item



(d) $C''(x) = -0.0012 + 0.000018x = 0$

$x = \frac{0.0012}{0.000018} = 66.667$
 $(\frac{200}{3})$ units

The marginal cost is increasing at a slower rate and $x = \frac{200}{3}$ is approached, after this point the marginal cost starts to increase at a faster rate.