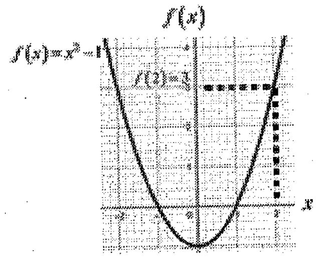
Unit 2-1: Limit Definition of Derivative

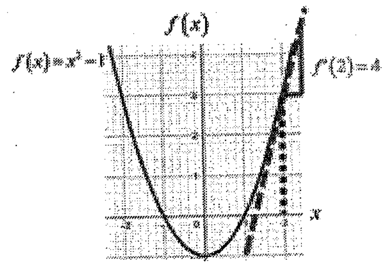
Concept of the Derivative - instantaneous rate of change

Precalculus is mainly concerned with the value of a function at a given *x*:



$f(x)$ gives us the y-value at x.

But calculus allows us to know how fast the value of a function is changing at *x* (the 'instantaneous rate of change'):



$f'(x)$ gives us the slope of the tangent line to the curve at x (the instantaneous rate of change of the curve at x)...and is called the ***derivative*** of f(x) at x.

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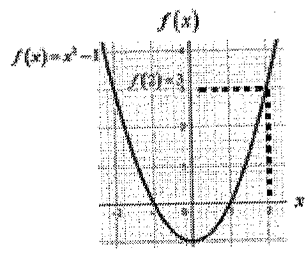
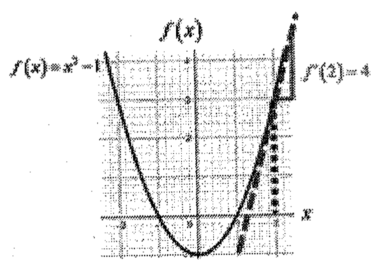


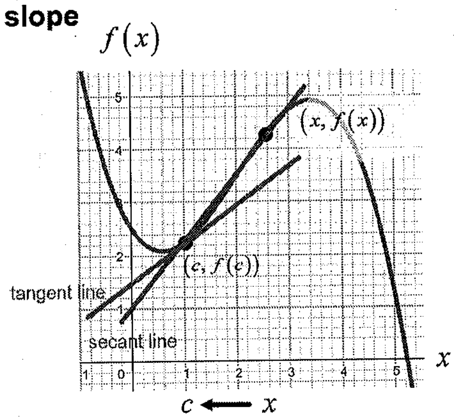


Computing the derivative as the limit of a secant line slope

If you already know ahead of time which x value at which you want to know the derivative, you can use a second, sometimes simpler, limit structure to compute the numerical value of the derivative at that x.

We then take the limit to allow x to approach the value c:

$$\boxed{f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}}$$



The result is a not a function of x, but a numerical value, the value of the derivative at that known x = c.

#1. Find $f'(x)$ for $f(x)=x^2-x$

...first, using $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, then plug in to find $f(2)$ and $f'(2)$

...Now start over, using $f'(c)=\lim_{x\to c}\frac{f(c+x)-f(c)}{x-c}$ using c = 2

When you are done, discuss and try the following:

- What does $f(2)$ tell you about $f(x)$?
- What does $f'(2)$ tell you about $f(x)$?
- Sketch the function (use your calculator to get the graph).
- Find the equation of a tangent line to $f(x)$ at x = 2 and add it to your sketch.

$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{[(x+h)^2-(x+h)]-[x^2-x]}{h}$

$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x}-h-\cancel{x^2}+\cancel{x}}{h}$

$=\lim_{h\to 0}\frac{2xh+h^2-h}{h}$

$=\lim_{h\to 0}\frac{h(2x+h-1)}{h(1)}$

$=\lim_{h\to 0}(2x+h-1)$

$=2x+(0)-1$

$f'(x)=2x-1$

$\boxed{f'(2)=2(2)-1=3}$

$\boxed{f(2)=(2)^2-(2)=2}$

$f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$

$=\lim_{x\to 2}\frac{[x^2-x]-[(2)^2-2]}{x-2}$

$=\lim_{x\to 2}\frac{x^2-x-2}{x-2}$

$=\lim_{x\to 2}\frac{(x-2)(x+1)}{x-2}$

$=\lim_{x\to 2}(x+1)$

$\boxed{f'(2)=(2)+1=3}$

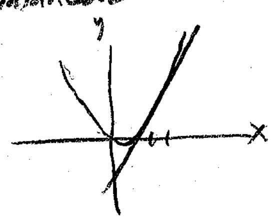
$f(2)=2$ tells us the y-coordinate of $f(x)$ at $x=2$ is 2.

$f'(2)=3$ tells us the slope of the line tangent to $f(x)$ at $x=2$ is 3 (the instantaneous rate of change of $f(x)$ is 3 at $x=2$

tangent line:

$\boxed{(y-2)=3(x-2)}$

$y=3x-4$

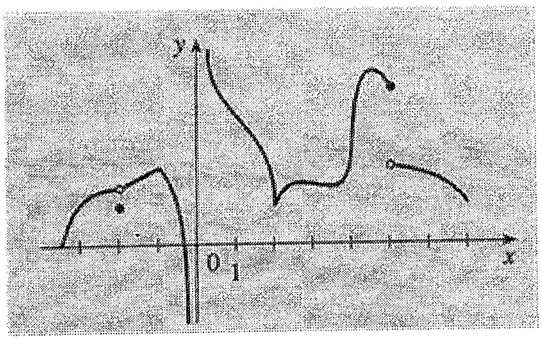


Does the derivative of a function always exist?

The derivative of a function has a potentially unique value at every x value in the function's domain, and it is possible that for some values of x, the derivative will not exist. The things that can cause a function's derivative not to exist at an x value are:

- If the function is discontinuous at the x (the tangent line must have a point to attach to).
- If the function's curve shape abruptly changes directly at the x instead of moving smoothly through the x value.
- If the tangent line is perfectly vertical (slope is infinite, therefore not a number and DNE).

#2: Find the x-values for which the derivative does not exist.



y' does not exist at...

$x=-2$ (discontinuity)

$x=0$ (discontinuity)

$x=2$ ('corner')

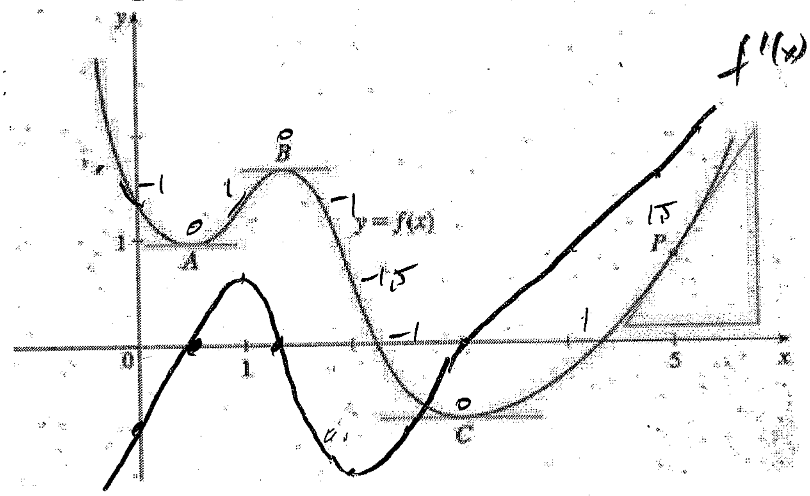
$x=4$ (vertical tangent)

$x=5$ (discontinuity)

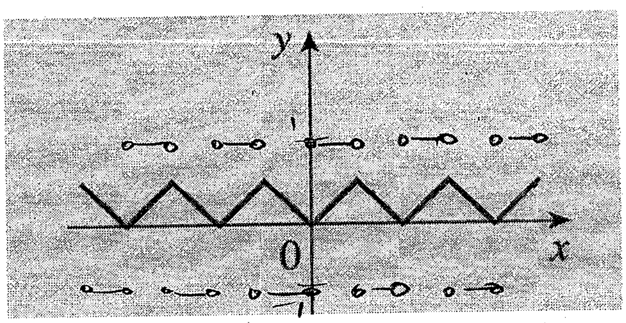
Sketching a derivative curve from a function curve

If we are given the graph of a function, we can sketch an approximate function curve for the derivative function. Remember that the value of the derivative equals the instantaneous rate of change (the 'slope') of the original function curve at that x-value.

#3. Given the graph of $f(x)$, sketch the graph of $f'(x)$



#4. Given the graph of $f(x)$, sketch the graph of $f'(x)$



When given a limit expression, it is sometimes helpful to be able to recognize that the limit is actually the limit definition of the derivative for a particular function at a given x.

Each of the limits below represents the limit definition of a derivative of a particular function at a given x.
Can you determine what the function is and the given x?

#5. $\lim_{h \to 0} \frac{\sqrt{h+1}-1}{h}$

$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$\boxed{\text{if } f(x)=\sqrt{x} \text{ and } x=1}$

#6. $\lim_{x \to 3\pi} \frac{\cos(x)+1}{x-3\pi}$

$\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

$\boxed{\text{if } f(x)=\cos x \text{ and } x=c=3\pi}$

#7. Find $f'(x)$ for $f(x)=\frac{1}{\sqrt{x}}$, then find $f(1)$ and $f'(1)$

$$f'(x)=\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h \to 0} \frac{\left[\frac{1}{\sqrt{x+h}}\right]-\left[\frac{1}{\sqrt{x}}\right]}{h}$$

$$=\lim_{h \to 0} \frac{\left(\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right)\left(\sqrt{x}\sqrt{x+h}\right)}{h\left(\sqrt{x}\sqrt{x+h}\right)}$$

$$=\lim_{h \to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$$

$$=\lim_{h \to 0} \frac{\left(\sqrt{x}-\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$=\lim_{h \to 0} \frac{x-(x-h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$=\lim_{h \to 0} \frac{h(-1)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$=\lim_{h \to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$=\frac{-1}{\sqrt{x}\sqrt{x+0}\left(\sqrt{x}+\sqrt{x+0}\right)}$$

$$=\frac{-1}{2x\sqrt{x}}$$

$\boxed{f'(x)=\frac{-1}{2x\sqrt{x}}}$

$\boxed{f'(1)=\frac{-1}{2(1)\sqrt{1}}=\frac{-1}{2}}$

$\boxed{f(1)=\frac{1}{\sqrt{1}}=1}$

now we could build a tangent line at $(1,1)$ with $m=\frac{-1}{2}$:

$(y-1)=\frac{-1}{2}(x-1)$

Unit 2-2: Derivative 'Shortcuts'

We can evaluate a derivative of a given function at a generic x to get a 'shortcut'

The limit definition of the derivative is the official way to determine a derivative equation for a given function, but it can be very time-consuming to compute. To accelerate finding derivatives so that we can focus later on how derivatives are *used*, we employ 'derivative shortcuts' (not an official term).

Derivations of selected derivative shortcuts

Derive the derivative shortcut for $f(x)=c \quad \text{where } c \text{ is any constant}$

proof

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{[c]-[c]}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

shortcut

$$\boxed{\frac{d}{dx}[c]=0}$$

Derive the derivative shortcut for $f(x)=x^n \quad \text{where } n \text{ is any real number}$

proof

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\left[(x+h)^n\right]-\left[(x)^n\right]}{h} \end{aligned}$$

(binomial theorem)

$$\begin{aligned} &= \lim_{h\to 0}\frac{\left[x^n+nx^{n-1}h+\binom{n}{2}x^{n-2}h^2+\binom{n}{3}x^{n-3}h^3\ldots+\binom{n}{n-1}x^1h^{n-1}+h^n\right]-\left[(x)^n\right]}{h} \\ &= \lim_{h\to 0}\frac{nx^{n-1}h+\binom{n}{2}x^{n-2}h^2+\binom{n}{3}x^{n-3}h^3\ldots+\binom{n}{n-1}x^1h^{n-1}+h^n}{h} \\ &= \lim_{h\to 0}\frac{h\left[nx^{n-1}+\binom{n}{2}x^{n-2}h+\binom{n}{3}x^{n-3}h^2\ldots+\binom{n}{n-1}x^1h^{n-2}+h^{n-1}\right]}{h} \\ &= \lim_{h\to 0}\left(nx^{n-1}+\binom{n}{2}x^{n-2}h+\binom{n}{3}x^{n-3}h^2\ldots+\binom{n}{n-1}x^1h^{n-2}+h^{n-1}\right) \\ &= nx^{n-1} \end{aligned}$$

shortcut

$$\boxed{\frac{d}{dx}\left[x^n\right]=nx^{n-1}}$$

Derivations of selected derivative shortcuts

Derive the derivative shortcut for $f(x)=e^x$

proof

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\left[e^{x+h}\right]-\left[e^x\right]}{h}$$
$$=\lim_{h\to 0}\frac{e^xe^h-e^x}{h}$$
$$=\lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h}$$

definition of e: $e=\lim_{h\to 0}(1+h)^{\frac{1}{h}}$ *means that for small h*, $e\approx(1+h)^{\frac{1}{h}}$

therefore, $e^h\approx 1+h$ *Substituting this...*

$$=\lim_{h\to 0}\frac{e^x\left([1+h]-1\right)}{h}$$
$$=\lim_{h\to 0}\frac{e^xh}{h}$$
$$=\lim_{h\to 0}e^x$$
$$=e^x$$

shortcut

$$\boxed{\frac{d}{dx}\left[e^x\right]=e^x}$$

Derive the derivative shortcut for $f(x)=\sin(x)$

proof

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\left[\sin(x+h)\right]-\left[\sin(x)\right]}{h}$$

trig identity: $\sin(u+v)=\sin(u)\cos(v)+\cos(u)\sin(v)$

Substituting this...

$$=\lim_{h\to 0}\frac{\left[\sin(x)\cos(h)+\cos(x)\sin(h)\right]-\left[\sin(x)\right]}{h}$$
$$=\lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)+\cos(x)\sin(h)}{h}$$
$$=\lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}$$
$$=\lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h}+\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}$$

The functions not containing h can be moved outside the limits...

$$=\sin(x)\lim_{h\to 0}\frac{(\cos(h)-1)}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

Special trig limits: $\lim_{h\to 0}\frac{(\cos(h)-1)}{h}=0,\ \lim_{h\to 0}\frac{\sin(h)}{h}=1...$

$$=\sin(x)[0]+\cos(x)[1]$$
$$=\cos(x)$$

shortcut

$$\boxed{\frac{d}{dx}\left[\sin(x)\right]=\cos(x)}$$

Derivations of selected derivative properties / 'rules'

What is the derivative of a constant times a function? $\frac{d}{dx}\left[cf(x)\right]$

proof

$$\begin{aligned}\frac{d}{dx}\left[cf(x)\right] &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to0}\frac{\left[cf(x+h)\right]-\left[cf(x)\right]}{h}\\ &= \lim_{h\to0}\frac{c\left(f(x+h)-f(x)\right)}{h}\\ &= c\lim_{h\to0}\frac{f(x+h)-f(x)}{h}\\ &= cf'(x)\end{aligned}$$

property / rule

$$\boxed{\frac{d}{dx}\left[cf(x)\right]=c\frac{d}{dx}\left[f(x)\right]}$$

"constant multiple rule"

What is the derivative of the sum of two different functions? $\frac{d}{dx}\left[f(x)+g(x)\right]$

proof

$$\begin{aligned}&\frac{d}{dx}\left[f(x)+g(x)\right]\\ &= \lim_{h\to0}\frac{\left[f(x+h)+g(x+h)\right]-\left[f(x)+g(x)\right]}{h}\\ &= \lim_{h\to0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\ &= \lim_{h\to0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\ &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to0}\frac{g(x+h)-g(x)}{h}\\ &= \frac{d}{dx}\left[f(x)\right]+\frac{d}{dx}\left[g(x)\right]\end{aligned}$$

property / rule

$$\boxed{\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}\left[f(x)\right]+\frac{d}{dx}\left[g(x)\right]}$$

"sum rule"

List of derivative shortcuts (so far) that you must memorize

$\frac{d}{dx}\left[c\right]=0$

$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$ "power rule"

$\frac{d}{dx}\left[e^x\right]=e^x$

$\frac{d}{dx}\left[a^x\right]=(\ln a)a^x$

$\frac{d}{dx}\left[\sin(x)\right]=\cos(x)$

$\frac{d}{dx}\left[\cos(x)\right]=-\sin(x)$

List of derivative properties / rules (so far) that you must memorize

$\frac{d}{dx}\left[cf(x)\right]=c\frac{d}{dx}\left[f(x)\right]$ "constant multiple rule"

$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}\left[f(x)\right]+\frac{d}{dx}\left[g(x)\right]$ "sum rule"

$\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}\left[f(x)\right]-\frac{d}{dx}\left[g(x)\right]$ "difference rule"

(The sum and difference rules are what allow us to take derivatives of each term separately in a function with multiple terms...each terms is being treated as a separate function.)

Examples

Find the derivatives of the given functions.

#1. $f(x)=x^5$

$\boxed{f'(x)=5x^4}$

#2. $f(x)=x^{\frac{2}{3}}$

$\boxed{f'(x)=\frac{2}{3}x^{-1/3}}$

#3. $g(t)=\frac{2}{t^5}=2t^{-5}$

$g'(t)=2(-5t^{-6})$

$\boxed{g'(t)=\frac{-10}{t^6}}$

Examples

Find the derivatives of the given functions.

#4. $f(x)=3x^5+4e^x+7$

$\boxed{f'(x)=15x^4+4e^x}$

#5. $g(x)=\dfrac{x^4-3x^2}{2x}=\dfrac{x^4}{2x}-\dfrac{3x^2}{2x}$

$g(x)=\frac{1}{2}x^3-\frac{3}{2}x$

$\boxed{g'(x)=\frac{3}{2}x^2-\frac{3}{2}}$

#6. For what values of a and b is the line $2x+y=b$ tangent to the parabola $y=ax^2$ when $x=2$?

y and y' must both match (at $x=2$)!

$y=b-2x$ and $y=ax^2$

$y'=-2$ $\quad$ $y'=2ax$

$$\begin{cases} b-2x=ax^2 \\ -2=2ax \end{cases}$$

at $x=2$:

$b-2(2)=a(2)^2 \quad -2=2a(2)$

$b-4=4a \quad -2=4a$

$$\begin{cases} b-4=4a \\ -2=4a \end{cases} \rightarrow \quad a=\frac{-2}{4}=\frac{-1}{2}$$

$b-4=4(\frac{-1}{2})$

$b-4=-2$

$b=2$

$$\boxed{\begin{aligned} a&=\tfrac{-1}{2} \\ b&=2 \end{aligned}}$$

#7. Find a cubic function $y=ax^3+bx^2+cx+d$ whose graph has horizontal tangents at the points $(-2,6)$ and $(2,0)$.

horizontal tangents when $y'=0$

$y'=3ax^2+2bx+c$

at $x=-2$: $\quad y'=3a(-2)^2+2b(-2)+c=0$ ①

at $x=2$: $\quad y'=3a(2)^2+2b(2)+c=0$ ②

also y's must match!

at $x=-2$: $\quad y=a(-2)^3+b(-2)^2+c(-2)+d=6$ ③

at $x=2$: $\quad y=a(2)^3+b(2)^2+c(2)+d=0$ ④

system:
$$\begin{cases} 12a-4b+1c+0d=0 \\ 12a+4b+1c+0d=0 \\ -8a+4b-2c+1d=6 \\ 8a+4b+2c+1d=0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 12 & -4 & 1 & 0 & 0 \\ 12 & 4 & 1 & 0 & 0 \\ -8 & 4 & -2 & 1 & 6 \\ 8 & 4 & 2 & 1 & 0 \end{array}\right]$$

rref

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/16 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -9/4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$\boxed{y=\frac{3}{16}x^3-\frac{9}{4}x+3}$

Unit 2-3: Derivative Product and Quotient Rules

Derivations of selected derivative properties (rules)

What is the derivative of the product of two functions? $\frac{d}{dx}\left[f(x)g(x)\right]$

proof

$$\frac{d}{dx}\left[f(x)g(x)\right]=$$

$$\lim_{h\to 0}\frac{\left[f(x+h)g(x+h)\right]-\left[f(x)g(x)\right]}{h}$$

add and subtract a specific term in the middle...

$$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

factor each pair...

$$\lim_{h\to 0}\frac{f(x+h)\left(g(x+h)-g(x)\right)+g(x)\left(f(x+h)-f(x)\right)}{h}$$

$$\lim_{h\to 0}\frac{f(x+h)\left(g(x+h)-g(x)\right)+g(x)\left(f(x+h)-f(x)\right)}{h}$$

$$\lim_{h\to 0}f(x+h)\frac{\left(g(x+h)-g(x)\right)}{h}+\lim_{h\to 0}g(x)\frac{\left(f(x+h)-f(x)\right)}{h}$$

$$\lim_{h\to 0}f(x+h)\lim_{h\to 0}\frac{\left(g(x+h)-g(x)\right)}{h}+\lim_{h\to 0}g(x)\lim_{h\to 0}\frac{\left(f(x+h)-f(x)\right)}{h}$$

$$f(x)g'(x)+g(x)f'(x)$$

property / rule

$$\boxed{\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+g(x)f'(x)}$$

"product rule"
(first times derivative of the second, plus second times derivative of the first)

What is the derivative of the quotient of two functions? $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$

proof

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=$$

$$\lim_{h\to 0}\frac{\left[\frac{f(x+h)}{g(x+h)}\right]-\left[\frac{f(x)}{g(x)}\right]}{h}$$

$$\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x)g(x+h)}$$

factor each pair...

$$\lim_{h\to 0}\frac{g(x)\left(f(x+h)-f(x)\right)+f(x)\left(g(x)-g(x+h)\right)}{hg(x)g(x+h)}$$

$$\lim_{h\to 0}\frac{g(x)\left(f(x+h)-f(x)\right)-f(x)\left(g(x+h)-g(x)\right)}{hg(x)g(x+h)}$$

$$\lim_{h\to 0}\frac{g(x)\frac{\left(f(x+h)-f(x)\right)}{h}-f(x)\frac{\left(g(x+h)-g(x)\right)}{h}}{g(x)g(x+h)}$$

$$\frac{g(x)\lim_{h\to 0}\frac{\left(f(x+h)-f(x)\right)}{h}-f(x)\lim_{h\to 0}\frac{\left(g(x+h)-g(x)\right)}{h}}{\lim_{h\to 0}g(x)g(x+h)}$$

$$\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

property / rule

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}}$$

"quotient rule"
(low d-high minus high d-low divided by low squared)

Differentiate each function:

#1. $f(x) = x^3 e^x$

$\boxed{f'(x) = x^3 e^x + e^x(3x^2)}$

#2. $f(x) = 2x^3$

$f'(x) = 2(3x^2)$

$\boxed{f'(x) = 6x^2}$

(no need for product rule but could use it!

$f'(x) = 2(3x^2) + x^3(0)$)

#3. $f(x) = 2x^3 e^x$

$\boxed{f'(x) = 2x^3 e^x + e^x(6x^2)}$

#4. $f(x) = \dfrac{2x^5 - x^3}{x^2 + 3x}$

$$\boxed{f'(x) = \frac{(x^2+3x)(10x^4 - 3x^2) - (2x^5 - x^3)(2x+3)}{(x^2+3x)^2}}$$

#5. $y = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1} = \dfrac{x^{1/2} - 1}{x^{1/2} + 1}$

$$\boxed{y' = \frac{(\sqrt{x}+1)(\frac{1}{2}x^{-1/2}) - (\sqrt{x}-1)(\frac{1}{2}x^{-1/2})}{(\sqrt{x}+1)^2}}$$

#6. $y = \sqrt{x}\,(x^2 - 2x)$

$\boxed{y' = \sqrt{x}(2x-2) + (x^2 - 2x)\frac{1}{2}x^{-1/2}}$

#7. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$

$$\boxed{y' = \frac{\sqrt{x}(2x+4) - (x^2+4x+3)(\frac{1}{2}x^{-1/2})}{(\sqrt{x})^2}}$$

— or —

$y = \dfrac{x^2}{\sqrt{x}} + \dfrac{4x}{\sqrt{x}} + \dfrac{3}{\sqrt{x}}$

$y = x^{3/2} + 4x^{1/2} + 3x^{-1/2}$

$\boxed{y' = \frac{3}{2}x^{1/2} + 2x^{-1/2} - \frac{3}{2}x^{-3/2}}$

Shortcuts

$\frac{d}{dx}[c]=0$

$\frac{d}{dx}[x^n]=nx^{n-1}$

$\frac{d}{dx}[e^x]=e^x$

$\frac{d}{dx}[a^x]=(\ln a)a^x$

$\frac{d}{dx}[\sin(x)]=\cos(x)$

$\frac{d}{dx}[\cos(x)]=-\sin(x)$

Properties

$\frac{d}{dx}[cf(x)]=c\frac{d}{dx}[f(x)]$

$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)]$

$\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}[f(x)]-\frac{d}{dx}[g(x)]$

$\frac{d}{dx}[f(x)g(x)]=f(x)g'(x)+g(x)f'(x)$

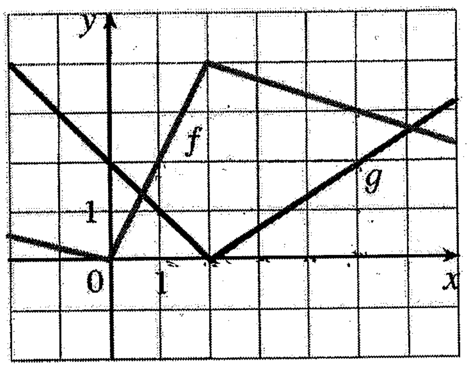
$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

Examples

If f and g are the functions whose graphs are show, let
$u(x)=f(x)g(x)$ and $v(x)=f(x)/g(x)$.

#8. Find $u'(1)$.

#9. Find $v'(5)$.



#8: $u'(x)=f(x)g'(x)+g(x)f'(x)$

$u'(1)=f(1)g'(1)+g(1)f'(1)$

$u'(1)=(2)(-1)+(1)(2)=-2+2=\boxed{0}$

#9. $v'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

$v'(5)=\frac{g(5)f'(5)-f(5)g'(5)}{[g(5)]^2}$

$\boxed{v'(5)=\frac{(2)(-\frac{1}{3})-(3)(\frac{2}{3})}{[2]^2}}=\frac{-\frac{2}{3}-\frac{6}{3}}{4}=\frac{-4}{12}=\boxed{\frac{-1}{3}}$

Unit 2-4: The Chain Rule

The Chain Rule - Derivatives of a Composition of Functions

When a function can be defined as a composition of two other functions (one 'inside' the other), we have to use the **Chain Rule** when computing the derivative.

#1. $y = f(g(x)) = e^{x^2+3}$ $f(x) = e^x$ $g(x) = x^2 + 3$

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)}$$

derivative of the outside function (with inside just copied as is)

derivative of the inside function

$y' = e^{(x^2+3)}(2x)$

This applies to any number of nested functions

The reason this is called the chain rule is that if you have many nested functions, you always start with the outer most function, take its derivative, then mulitply by ('chain on') the derivative of the inside, repeating until you finally stop when the inside function is an *x*:

#2. $y = \ln\left(\sin\left(e^{x^5}\right)\right)$

$y' = \frac{1}{\sin(e^{x^5})}(-\cos(e^{x^5}))e^{(x^5)}5x$

#3. $y = \sin^3\left(x^3 + 7x\right) = (\sin(x^3+7x))^3$

$y' = 3(\sin(x^3+7x))^2\cos(x^3+7x)(3x^2+7)$

Shortcut for derivative of logarithmic function

We can use the chain rule to develop a shortcut for derivative of a log function.

$y = \ln(x)$

$e^y = x$

taking derivative of each side

(using chain rule on left)

$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$

$e^y \frac{d}{dx}[y] = 1$

$e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y}$

$\frac{dy}{dx} = \frac{1}{x}$

$$\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x} \qquad \frac{d}{dx}[\log_b(x)] = \frac{1}{x\ln(b)}}$$

The extended (general) derivative shortcuts

With the chain rule we can make more general (extended) forms of each of our derivative shortcuts:

$$\frac{d}{dx}\left[x^n\right]=nx^{n-1} \Rightarrow \frac{d}{dx}\left[(something)^n\right]=n(something)^{n-1}\cdot\frac{d(something)}{dx}$$

#4. $y=(x^3+4x)^5$ $y'=5(x^3+4x)^4(3x^2+4)$

$$\frac{d}{dx}\left[\sin(x)\right]=\cos(x) \Rightarrow \frac{d}{dx}\left[\sin(something)\right]=\cos(something)\cdot\frac{d(something)}{dx}$$

#5. $y=\sin(3x^2+x)$ $y'=\cos(3x^2+x)(6x+1)$

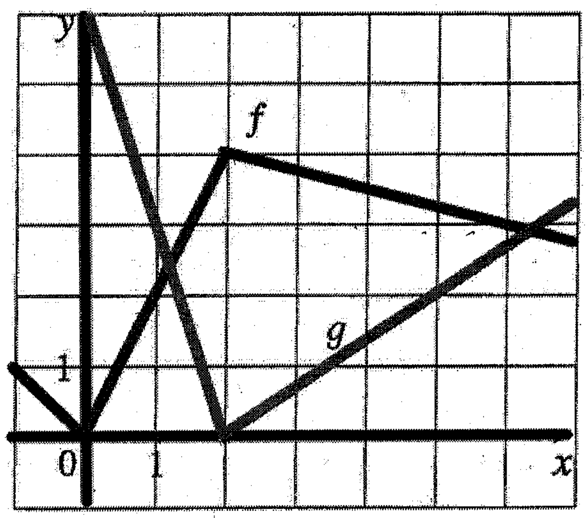
If f and g are the functions whose graphs are shown, let $u(x)=f(g(x))$, $v(x)=g(f(x))$, and $w(x)=g(g(x))$.

Find each derivative, if it exists.
If it does not exist, explain why.

#6. $u'(1)$

#7. $v'(1)$

#8. $w'(1)$



#6. $u'(x)=f'(g(x))g'(x)$

$u'(1)=f'(g(1))g'(1)$

$u'(1)=f'(3)g'(1)$

$\boxed{u'(1)=(-\frac{1}{4})(\frac{-1}{3})}$

$\boxed{u'(1)=\frac{1}{12}}$

#7. $v'(x)=g'(f(x))f'(x)$

$v'(1)=g'(f(1))f'(1)$

$v'(1)=\underbrace{g'(2)}_{\text{DNE}}f'(1)$

$v'(1)$ DNE because $g'(2)$ DNE ("corner")

#8. $w'(x)=g'(g(x))g'(x)$

$w'(1)=g'(g(1))g'(1)$

$w'(1)=g'(3)g'(1)$

$\boxed{w'(1)=(\frac{2}{3})(\frac{-1}{3})}$

$\boxed{w'(1)=\frac{-2}{9}}$

Unit 2-5: Implicit and Logarithmic Differentiation

Implicit Differentiation: This procedure is useful when we have an equation which is not easy to solve for y in terms of x. We use this procedure to take the derivative of each term where it is in the original equation first, then solve for the derivative at the end of the procedure.

#1. $x^3 + \underline{xy} + y^3 = 5$ Find the slope of the line tangent to this curve at the point (-1, 2)?

$\frac{d}{dx}[x^3] + x\frac{d}{dx}[y] + y\frac{d}{dx}[x] + \frac{d}{dx}[y^3] = \frac{d}{dx}[5]$

$3x^2 + x(1 \cdot \frac{dy}{dx}) + y(1) + 3y^2\frac{dy}{dx} = 0$

$x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -3x^2 - y$

$(x+3y^2)\frac{dy}{dx} = -3x^2 - y$

$\boxed{\frac{dy}{dx} = \frac{-3x^2-y}{x+3y^2}}$

#2. $\underline{x^2}\,\underline{y} + \underline{y^2}\,\underline{x} = -2$ Find $\frac{dy}{dx}$

$x^2\frac{d}{dx}[y] + y\frac{d}{dx}[x^2] + y^2\frac{d}{dx}[x] + x\frac{d}{dx}[y^2] = \frac{d}{dx}[-2]$

$x^2(1\frac{dy}{dx}) + y(2x) + y^2(1) + x(2y\frac{dy}{dx}) = 0$

$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$

$(x^2+2xy)\frac{dy}{dx} = -2xy - y^2$

$\boxed{\frac{dy}{dx} = \frac{-2xy-y^2}{x^2+2xy}}$

Logarithmic Differentiation: Is useful when we have an equation which employs exponential expressions or many factors multiplied/divided. By taking the logarithm of both sides of the equation, we can use log properties to simplify the expression before taking the derivative.

Logarithmic Differentiation is useful for something like: $y=\frac{(x^2+3x)^4(5x^3)}{(2x^4-4x)^2}$

...or: $y=x^{5x^3+2x}$

Logarithmic Differentiation Procedure

1) Take the natural logarithm of both sides of the equation.

2) Use log properties to simplify the equation.

3) Take the derivative of both sides of the equation.

4) Solve algebraically for $\frac{dy}{dx}$

#3. $y=\frac{(x^2+3x)^4(5x^3)}{(2x^4-4x)^2}$

$$\ln y=\ln\left[\frac{(x^2+3x)^4(5x^3)}{(2x^4-4x)^2}\right]$$

$$\ln y=4\ln(x^2+3x)+\ln(5x^3)-2\ln(2x^4-4x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{4}{x^2+3x}(2x+3)+\frac{1}{5x^3}15x^2-\frac{2}{2x^4-4x}(8x^3-4)$$

$$\boxed{\frac{dy}{dx}=\left[\frac{4(2x+3)}{x^2+3x}+\frac{15x^2}{5x^3}-\frac{2(8x^3-4)}{2x^4-4x}\right]\frac{(x^2+3x)^4(5x^3)}{(2x^4-4x)^2}}$$

#4. $y=x^{5x^3+2x}$

$$\ln y=\ln(x^{5x^3+2x})$$

$$\ln y=(5x^3+2x)\ln(x)$$

$$\frac{1}{y}\frac{dy}{dx}=(5x^3+2x)\frac{1}{x}+\ln(x)(15x^2+2)$$

$$\boxed{\frac{dy}{dx}=\left[\frac{5x^3+2x}{x}+\ln(x)(15x^2+2)\right]x^{5x^3+2x}}$$

#5. $y=x^{\left(\frac{2}{x}\right)}$

$$\ln y=\ln(x^{\frac{2}{x}})$$

$$\ln y=\left(\frac{2}{x}\right)\ln(x)=(2x^{-1})\ln(x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{2}{x}\frac{1}{x}+\ln(x)(-2x^{-2})$$

$$\boxed{\frac{dy}{dx}=\left[\frac{2}{x^2}-\frac{2\ln(x)}{x^2}\right]x^{\left(\frac{2}{x}\right)}}$$

Unit 2-6: More Trig, Inverse function and higher-order derivatives

Derivations of selected derivative shortcuts (rules)

What is the derivative of tangent? $f(x)=\tan(x)$

proof

$$\frac{d}{dx}[\tan(x)]$$
$$=\frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right]$$
$$=\frac{\cos(x)\frac{d}{dx}[\sin(x)]-\sin(x)\frac{d}{dx}[\cos(x)]}{[\cos(x)]^2}$$
$$=\frac{\cos(x)\cos(x)-\sin(x)(-\sin(x))}{[\cos(x)]^2}$$
$$=\frac{\cos^2(x)+\sin^2(x)}{[\cos(x)]^2}$$
$$=\frac{1}{[\cos(x)]^2}$$
$$=\sec^2(x)$$

shortcut

$$\boxed{\frac{d}{dx}[\tan(x)]=\sec^2(x)}$$

What is the derivative of cotangent? $f(x)=\cot(x)$

proof

$$\frac{d}{dx}[\cot(x)]$$
$$=\frac{d}{dx}\left[\frac{\cos(x)}{\sin(x)}\right]$$
$$=\frac{\sin(x)\frac{d}{dx}[\cos(x)]-\cos(x)\frac{d}{dx}[\sin(x)]}{[\sin(x)]^2}$$
$$=\frac{\sin(x)(-\sin(x))-\cos(x)\cos(x)}{[\sin(x)]^2}$$
$$=\frac{-(\sin^2(x)+\cos^2(x))}{[\sin(x)]^2}$$
$$=\frac{-1}{[\sin(x)]^2}$$
$$=-\csc^2(x)$$

shortcut

$$\boxed{\frac{d}{dx}[\cot(x)]=-\csc^2(x)}$$

Derivations of selected derivative shortcuts (rules)

What is the derivative of secant? $f(x) = \sec(x)$

proof

$$\frac{d}{dx}[\sec(x)]$$
$$= \frac{d}{dx}\left[\frac{1}{\cos(x)}\right]$$
$$= \frac{\cos(x)\frac{d}{dx}[1] - 1\frac{d}{dx}[\cos(x)]}{[\cos(x)]^2}$$
$$= \frac{\cos(x)(0) - 1(-\sin(x))}{[\cos(x)]^2}$$
$$= \frac{\sin(x)}{[\cos(x)]^2}$$
$$= \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)}$$
$$= \sec(x)\tan(x)$$

shortcut

$$\boxed{\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)}$$

What is the derivative of cosecant? $f(x) = \csc(x)$

proof

$$\frac{d}{dx}[\csc(x)]$$
$$= \frac{d}{dx}\left[\frac{1}{\sin(x)}\right]$$
$$= \frac{\sin(x)\frac{d}{dx}[1] - 1\frac{d}{dx}[\sin(x)]}{[\sin(x)]^2}$$
$$= \frac{\sin(x)(0) - 1(\cos(x))}{[\sin(x)]^2}$$
$$= \frac{-\cos(x)}{[\sin(x)]^2}$$
$$= -\frac{1}{\csc(x)}\frac{\cos(x)}{\sin(x)}$$
$$= -\csc(x)\cot(x)$$

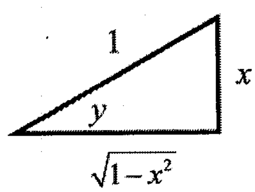
shortcut

$$\boxed{\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)}$$

We also need to know the derivatives of the inverse trig functions.

Here is a clever way we can derive the derivative for the arcsin(x) function:

Find y', if y = arcsin(x).



$$\sin(y) = x$$
$$\frac{d}{dx}[\sin(y)] = \frac{d}{dx}[x]$$
$$\cos(y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\left(\frac{\sqrt{1-x^2}}{1}\right)} = \boxed{\frac{1}{\sqrt{1-x^2}}}$$

...the other 5 inverse trig function derivatives are similarly derived...

$$\frac{d}{dx}[c]=0$$

$$\frac{d}{dx}[x^n]=nx^{n-1}$$

$$\frac{d}{dx}[e^x]=e^x$$

$$\frac{d}{dx}[a^x]=a^x\ln(a)$$

$$\frac{d}{dx}[\ln(x)]=\frac{1}{x}$$

$$\frac{d}{dx}[\log_b(x)]=\frac{1}{x\ln(b)}$$

$$\frac{d}{dx}[\sin(x)]=\cos(x)$$

$$\frac{d}{dx}[\cos(x)]=-\sin(x)$$

$$\frac{d}{dx}[\tan(x)]=\sec^2(x)$$

$$\frac{d}{dx}[\sec(x)]=\sec(x)\tan(x)$$

$$\frac{d}{dx}[\csc(x)]=-\csc(x)\cot(x)$$

$$\frac{d}{dx}[\cot(x)]=-\csc^2(x)$$

$$\frac{d}{dx}[\sin^{-1}(x)]=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\tan^{-1}(x)]=\frac{1}{1+x^2}$$

$$\frac{d}{dx}[\cos^{-1}(x)]=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\cot^{-1}(x)]=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}[\sec^{-1}(x)]=\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\csc^{-1}(x)]=\frac{-1}{|x|\sqrt{x^2-1}}$$

(rarely used)

Memorize these!

Also memorize the derivative properties/rules

$$\frac{d}{dx}[cf(x)]=c\frac{d}{dx}[f(x)]$$ **constants can move in/out of derivatives**

$$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)]$$
$$\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}[f(x)]-\frac{d}{dx}[g(x)]$$ **sum and difference rules**

$$\frac{d}{dx}[f(x)g(x)]=f(x)g'(x)+g(x)f'(x)$$ **product rule**

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$ **quotient rule**

should also know... $\left(\tan(x)=\frac{\sin(x)}{\cos(x)}\ \ \textit{and}\ \ \sin^2(x)+\cos^2(x)=1\right)$

Higher-order Derivatives

Because the derivative of a function produces another function, you can then take the derivative of that derivative function. This is called a **second-derivative**, and this process can repeat to create third or other **higher-order derivatives**.

Notations

original function	$y=f(x)$
first derivative	$y',\ f'(x),\ \frac{dy}{dx},\ \frac{d}{dx}[f(x)]$
second derivative	$y'',\ f''(x),\ \frac{d^2y}{dx^2},\ \frac{d^2}{dx^2}[f(x)]$
third derivative	$y''',\ f'''(x),\ \frac{d^3y}{dx^3},\ \frac{d^3}{dx^3}[f(x)]$
fourth derivative	$y^{(4)},\ f^{(4)}(x),\ \frac{d^4y}{dx^4},\ \frac{d^4}{dx^4}[f(x)]$
n-th derivative	$y^{(n)},\ f^{(n)}(x),\ \frac{d^ny}{dx^n},\ \frac{d^n}{dx^n}[f(x)]$

There is also a theorem about derivatives of inverse functions (sometimes needed on AP Exam)...

This theorem doesn't seem intuitive, but is easy to derive if you need it using the Chain Rule:

If $f(x)$ *and* $g(x)$ are <u>inverses</u> of each other, then: $f(g(x))=x$

Differentiate on both sides (Chain Rule on left): $\frac{d}{dx}[f(g(x))]=\frac{d}{dx}[x]$

$$f'(g(x))\,g'(x)=1$$

Solve for g'(x): $g'(x)=\dfrac{1}{f'(g(x))}$

$$\boxed{\text{If } f(x) \text{ and } g(x) \text{ are inverses of each other, then: } g'(x)=\dfrac{1}{f'(g(x))}}$$

Examples

#1. *If* $f(x)=x^3+x$ *and* $g(x)=f^{-1}(x)$ *and* $g(2)=1$ *what is the value of* $g'(2)$?

$g'(x)=\dfrac{1}{f'(g(x))}$ $\quad f'(x)=3x^2+1$

$g'(2)=\dfrac{1}{f'(g(2))}$ $\quad g(2)=1$

$f'(g(2))=3(1)^2+1=4$

$$\boxed{g'(2)=\frac{1}{4}}$$

#2. Find $f'(x)$ and $f''(x)$: $f(x)=\dfrac{\sin x}{1+\cos x}$

$f'(x)=\dfrac{(1+\cos x)(\cos x)-\sin x(-\sin x)}{(1+\cos x)^2}$

$=\dfrac{1+\cos^2 x+\sin^2 x}{(1+\cos x)^2}$ $\quad \cos^2 x+\sin^2 x=1$

$\boxed{f'(x)=\dfrac{2}{(1+\cos x)^2}}=2(1+\cos x)^{-2}$

$f''(x)=-4(1+\cos x)^{-3}(-\sin x)$

$$\boxed{f''(x)=\frac{4\sin x}{(1+\cos x)^3}}$$

Examples

#3. Find the equation of the tangent line to the curve

$$y = x + \cos x \quad \text{at} \quad (0,1)$$

$y' = 1 - \sin x$

$y'(0) = 1 - \sin 0 = 1 = m$

$$\boxed{(y-1) = 1(x-0)}$$

#4. *Find* y' *for* $y = \ln(t^2+4) - \frac{1}{2}\arctan\left(\frac{t}{2}\right)$

$$\boxed{y' = \frac{1}{t^2+4}(2t) - \frac{1}{2}\frac{1}{1+\left(\frac{t}{2}\right)^2}\left(\frac{1}{2}\right)}$$