AP Calc BC – Lesson Notes – Unit 2: Limits

Unit 2-1: Average vs. Instantaneous Velocity, Derivatives and Tangents
Larsen: 1.1 (Stewart: 2.1)

Algebra/Precalculus vs. Calculus

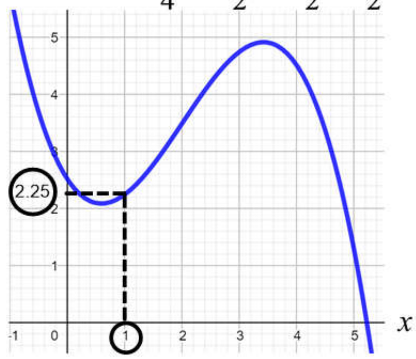
Algebra/Precalculus is great for determining the value of a function at a given input value...

$$f(x) = -\frac{1}{4}x^3 + \frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{2}$$

$$f(1) = -\frac{1}{4}(1)^3 + \frac{3}{2}(1)^2 - \frac{3}{2}(1) + \frac{5}{2}$$

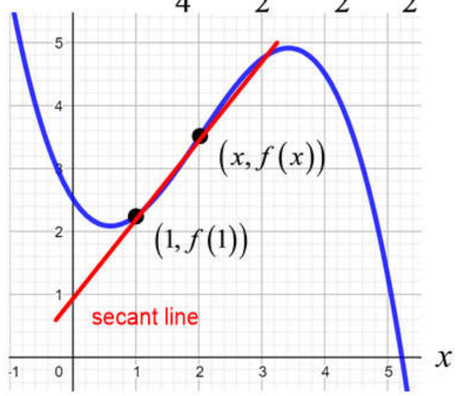
$$f(1) = \frac{9}{4} = 2.25$$

$$f(x) = -\frac{1}{4}x^3 + \frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{2}$$



But if we want to know *how fast the function is increasing* (the rate of change of the function) at a given value, the best we can do with precalculus/algebra is to find the **average rate of change**

$$f(x) = -\frac{1}{4}x^3 + \frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{2}$$



(a secant line intersects a curve in two places)

$$average\ rate\ of\ change = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\underset{at\ x=1}{average\ rate\ of\ change} = \frac{f(x) - f(1)}{x - 1}$$

If we change x, we change where the 2nd point is located, and we get a different average rate of change.

If we choose $x = 2$: $$\underset{at\ x=1}{average\ rate\ of\ change} = \frac{f(2) - f(1)}{2-1} = \frac{3.5 - 2.25}{2-1} = \frac{1.25}{1} = \boxed{1.25}$$

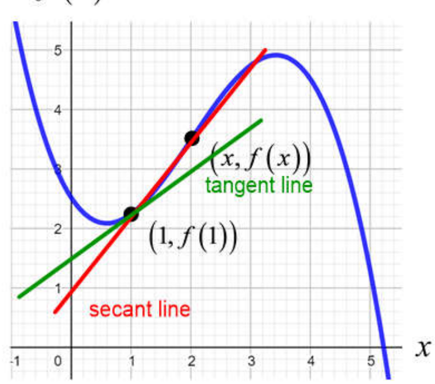
and this is the slope of the secant line

But if we look at how fast the curve is increasing at x=1 by drawing a **tangent line** touching the curve at x=1 and having the same rate of change as the curve at this point, the slope of the tangent line is not the same as the slope of the average rate of change secant line.

The slope of the tangent line represents how fast the curve is changing at exactly x=1, which is called the **instantaneous rate of change**.

The secant line represents an approximation to the tangent line, and the closer we choose x to 1, the closer the secant line will be to the tangent line.

$f(x)$



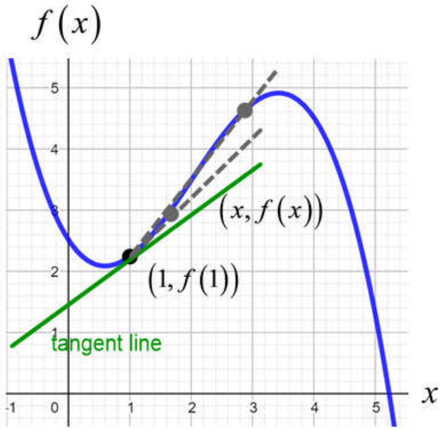
(a secant line intersects a curve in two places)

(a tangent line intersects a curve at one place and is in the same direction as the curve at this point)

Algebra/Precalculus vs. Calculus

We can use the idea of a **limit** to express the concept of moving our point $(x, f(x))$ closer and closer to the point $(1, f(1))$ but not quite touching this point (because we don't want the denominator of the slope calculator to go all the way to zero):

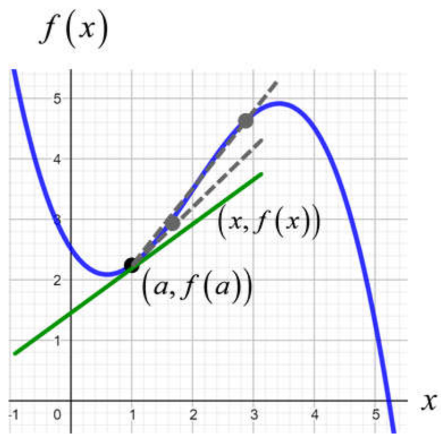
$$f'(x) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$



By using the calculation for average rate of change, but taking a limit as x moves closer and closer to 1, we get the **instantaneous rate of change**, which is also called **the derivative of the function at 1**.

We could do this calculation for any input value, so instead of specifying it, say, as 1, ahead of time, we can use a constant *a*:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

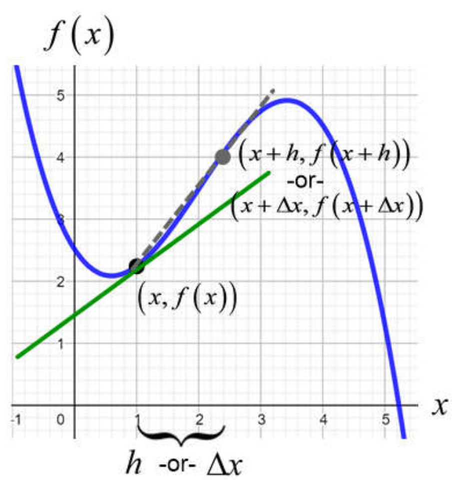


By using the calculation for average rate of change, but taking a limit as x moves closer and closer to *a*, we get the **instantaneous rate of change**, which is also called **the derivative of the function at a**.

Another way this is expressed is by using the variable *x* as the input value where we want to find the instantaneous rate of change and indicating the 2nd point for the secant line as being a distance away from the *x* value. This distance is sometimes notated as h or as :

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



By using the calculation for average rate of change, but taking the limit as the difference h *or* Δx approaches zero, we get the **instantaneous rate of change at x**, which is also called **the derivative of the function at x**.

Average Velocity and Instantaneous Velocity

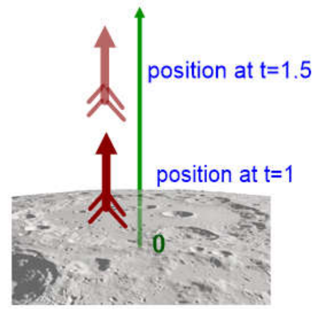
These concepts of average rate of change and instantaneous rate of change are particularly useful when we talk about how fast an object is moving. We'll use an example to illustrate.

Ex: If an arrow is shot upward on the moon with a velocity of 58 m/s, its height in meters after t seconds is given by $h(t) = 58t - 0.83t^2$

(a) Find the average velocity over [1, 1.5].

(b) Find the instantaneous velocity after one second.

First, some definitions...



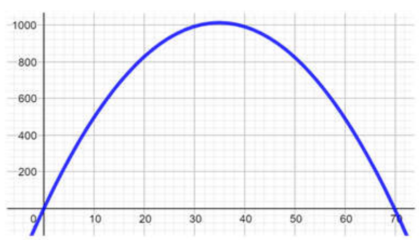
Frame of Reference: A physics term meaning that we define the direction we consider 'positive motion' along with a place considered the zero of the coordinate system. Here, let's define 'upward' to be positive and zero to be the surface of the moon:

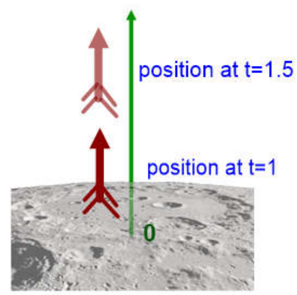
Velocity: The rate of change of a position function in a specified direction. In this course, motion will be in 1-dimension (take Honors CalcIII next year if you want to know how we handle arbitrary motion :) So velocity is the instantaneous rate of change of a function which specified position as a function of time. Velocity can be positive (in the direction of positive motion) or negative (in the reverse direction).

Speed: The magnitude (absolute value) of the velocity. Speed can never be negative.

Ex: If an arrow is shot upward on the moon with a velocity of 58 m/s, its height in meters after t seconds is given by $h(t) = 58t - 0.83t^2$

It we graphed the given position function vs. time...





...we see that, even on the moon, the arrow would eventually reach some maximum height, then fall back to the surface of the moon, but that takes a long time (70 seconds).

(a) Find the average velocity over [1, 1.5].

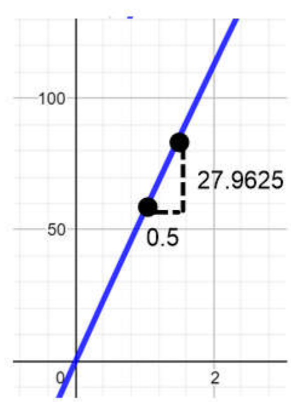
This question is asking about what happens in the first few seconds of flight and in this region, the position curve is almost a straight line:

For **<u>average velocity</u>** we use **<u>algebra</u>**, and just find the change in y over the change in x for a defined x interval:

average velocity over [1, 1.5] = $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta h}{\Delta t} = \dfrac{h(1.5) - h(1)}{1.5 - 1}$

$$= \frac{85.1325 - 57.17}{1.5 - 1} = \frac{27.9625}{0.5}$$

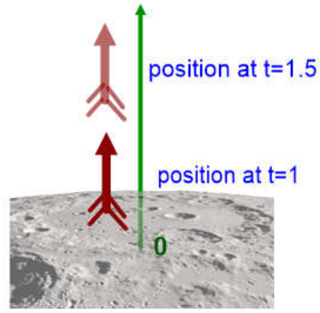
$$\boxed{= 55.925 \frac{m}{s}}$$



Average Velocity and Instantaneous Velocity

Ex: If an arrow is shot upward on the moon with a velocity of 58 m/s, its height in meters after *t* seconds is given by $h(t) = 58t - 0.83t^2$

(b) Find the instantaneous velocity after one second.



For **instantaneous velocity** we need to use **calculus**, specifically, we need to find the limit of the expression for average velocity as t approaches t = 1:

instantaneous velocity after one second $= \lim_{t \to 1} \dfrac{f(t) - f(1)}{t - 1}$

If we select a value for *t* which is above 1 second (but closer to 1 than 1.5 seconds) and then recompute the average velocity, this will be a closer estimate to the instantaneous velocity:

$$\textit{for } t = 1.1: \quad \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{62.7957 - 57.17}{0.1} = \frac{5.6257}{0.1} = 56.257$$

$$\textit{for } t = 1.01: \quad \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{57.733317 - 57.17}{0.01} = \frac{0.563317}{0.01} = 56.3317$$

$$\textit{for } t = 1.001: \quad \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{57.22633917 - 57.17}{0.001} = \frac{0.05633917}{0.001} = 56.33917$$

We are getting closer and closer to the actual instantaneous velocity.

$$\lim_{t \to 1} \frac{f(t) - f(1)}{t - 1} \approx 56.33917 \frac{m}{s}$$

(In later sections, we will learn more efficient ways of computing limits than plugging in numbers closer and closer to the target *x* value.)

Can find average or instantaneous rate of change for any function

The idea of average rate of change and instantaneous rate of change is applicable to all functions, not just position functions. We just don't have a word like 'velocity' to name this rate of change. Here is another example to illustrate:

Ex. A piece of chocolate is pulled from a refrigerator $\left(6^{\circ}C\right)$ and placed on a counter $\left(22^{\circ}C\right)$. The temperature of the chocolate is given by:

min	0	4	8	12	16	20	24	28	32	36
temp	6.00	9.87	12.81	15.04	16.72	18.00	18.97	19.70	20.26	20.68

a) What is the average rate of change in the temperature of the chocolate from 8 to 20 minutes?

b) Estimate the instantaneous rate of change of temperature at time equals 22 minutes.

4. The point $P(0.5, 2)$ lies on the curve $y = 1/x$.
If Q is the point $(x, 1/x)$, use your calculator to find the slope of the secant line PQ.

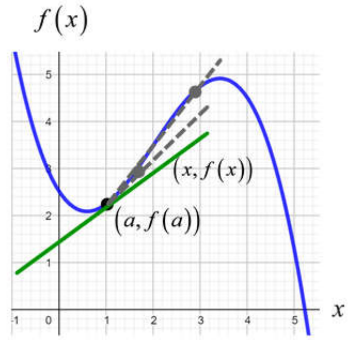
iv) 1.8 viii) 0.51 ix) 0.45

Unit 2-2: Finding Limits Graphically and Numerically
Larsen: 1.2/1.4 (Stewart: 2.2)

The idea of a limit

In the last section, we said to find instantaneous rate of change of a function at a given *x* value, *a*, we needed to select another point on the curve (*x*) and then bring this point closer and closer to *a*:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



We use a **limit** to carry out this 'closer and closer' idea. Here is a more formal definition of a limit:

$$\lim_{x \to c} f(x) = N$$

Read:
"The limit of f of x as x approaches c equals the number N."

This means:
For all x approximately equal to c, but not equal to c, the value f(x) is approximately equal to N.

The 3 methods for evaluating limits

There are 3 ways to evaluate a limit. Let's evaluate this limit using all 3 methods:

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$$

1) **Numerically**: This what we did in the last section - plug numbers into the function getting closer and closer to the target *x*-value. Note: you must plug in numbers from both below and above the number, because they may not result in the same value.

Seems to be approaching

From the left (from lower values):

x	4	4.5	4.9	4.99	4.999
$\frac{x^2-25}{x-5}$	9.0	9.5	9.9	9.99	9.999

$$\lim_{x \to 5^-} \frac{x^2 - 25}{x - 5} =$$

From the right (from higher values):

x	6	5.5	5.1	5.01	5.001
$\frac{x^2-25}{x-5}$	11.0	10.5	10.1	10.01	10.001

$$\lim_{x \to 5^+} \frac{x^2 - 25}{x - 5} =$$

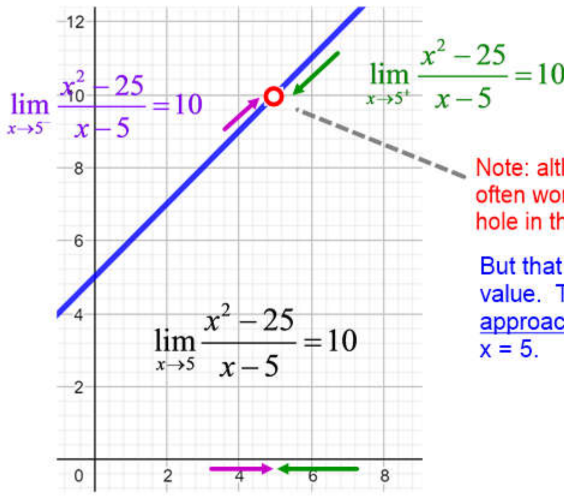
$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5} =$$

The 3 methods for evaluating limits

There are 3 ways to evaluate a limit. Let's evaluate this limit using all 3 methods:

$$\lim_{x\to 5}\frac{x^2-25}{x-5}$$

2) **<u>Graphically</u>**: If you graph the function, you can visually see what number is being approached from the left and the right:



Note: although graphing technology often won't show it, there is actually a hole in the domain right at x = 5.

But that doesn't affect the limit value. The limit is the value being <u>approached</u>, not the value right at x = 5.

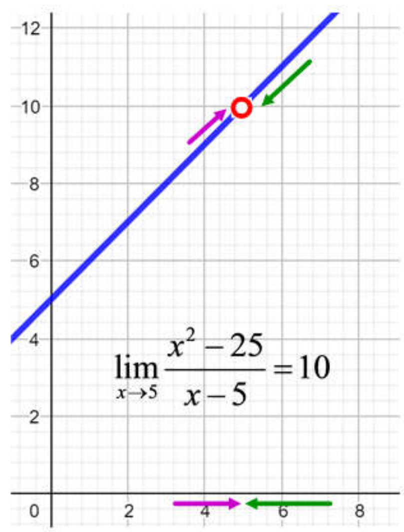
3) **<u>Analytically</u>**: Since there is a single *y* value that is being approached (10) we may be able to find this value by simply plugging the target *x* value into the function:

If we just plug 5 in for x...

$$\frac{(5)^2-25}{(5)-5}=\frac{0}{0}$$

...the function is undefined (because x=5 is not in the domain of the function)

But the graph doesn't look like there is a 'problem' at x = 5 (there is no vertical asymptote here). So before we plug in x=5, we can try any algebraic simplification techniques we can think of to find an equivalent algebraic expression for the function. Here, factoring would work:



$$\frac{x^2-25}{x-5}=\frac{(x-5)(x+5)}{(x-5)}=x+5$$

$$\lim_{x\to 5}\frac{x^2-25}{x-5}=\lim_{x\to 5}x+5=(5)+5=10$$

This is the fastest and most convenient way to find a limit, so we usually try this first and only resort to using graphing or numerical methods when needed (or when directed to by the problem). In the next section, we will learn additional algebraic simplifying methods we can use besides factoring before plugging in.

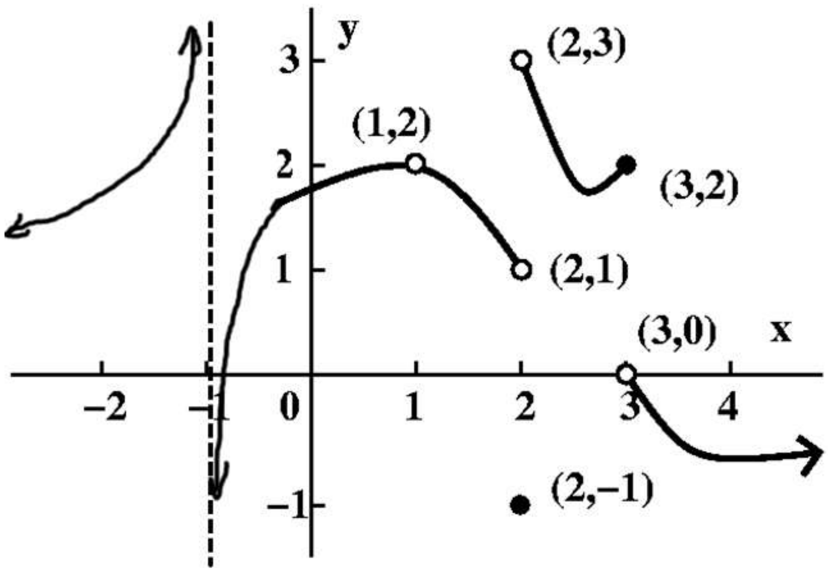
Some limits do not exist

For a given function, it is possible that for every *x* there will be a limit for the function at that *x*. But it is very common for functions to have some *x* values for which there is no limit (the limit does not exist).

Things that cause a limit not to exist at a given *x* value:

1) If the value being approach from either left or right is not a number (is, for example, infinity because we are approaching a vertical asymptote).

2) If the values being approached from the left and right are both numbers, but they are not the same number.



$$\lim_{x\to 2^-} f(x)$$

$$\lim_{x\to 2^+} f(x)$$

$$\lim_{x\to 2} f(x)$$

$$f(2)$$

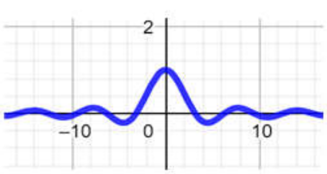
$$\lim_{x\to -1^+} f(x)$$

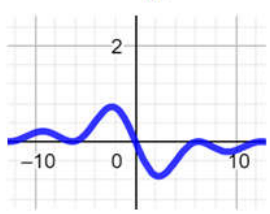
$$\lim_{x\to -1} f(x)$$

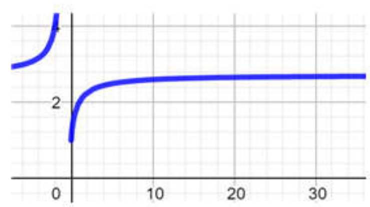
Evaluating Limits by using the 'special limits'

In some cases, we can rearrange an expression to factor it into one or more 'special limit' form where we have 'pre-evaluated' the limit using a graph...these special limits are:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0 \qquad \lim_{x\to \infty} \left(1 + \frac{1}{x}\right)^x = e$$



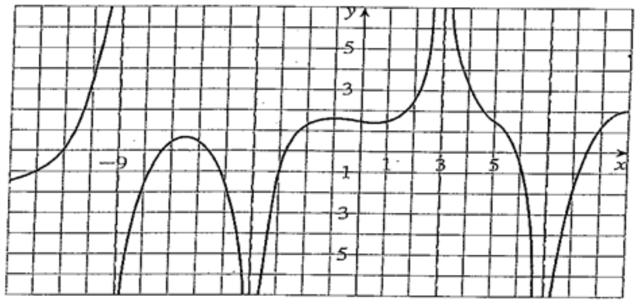




(You need to memorize these 3 special limit values)

Example: Evaluate the limit $\lim_{x\to 0} \dfrac{\tan(3x)}{\tan(5x)}$

Examples



For the function f whose graph is shown, state the following:

a) $\lim\limits_{x \to 3} f(x)$ b) $\lim\limits_{x \to 7} f(x)$ c) $\lim\limits_{x \to -4} f(x)$

d) $\lim\limits_{x \to -9^-} f(x)$ e) $\lim\limits_{x \to -9^+} f(x)$

f) The equations of the vertical asymptotes

15. Evaluate the function at the given numbers. Use the results to guess the value of the limit, or explain why it does not exist.

$g(x) = \dfrac{x-1}{x^3-1}$;

$x = 0.2, 0.4, 0.6, 0.8, 0.9, 0.99, 1.8, 1.6, 1.4, 1.2, 1.1, 1.01$

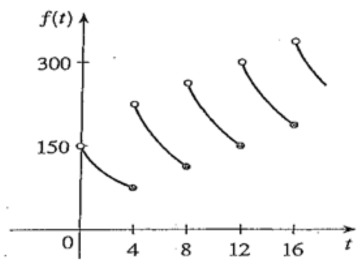
$\lim\limits_{x \to 1} \dfrac{x-1}{x^3-1}$

Numerically... Analytically...

10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. Find

$$\lim_{t \to 12^-} f(t) \text{ and } \lim_{t \to 12^+} f(t)$$

and explain the significance of these one-sided limits.



14. Sketch the graph of an example of a function that satisfies all of the given conditions.

$$\lim_{x \to 0^-} f(x) = 1, \quad \lim_{x \to 0^+} f(x) = -1, \quad \lim_{x \to 2^-} f(x) = 0$$

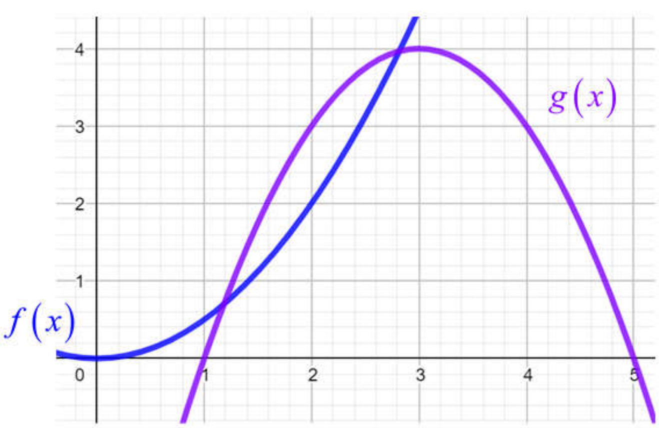
$$\lim_{x \to 2^+} f(x) = 1, \quad f(2) = 1, \quad f(0) \text{ is undefined}$$

Evaluate the limit $\lim\limits_{x\to(\pi/2)^-} \sec x$

50. Find the limit $\lim\limits_{x\to1^-} f(x)$ and $\lim\limits_{x\to1^+} f(x)$ for the function

$$f(x)=\begin{cases} 3x-1 & \text{if} \quad x<1 \\ 2 & \text{if} \quad x=1 \\ 3x & \text{if} \quad x>1 \end{cases}$$

Does $\lim\limits_{x\to1} f(x)$ exist?



Evaluate the limit

$$\lim_{x\to2}\frac{2f(x)+1}{4-g(x)}$$

Special Limits that Students Need to Memorize

The four following special limits are not special because of the warm way they make you feel all giddy inside. By "special," I really mean they cannot be evaluated by the means we've discussed so far, but yet you will see them frequently so you should probably memorize them, even though that stinks. Now that we're on the same page, so to speak, here they are with no further ado.

$$\lim_{x\to 0}\frac{\sin\alpha}{\alpha}=1$$

This is only true when you approach zero, so don't use this formula under any other circumstance. The α can be any quantity. This limit is used the most so many of the trig limit problems are simply about doing some algebraic manipulation so that you can introduce this form.

$$\lim_{x\to 0}\frac{\cos\alpha-1}{\alpha}=0$$

Just like the first special limit, this is only true when approaching zero. Sometimes, you'll also see this formula written as $\frac{1-cos\alpha}{\alpha}$; the limit is still 0 either way.

$$\lim_{x\to\infty}\frac{any\ real\ number}{x^{any\ integer\ \geq 1}}=0$$

If any real number is divided by *x*, and we let that *x* get infinitely large, the result is 0. Think about that – it makes good sense. What is 4 divided by 900 kajillion? Who knows, but it is definitely very, very small. So small, in fact, that it's basically 0.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$$

This basically says that 1 plus an extremely small number, when raised to an extremely high power, is exactly equal to Euler's number (2.71828…). You will see this very infrequently, but it's important to recognize it when you do.

Infinite Limits:

Infinite limits are very easy. You are usually asked to find $\lim_{x\to\infty}\frac{P(x)}{Q(x)}$ for some rational function. All you need to do is look at the degree of $P(x)$ and the degree of $Q(x)$. This is what can happen:

1. If the degree of $P(x)$ is > the degree of $Q(x)$ the limit is infinite.
2. If the degree of $P(x)$ is < the degree of $Q(x)$ the limit is zero.
3. If the degree of $P(x)$ is = the degree of $Q(x)$ the limit is the ratio of the coefficients of the highest order terms.

So, most of the time you should know these limits on sight.

Examples:

$$\lim_{x\to\infty}\frac{x^3-2x+3}{x^2+1}=\infty \qquad \lim_{x\to\infty}\frac{2x^2+1}{3x^2-5}=\frac{2}{3} \qquad \lim_{x\to\infty}\frac{x^3+2x}{x^5-1}=0$$

Unit 2-3: Finding Limits Analytically
Larsen: 1.3 (Stewart: 2.3)

You can simplify the function expression before 'plugging in'

In the last section, the 3rd method for evaluating a limit was to just try 'plugging in' the x into the expression. More formally, this is known as '**direct substitution**'.

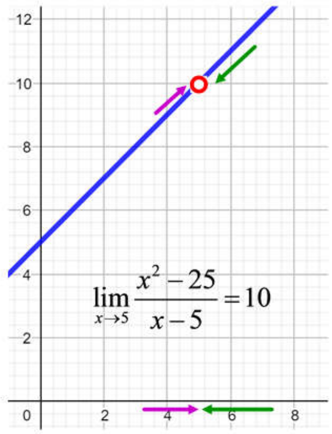
$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$$

If we just plug 5 in for x...

$$\frac{(5)^2 - 25}{(5) - 5} = \frac{0}{0}$$

...the function is undefined (because x=5 is not in the domain of the function)

But the graph doesn't look like there is a 'problem' at x = 5 (there is no vertical asymptote here). So before we plug in x=5, we can try any algebraic simplification techniques we can think of to find an equivalent algebraic expression for the function. Here, factoring would work:



$$\frac{x^2 - 25}{x - 5} = \frac{(x-5)(x+5)}{(x-5)} = x + 5$$

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} x + 5 = (5) + 5 = 10$$

Factoring and canceling before plugging in is the most common way to simplify before evaluating analytically. But there are also other tactics...

Tactics for simplifying before evaluating a limit by direct substitution

1) **Factoring/Dividing Out** This tactic works when you can factor the numerator and/or denominator of a rational function and cancel the term in the denominator which is causing the denominator to go to zero. For higher-order polynomials, **synthetic division** is helpful to carry out the factorization.

2) **Rationalizing** This tactic works when you have a rational function containing a radical. If you multiply numerator and denominator by the conjugate (of whichever side of the fraction contains the radical), then the simplification often results in eliminating the factor in the denominator which causes divide-by-zero.

3) **Using special limits** This tactic works when you have specific forms which match memorized limits. The most common is $\lim_{x \to 0} \frac{\sin ax}{ax} = 1$

Properties of Limits

Our textbook defines a number of useful properties about limits. Here is a sample of the most important ones:

$\lim_{x \to c} b = b$ If you are taking a limit of a constant, the limit is the constant for limits at any x.

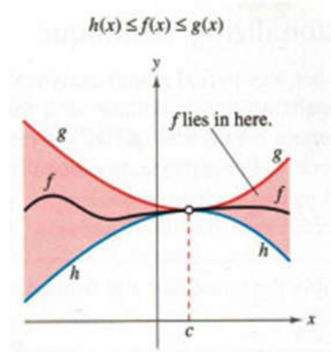
$\lim_{x \to c} x = c$ You can plug in the x-value to the function to determine the limit (if well-behaved).

$$\lim_{x \to c} \left[f(x) \pm g(x) \right] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

$$\lim_{x \to c} \left[f(x) \cdot g(x) \right] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$$

When evaluating limits with multiple functions, you can combine the results of the separate limit calculations algebraically. This also applies to algebraic operations on a single limit...compute the limit, and apply the algebraic operations.

The Squeeze Theorem



Sometimes it is difficult to evaluate a limit directly, but you can show that the function's value in an interval is always between two other functions whose limits are easier to evaluate. If these other two functions evaluate to the same value, then the given function in the middle is 'squeezed' in between, and must also have the same limit.

Here is the formal theorem:

> ***If*** $h(x) \le f(x) \le g(x)$ ***for all*** x ***in an open interval containing*** c***,***
> ***except possibly at*** c ***itself, and if***
> $$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$
> ***then*** $\lim_{x \to c} g(x)$ ***exists and is equal to*** L***.***

Examples

16. $\displaystyle \lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$

11. $\displaystyle \lim_{x \to -3} \frac{x^2 - x + 12}{x + 3}$

18. $\lim_{h\to 0}\frac{(2+h)^3-8}{h}$

23. $\lim_{x\to 9}\frac{x^2-81}{\sqrt{x}-3}$

44. Let $f(x)=\begin{cases} x^2-2x+2 & \text{if } x<1 \\ 3-x & \text{if } x\geq 1 \end{cases}$

a) Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

b) Does $\lim_{x\to 1} f(x)$ exist?

c) Sketch the graph of *f*.

31. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^2 \cos 20\pi x = 0$.

Illustrate by graphing the functions

$f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$, and $\mathrm{h}(\mathrm{x})=x^2$

on the same screen.

39. Find the limit, if it exists.
If the limit does not exist, explain why.

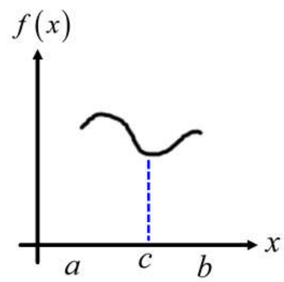
$$\lim_{x \to 2} \frac{|x-2|}{x-2}$$

28. $\lim_{x \to 1} \frac{\sqrt{x} - x^2}{1 - \sqrt{x}}$

Unit 2-4: Continuity and One-sided Limits
Larsen: 1.4 (Stewart: 2.5)

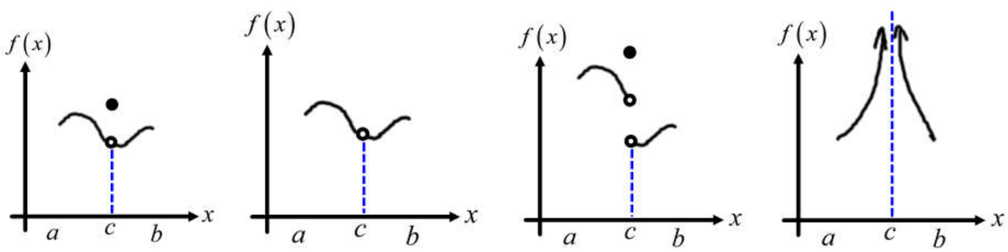
Definition of Continuity of a Function

A function can be determined to be **continuous** or **discontinuous** at every *x*-value in its domain. The general idea of continuity is that a function is continuous at an *x*-value if, as you are drawing the function from left to right your pencil stays on the paper as you go through this *x*-value:



This function is said to be **continuous over the interval (a, b)** because, for every value *c* in the interval, there is no 'break' in the function.

A function can be discontinuous at an input value in various ways which are labelled as follows:



jump discontinuity

infinite discontinuity

removable discontinuities

nonremovable discontinuities

A discontinuity is called '**removable**' if the function can be made continuous by appropriately defining or re-defining *f* at *c*. (If you could change things so that the 'hole is filled in'.)

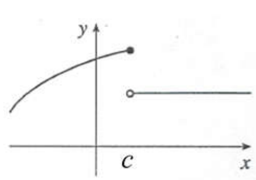
Verifying/Proving the Continuity of a Function at *x* = *c*

Conditions for a Function to Be Continuous at c
To summarize, a function f is continuous at c provided that three conditions are met:

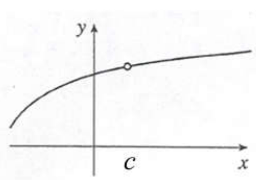
Condition 1 $f(c)$ is defined;
that is, c is in the domain of the function

Condition 2 $\lim_{x \to c} f(x)$ exists

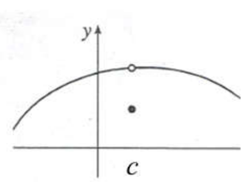
Condition 3 $\lim_{x \to c} f(x) = f(c)$



f is discontinuous at *c*
(violates condition 2)



f is discontinuous at *c*
(violates condition 1)



f is discontinuous at *c*
(violates condition 3)

Verifying/Proving the Continuity of a Function at *x* = *c*

Conditions for a Function to Be Continuous at c
To summarize, a function f is continuous at c provided that three conditions are met:

Condition 1 $f(c)$ is defined;
that is, c is in the domain of the function

Condition 2 $\lim_{x \to c} f(x)$ exists

Condition 3 $\lim_{x \to c} f(x) = f(c)$

Determine if the function is continuous at *a*:

$$f(x) = \ln|x-2| \qquad a = 2$$

Determine if the function is continuous at *a*:

$$f(x) = \begin{cases} \dfrac{x^2 - 2x - 8}{x-4} & \text{if } x \neq 4 \\ 3 & \text{if } x = 4 \end{cases} \qquad a = 4$$

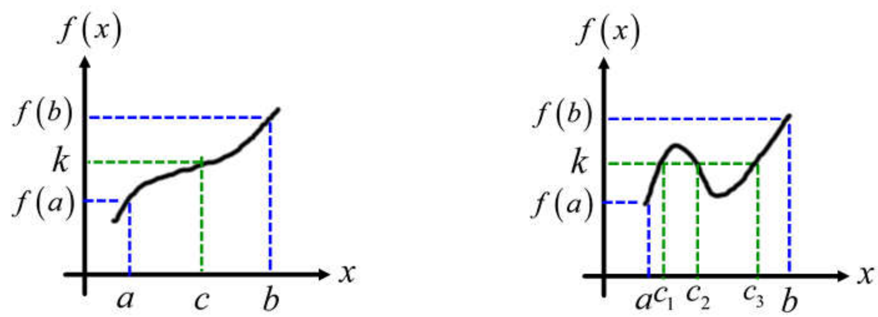
Adjusting a function to make it continuous

40. Find the constant c that makes g continuous on $(-\infty, \infty)$

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases}$$

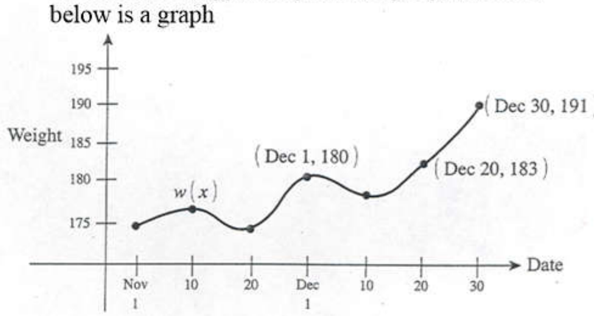
The Intermediate Value Theorem

If f is continuous on the closed interval $[a,b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c) = k$.



Note: This theorem doesn't provide a method for finding the value(s) *c*, and doesn't indicate the number of *c* values which map to *k*, it only guarantees the existence of at least one number *c* such that f(c) = k.

example:
A man enjoyed a little too much holiday dining during the winter months of November and December. Exaggerating his weight gain a little, below is a graph



From the graph, the man weighed 180 pounds on December 1, and 191 pounds on December 30. Comparing this to the Intermediate Value Theorem, $a =$ Dec 1, $b =$ Dec 30.
$f(a) = w(Dec\ 1) = 180$,
and $f(b) = w(Dec\ 30) = 191$.

According to the theorem, we can choose any value between 180 and 191 (183, for example) and be guaranteed that at some time between December 1 and December 30, the man actually weighed that much.

The Intermediate Value Theorem

Example: Use the Intermediate Value Theorem to show that the following polynomial has a zero in the interval [0,1].

$$f(x) = x^3 + 2x - 1$$

More properties of limits

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$$

If you are taking the limit of a composition of functions, you can take the limit of the inner function and then apply the outer function to the result.

$$\lim_{x \to 0} \sqrt{\frac{1}{x^2}} = \sqrt{\lim_{x \to 0}\left(\frac{1}{x^2}\right)}$$

"The limit of the square root is the square root of its limit."

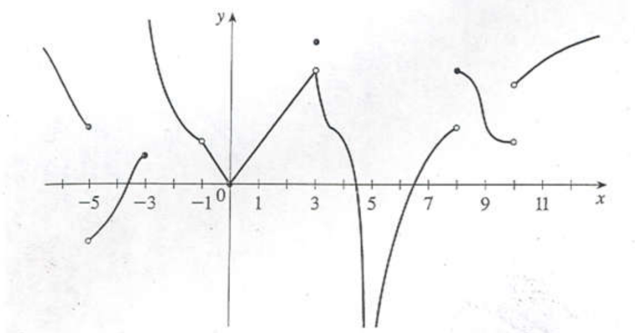
Examples

Find the limit and any asymptotes: $\lim_{x \to 5} \frac{x+2}{x-5}$

Examples

a) State the numbers at which f is discontinuous and explain why.

b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither



Explain why the function is continuous at every number in its domain. State the domain.

$$F(x) = \frac{x}{x^2 + 5x + 6}$$

Explain why the function is continuous at every number in its domain. State the domain.

$$f(t) = 2t + \sqrt{25 - t^2}$$

30. Locate the discontinuities of the function and illustrate by graphing

$$y = \ln\left(\tan^2 x\right)$$

31. Use continuity to evaluate the limit.

$$\lim_{x \to 4} \frac{5+\sqrt{x}}{\sqrt{5+x}}$$

Which of the following functions described represent continuous functions? Explain your choice.

The temperature on a specific day at a given location considered as a function of time

The selling price at ATT stock on a specific day considered as a function of time

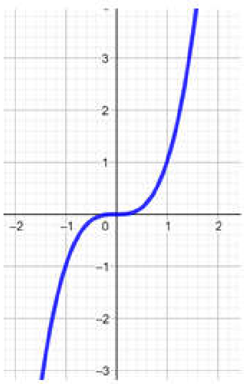
The altitude above sea level as a function of the distance due west from NYC.

Infinite Limits

An **infinite limit** is a limit that, when evaluated, is increasing without bound to $+\infty$ or decreasing without bound to $-\infty$. Technically, these limits Do Not Exist because they are not numbers, however we usually do indicate whether the value is approaching positive or negative infinity.

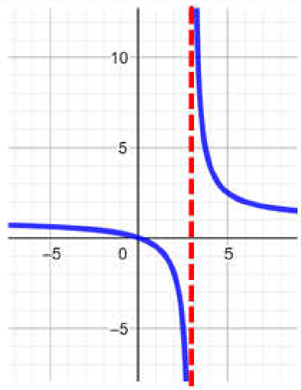
There are three general situations where we see infinite limits:

$x \to \infty \; or \; x \to -\infty$
in Polynomials



$\lim_{x\to-\infty} x^3 = -\infty \qquad \lim_{x\to\infty} x^3 = \infty$

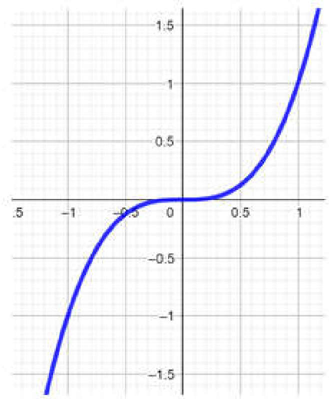
$x \to c$
in some Rational Functions
where denominator goes to zero



$\lim_{x\to3^-} \frac{x}{x-3} = -\infty \qquad \lim_{x\to3^+} \frac{x}{x-3} = \infty$

at a Vertical Asymptote
x = 3

$x \to \infty \; or \; x \to -\infty$
in Rational Functions
where numerator gets larger
faster than the denominator



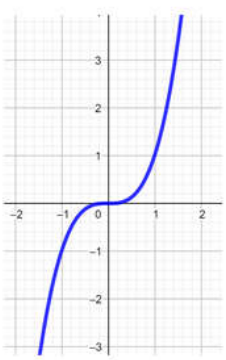
$\lim_{x\to-\infty} \frac{x^5}{x^2} = -\infty \qquad \lim_{x\to\infty} \frac{x^5}{x^2} = \infty$

Limits at Infinity

A **limit at infinity** is a limit where *x* is approaching either $+\infty$ or $-\infty$.

When evaluate limits at infinity by imagining what will happen as *x* gets very large (either in the positive or negative direction). The resulting limit value may be zero, a constant, or an infinite limit, depending upon the function:

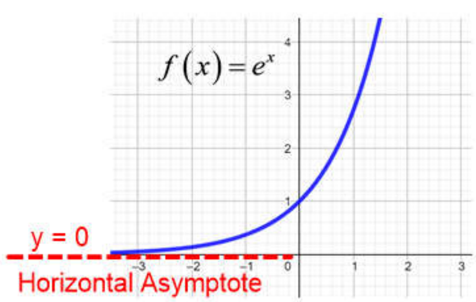
$x \to \infty \; or \; x \to -\infty$
in Polynomials



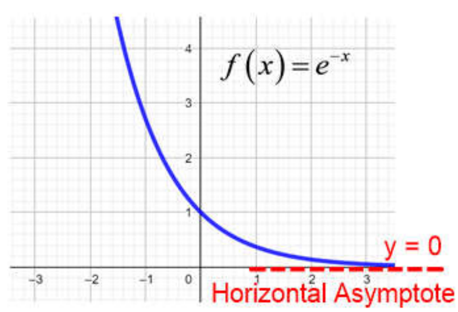
$\lim_{x\to-\infty} x^3 = -\infty \qquad \lim_{x\to\infty} x^3 = \infty$

you get an infinite limit

$x \to \infty \; or \; x \to -\infty$
in Exponentials



$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to\infty} e^x = \infty$



$\lim_{x\to-\infty} e^{-x} = \infty \qquad \lim_{x\to\infty} e^{-x} = 0$

If, as x approaches either positive or negative infinity the value of the limit approaches zero or any other constant, then there is a **horizontal asymptote** at this y value.

Evaluating limits without graphing

For many limits, you can evaluate without resorting to graphing the function...just consider what happens as x approaches the target value.

$$\lim_{x\to\infty}\frac{x^5}{e^x}$$

The numerator and denominator are both getting very large:

$\dfrac{\infty}{\infty}$ this is an **indeterminant form** (the numerator and denominator are 'fighting for control')

...in this case, the exponential increases faster than the power, so the denominator will eventually be much larger than the numerator:

$$\lim_{x\to\infty}\frac{x^5}{e^x}=0$$

$$\lim_{x\to-\infty}x^2\left(x-1\right)$$

Here, two number which are multiplied are both getting very large, but because x is negative and one is squared but the other is not, we have...

$$\infty\left(-\infty\right)$$

...so:

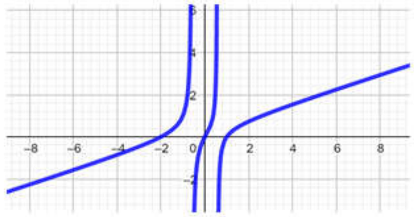
$$\lim_{x\to-\infty}x^2\left(x-1\right)=-\infty$$

Evaluating Rational Function limits

If you have a limit of a rational function with polynomials for numerator and denominator, three things can happen:

numerator degree is larger:

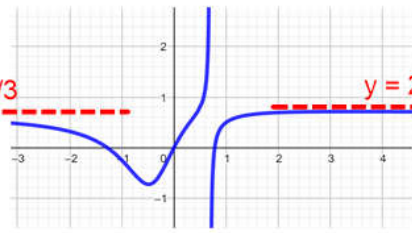
$$\lim_{x\to-\infty}\frac{x^5+x^4-2x^3}{3x^4-x^2}$$



$$\lim_{x\to\infty}\frac{x^5+x^4-2x^3}{3x^4-x^2}=\boxed{\infty}$$

numerator and denominator degrees are the same:

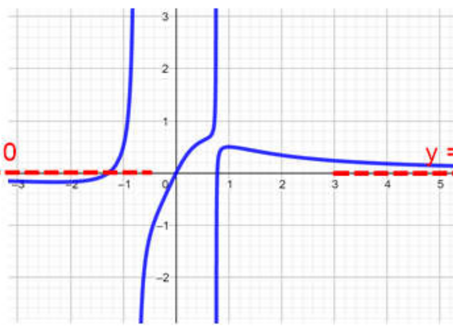
$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^5-x^2}$$



$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^5-x^2}=\boxed{\frac{2}{3}}$$

numerator degree is smaller:

$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^6-x^2}$$



$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^6-x^2}=\boxed{0}$$

Evaluating Rational Function limits without graphing

There is a procedure you can use to evaluate rational function limits without graphing.

- If numerator degree is higher than denominator, then the limit is an infinite limit (be careful of the sign).
- If denominator degree is higher than numerator, then the limit is 0.
- If the degrees of the numerator and denominator are the same, divide every term in both the numerator and denominator by $x^{\text{degree of denominator}}$. Then cancel within each term, and all terms with a constant over a power of x go to zero.

$$\lim_{x\to\infty}\frac{2x^5+x^4-2x^3}{3x^5-x^2}$$

divide everything by x^5

$$\lim_{x\to\infty}\frac{\frac{2x^5}{x^5}+\frac{x^4}{x^5}-\frac{2x^3}{x^5}}{\frac{3x^5}{x^5}-\frac{x^2}{x^5}}$$

$$\lim_{x\to\infty}\frac{2+\frac{1}{x}-\frac{2}{x^2}}{3-\frac{1}{x^3}}$$

$$\lim_{x\to\infty}\frac{2+0-0}{3-0}=\boxed{\frac{2}{3}}$$

Examples

a. $\lim_{x\to 2} f(x)$

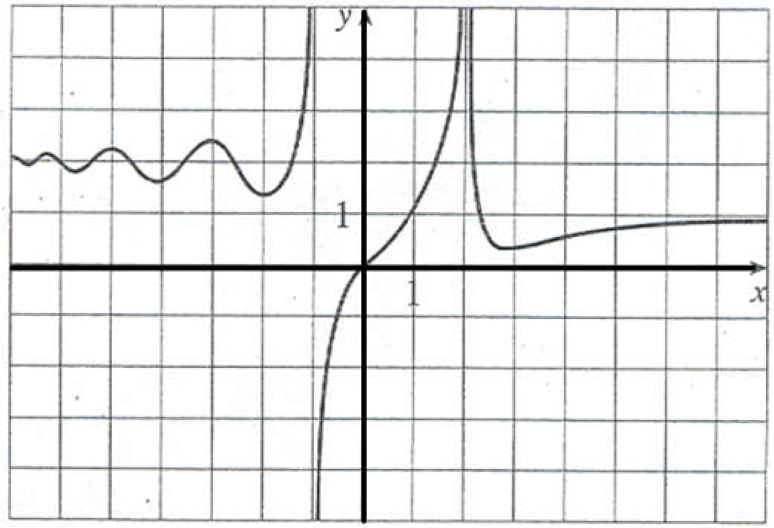
b. $\lim_{x\to -1^-} f(x)$

c. $\lim_{x\to -1^+} f(x)$

d. $\lim_{x\to \infty} f(x)$

e. $\lim_{x\to -\infty} f(x)$

f. asymptote equations



$$\lim_{r\to\infty}\frac{r^4 - r^2 + 1}{r^5 + r^3 - r}$$

$$\lim_{x\to\infty} e^{-x^2}$$

Find the horizontal and vertical asymptotes of the curve.
Check by graphing.

$$y = \frac{x^2 + 4}{x^2 - 1}$$

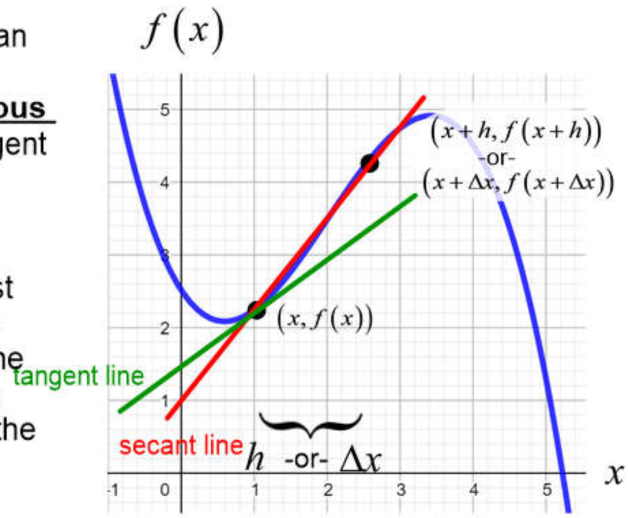
Find a formula for a function *f* that satisfies the following conditions:

$$\lim_{x \to \pm\infty} f(x) = 0 \qquad \lim_{x \to 0} f(x) = -\infty \qquad f(2) = 0$$

$$\lim_{x \to 3^-} f(x) = \infty \qquad \lim_{x \to 3^+} f(x) = -\infty$$

Tangent Line to a curve at a point

At the beginning of this unit, we said that we can ask how fast the values of a function are changing at a point by finding the **instantaneous rate of change**, which is the slope of the tangent line to the curve at this point.



We find this by first considering a secant line between two points: one at the point of interest and the other an arbitrary very small distance away. We then find the slope of the secant line and take the limit as the distance between the points gets closer to zero to find the slope of the tangent line:

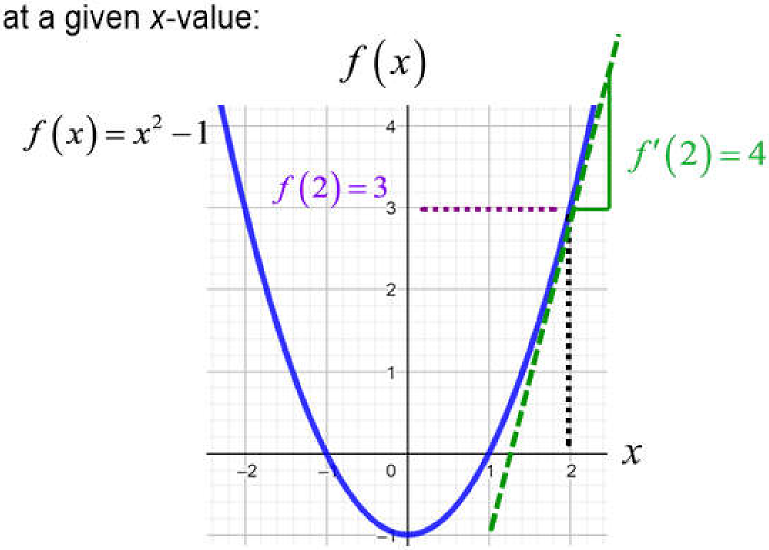
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

We now define this expression as the limit definition of the **derivative of the function at *x***.

What f(x) and f'(x) tell you about a function

It is very important that you understand the difference between what $f(x)$ and $f'(x)$ tell us about a function at a given *x*-value:



$f(x)$ gives us the **y-value** at x.

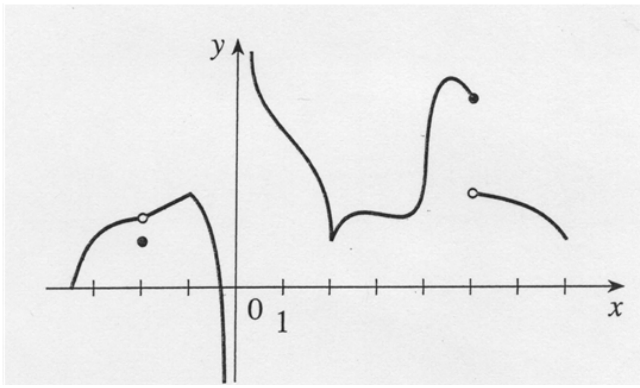
$f'(x)$ gives us the **slope of the tangent line** to the curve at x (the instantaneous rate of change of the curve at x).

Does the derivative of a function always exist?

The derivative of a function has a potentially unique value at every *x* value in the function's domain, and it is possible that for some values of *x*, the derivative will not exist. The things that can cause a function's derivative not to exist at an *x* value are:

- If the function is discontinuous at the *x* (the tangent line must have a point to attach to).
- If the function's curve shape abruptly changes directly at the *x* instead of moving smoothly through the *x* value.
- If the tangent line is perfectly vertical (slope is infinite, therefore not a number and DNE).

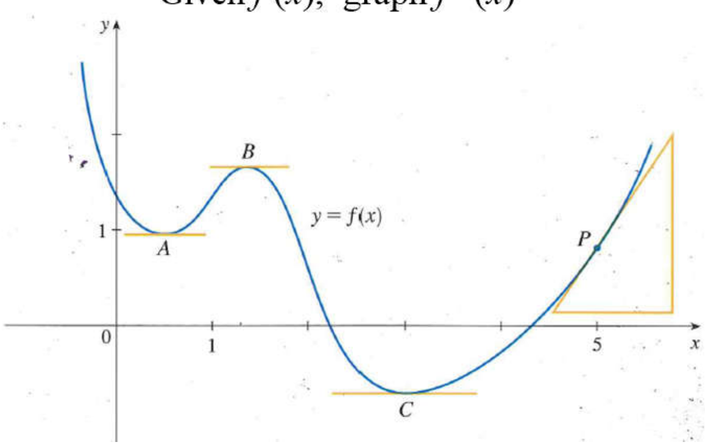
Ex: Find the x-values for which the derivative does not exist.



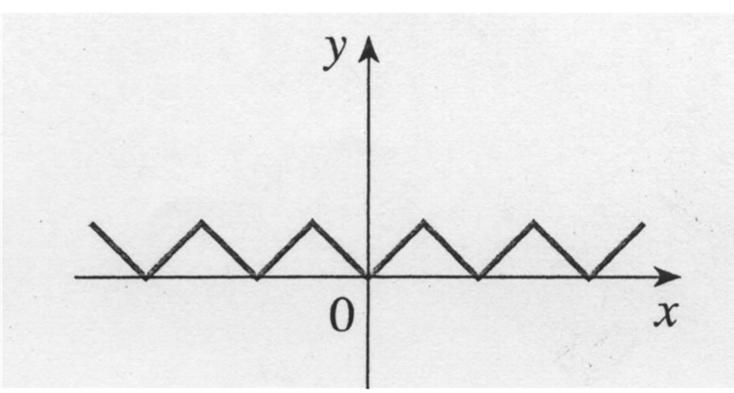
Sketching a derivative curve from a function curve

If we are given the graph of a function, we can sketch an approximate function curve for the derivative function. Remember that the value of the derivative equals the instantaneous rate of change (the 'slope') of the original function curve at that x-value.

Given $f(x)$, graph $f'(x)$

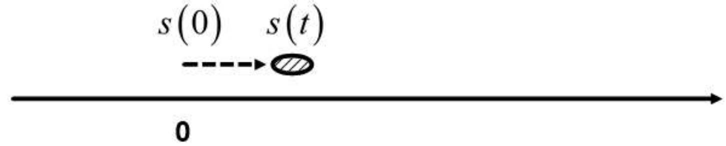


Given $f(x)$, graph $f'(x)$



For functions which represent position as a function of time, the derivative is the velocity of object motion

If you define a frame of reference (starting point and positive direction), and then have a function which gives the position of the object as a function of time...



...then the derivative of this position function with respect to time is the instantaneous rate of change of position vs. time, which is the **velocity** of the object. Velocity is a directed quantity...it can be positive or negative (negative meaning moving backwards).

Speed is the magnitude (absolute value) of the velocity, so speed is always positive.

$$V(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

If a problem asks for **average velocity**, then you don't use calculus...you just find the slope of the secant line between two points in time:

$$\textit{average velocity over an interval} = \frac{s_2 - s_1}{t_2 - t_1}$$

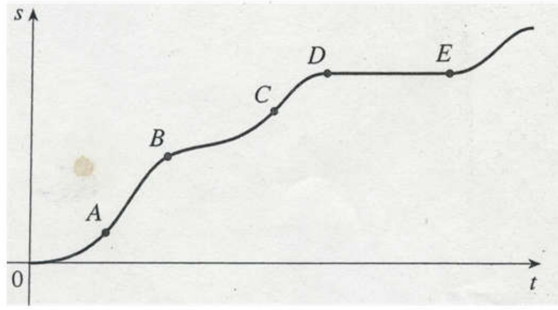
Examples

Ex. If you took 3 steps forward in 6 seconds and backwards 4 steps in 2 seconds, what was your average velocity?

Ex. You drive from LA to Phoenix: 400 miles in 8 hrs. @ 50 miles an hour; then turn around and return to LA. What was your average velocity?

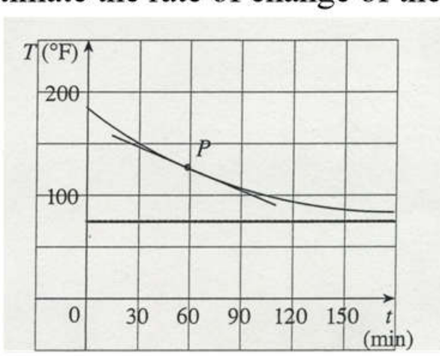
15. The graph shows the position function of a car.
Use the shape of the graph to explain your answers to the following questions.

a) What was the initial velocity of the car?
b) Was the car going faster at B or at C?
c) Was the car slowing down or speeding up at A, B, and C?
d) What happened between D and E?



19. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 4t^3 + 6t + 2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

22. A roast turkey is taken from an oven when its temperature has reached and $185^\circ F$ is placed on a table in a room where the temperature of the turkey decreases and eventually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



8. Find the equation of the tangent line to the curve.

$y = \frac{1}{\sqrt{x}}$, at $(1, 1)$

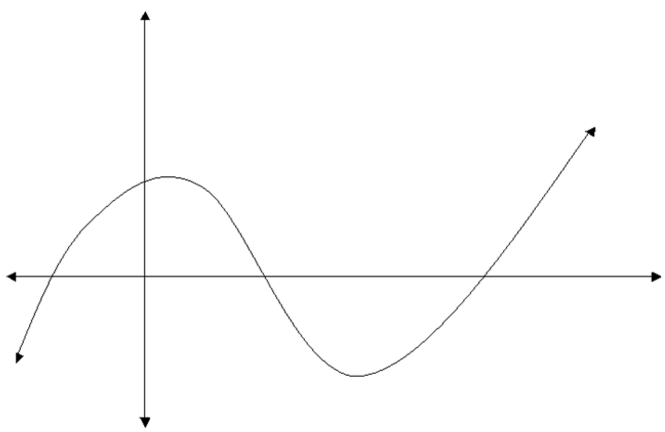
7. If $f(x) = 3x^2 - 5x$, find $f'(2)$ and use it to find an equation of the tangent line to the parabola $f(x) = 3x^2 - 5x$ at the point $(2, 2)$.

Each limit represents the derivative of some function f at some number a. State f and a in each case.

19. $\lim_{h \to 0} \frac{\sqrt{1+h}-1}{h}$

22. $\lim_{x \to 3\pi} \frac{\cos x + 1}{x - 3\pi}$

Given $f(x)$, graph $f'(x)$



20. Find the derivative of the function using the definition of the derivative. State the domain of the function and the domain of its derivative.

$f(x) = 5 - 4x + 3x^2$

20. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.

a) Find the average velocities over the following time intervals:

i) $[3, 4]$ ii) $[3.5, 4]$

iii) $[4, 5]$ iv) $[4, 4.5]$

b) Find the instantaneous velocity when $t = 4$.

c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).