Unit 10-1: Sequences
Larsen: 8.1

Sequences are functions with integers as the domain

A **sequence** is just a list of numbers in a particular order, but it formally defined as a function whose domain is the set of positive integers (usually starting at n=1, although sometimes starting at n=0):

n:	1	2	3	4	5	...
sequence:	5	10	17	26	37	...

For some sequences, it is possible to define the pattern by writing the output as a function of the value of *n*:

$f(n) = a_n = (n+1)^2 + 1$ (this is the expression for the *n*th term of the above sequence)

n is called the **index** which locates the position of a particular number in the sequence

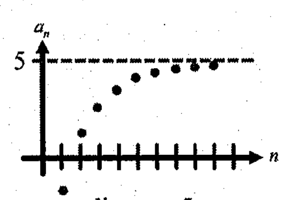
a_n is the expression defining the **n^{th} term** of the sequence

Given an expression for the *n*th term, you may be asked to write out some terms of the sequence:

Example: Write out the first 4 terms of the sequence defined by $a_n = (-1)^{n-1}\dfrac{n}{n+1}$ $\quad \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}$

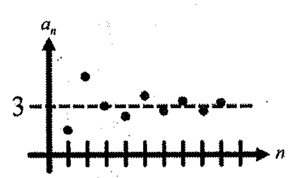
Limit of a sequence

In calculus, we are mainly concerned with whether or not a sequence approaches a limiting value as n approaches infinity. Some sequences approach a single numerical value and these are said to **converge**. Other sequences go off to positive or negative infinity or oscillate between values and these are said to **diverge**.



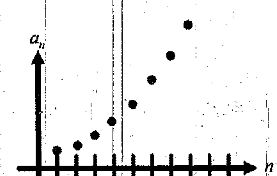
$\lim\limits_{n\to\infty} a_n = 5$

converges to a limit of 5



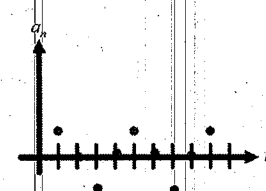
$\lim\limits_{n\to\infty} a_n = 3$

converges to a limit of 3



$\lim\limits_{n\to\infty} a_n = \infty$

diverges



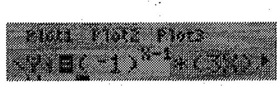
$\lim\limits_{n\to\infty} a_n = DNE$

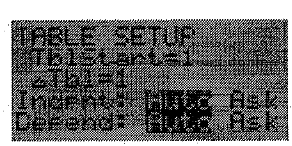
diverges

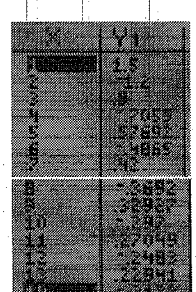
If a sequence converges, there are various methods we may need to employ to determine the limit of the sequence.

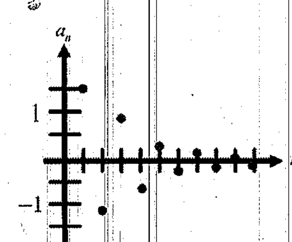
If you are allowed to use a calculator, you can enter the expression for a_n in as a function and use table features to quickly see if a sequence converges or diverges:

$a_n = (-1)^{n-1}\dfrac{3n}{n^2+1}$









...suggests this sequence is converging to 0

But to know for sure we need to find the limit analytically.

Finding the limit of a sequence analytically

$$a_n = (-1)^{n-1}\frac{3n}{n^2+1}$$

We first try evaluating the limit. After graphing, we can see that the sequence is converging to a number and the sign is oscillating, but the value around which it is converting is determined by: $\frac{3n}{n^2+1}$

We try evaluating this limit as n approaches infinity, initially by plugging in a large value:

$$\lim_{n\to\infty}\frac{3n}{n^2+1} = \frac{\infty}{\infty}$$

This is an indeterminant form, which allows us to use L'Hopital's Rule:

$$\lim_{n\to\infty}\frac{3n}{n^2+1} = \lim_{n\to\infty}\frac{3}{2n} = \frac{3}{\infty} = 0$$

So we verify that this sequence does indeed converge specifically to zero.

Properties of Limits of Sequences

A number of properties and theorems for functions and limits also apply to limits of sequences:

$$\textit{Let } \lim_{n\to\infty} a_n = L \textit{ and } \lim_{n\to\infty} b_n = K$$

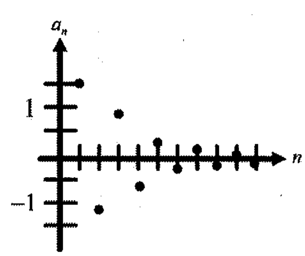
1) *Scalar multiple*: $\lim_{n\to\infty}(ca_n) = cL$

2) *Sum or difference*: $\lim_{n\to\infty}(a_n \pm b_n) = L \pm K$

3) *Product*: $\lim_{n\to\infty}(a_n b_n) = LK$

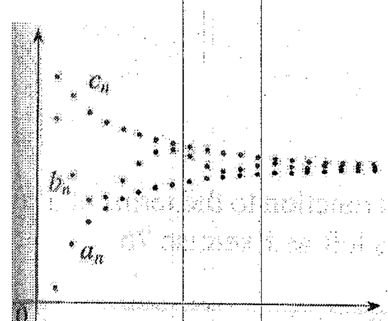
4) *Quotient*: $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{K}$ $(\textit{for } b_n \neq 0, K \neq 0)$

5) *Absolute Value*: If $\lim_{n\to\infty}|a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$



(If a series is oscillating around zero, then the sequence converges to zero)

6) *Squeeze Theorem*: If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ and there exists an integer N such that $a_n \le b_n \le c_n$ for all $n > N$, then $\lim_{n\to\infty} b_n = L$.



(If you can't evaluate the limit of the sequence directly but can find functions which bound the sequence above and below which both approach the same value, then the original sequence converges to this value - rarely used)

Monotonic and Bounded Sequences

If we just want to know if a sequence converges but do not necessarily need the limiting value we can use the following definitions and theorem:

A sequence a_n is **monotonic** when its terms are nondecreasing: $a_1 \le a_2 \le a_3 \le \ldots a_n \le \ldots$ or nonincreasing: $a_1 \ge a_2 \ge a_3 \ge \ldots a_n \ge \ldots$

A sequence a_n is **bounded above** when there is a real number *M* such that $a_n \le M$ for all *n*. The number *M* is called an **upper bound** of the sequence.
A sequence a_n is **bounded below** when there is a real number N such that $N \le a_n$ for all *n*. The number *N* is called a **lower bound** of the sequence.
A sequence a_n is **bounded** when it is bounded above and bounded below. $N \le a_n$

If a sequence a_n is **bounded and monotonic**, then it **converges**.

This example shows how these definitions are used...

Determine whether the sequence is monotonic and whether it is bounded: $a_n = \dfrac{3n}{n+2}$

Monotonic? Use the derivative to determine where increasing/decreasing:

$f(n) = \dfrac{3n}{n+2} \qquad f'(n) = \dfrac{(n+2)3 - 3n(1)}{(n+2)^2} = \dfrac{2}{(n+2)^2} \qquad f'(n) = 0 \text{ or DNE: nowhere and testing } n = 0,\ f'(0) > 0 \text{ (increasing)}$

$-\infty$ —— ∞

increasing everywhere

...so the sequence is **monotonic** (specifically, nondecreasing)

Bounded? Since it is increasing, the lower bound value would be at *n* = 1: $a_1 = \dfrac{3(1)}{(1)+2} = 1$ lower bound

The upper bound would be as n approaches infinity: $a_n = \lim\limits_{n\to\infty} \dfrac{3n}{n+2} = \dfrac{\infty}{\infty}$

$\text{by L'Hopital}: = \lim\limits_{n\to\infty} \dfrac{3}{1} = 3$ upper bound

...so the sequence is **bounded**

Therefore the sequence **converges**.

Finding an expression for the nth term

In order to analytically determine convergence, we must have an expression for the nth term, and some problems just directly ask us to find this for a given sequence.

First, you can determine if the sequence is arithmetic or geometric which each have defined formulas:

arithmetic: $10 \quad 7 \quad 4 \quad 1 \quad -2$

-3 -3 -3 -3 the common difference, *d*

$a_n = a_1 + (n-1)d$

$a_n = 10 + (n-1)(-3) = 10 - 3n + 3 = 13 - 3n$

geometric: $100 \quad 50 \quad 25 \quad \dfrac{25}{2} \quad \dfrac{25}{4}$

$*\tfrac{1}{2} \quad *\tfrac{1}{2} \quad *\tfrac{1}{2} \quad *\tfrac{1}{2}$ the common ratio, *r*

$a_n = a_1 (r)^{n-1}$

$a_n = 100\left(\dfrac{1}{2}\right)^{n-1}$

If the signs alternate, include multiplying by $(-1)^n$ or $(-1)^{n-1}$

$-100 \quad 50 \quad -25 \quad \dfrac{25}{2} \quad -\dfrac{25}{4} \qquad a_n = (-1)^n\, 100\left(\dfrac{1}{2}\right)^{n-1}$

Finding an expression for the n^{th} term

You can also try writing the values of n below each term, then try different operations on n:

sequence:	(5)	(10)	(17)	(26)	37	← these...
n:	1	2	3	4	5	
n^2:	1	(4)	(9)	(16)	(25)	← ...are one larger than these

$$a_n = (n+1)^2 + 1$$

More examples...

1) List the first 4 terms of the sequence:

$$a_n = 1 - (0.2)^n$$

[calculator function seq under list, OPS]

$seq(1-(0.2)^x, x, 1, 5, 1)$ — a_n, var, start, stop, step

0.8, 0.96, 0.992, 0.9984, 0.99968

2) Find an expression for the nth term:

$$\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27} \ldots\right\}$$ ← seems like powers of 2 and 3

but for '1' need to start at n = 0

n:	0	1	2	3
2^n	1	2	4	8
3^n	1	3	9	27

$$\boxed{a_n = (-1)^{n-1}\frac{2^n}{3^n}}$$ start at n = 0

(also could be $a_n = (-1)^{n-1}\left(\frac{2}{3}\right)^{n-1}$ start at n = 1)

3) Determine if the sequence converges and if so find the limiting value:

$$a_n = \frac{3+5n^2}{n+n^2}$$

$\lim_{n\to\infty} \frac{3+5n^2}{n+n^2}$ $\frac{\infty}{\infty}$

L'Hopital's: $\lim_{n\to\infty} \frac{10n}{1+2n}$ $\frac{\infty}{\infty}$

L'Hopital's! $\lim_{n\to\infty} \frac{10}{2} = 5$

converges (to 5)

$$a_n = \frac{n!}{2^n}$$

n:	1	2	3	4	5
$n!$	1	2	6	24	120
2^n	2	4	8	16	32

after $n=4$, $n! > 2^n$

so **diverges**

4) Determine if the sequence is monotonic, bounded, and if it converges:

$$a_n = \frac{n}{n^2+1}$$

$f'(n) = \frac{(n^2+1)(1) - (n)(2n)}{(n^2+1)^2}$

$= \frac{1-n^2}{(n^2+1)^2} = 0$ when $n = 1, -1$

$-\infty$ —— -1 —— 1 (2) —— ∞

$f'(2) = \frac{(-)}{(+)}$ – decreasing

(don't care about $n<1$)

always decreasing (for $n>1$) so

monotonic (non-increasing)

upper bound at $n=1$: $a_1 = \frac{1}{2}$

lower bound at $\lim_{n\to\infty} \frac{n}{n^2+1} \overset{\text{by L'Hop}}{=} \lim_{n\to\infty} \frac{1}{2n} = 0$

bounded so **converges**

Unit 10-2: Series and Convergence
Larsen: 8.2

Series vs. Sequences

A **sequence** is a set of numbers in a particular order but a **series** is the summation of the terms of a sequence. We typically use sigma notation along with the expression for the nth term of the series. Comparing a sequence and corresponding series:

Sequence: $a_n = \frac{1}{2^n} \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16} \quad \ldots$

Series: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \quad \ldots$

Finite Series: If there are a fixed number of terms in the series, we say the series is a finite series:

$$\sum_{n=1}^{4} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

Finite series always converge to a computed sum value.

Infinite Series: If there are an infinite number of terms, we say the series is an infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \quad \ldots = ??$$

Infinite series may or may not converge to a sum value (much of this unit is about how to determine if infinite series converge and if so to what sum).

Convergence of Series vs. Sequences

We say a **sequence converges** if the values of the terms approach a numerical value as n approaches infinity...

Sequence: $a_n = \frac{1}{2^n} \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16} \quad \ldots$

...and our main way to determine convergence is to compute the limit of the expression for the nth term as n approaches infinity:

$\lim_{n \to \infty} \frac{1}{2^n} = \frac{1}{\infty} = 0$ ...so this sequence converges to a value of 0.

We say a **series converges** if the sum of the terms approaches a numerical value as n approaches infinity.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \quad \ldots = ??$$

One way to determine convergence of a series is to consider the **sequence of partial sums** of a series. A **partial sum** is the sum of the terms up to a specified value of *n*:

Partial sums:

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

To determine if a series converges, instead of taking the limit of the expression for the nth term, we take the limit of **an expression for the nth partial sum** as n increases...

$$\lim_{n \to \infty} \frac{2^n - 1}{2^n} = 1$$

...since this limit approaches the number 1, this infinite series **converges** and its sum is 1.

Convergence of Series vs. Sequences

This particular series has an interesting geometrical interpretation as well...

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\ \ldots=??$$

...each term is a fraction which adds up to the total area of a 1 x 1 square:

$$S_1=\frac{1}{2}$$

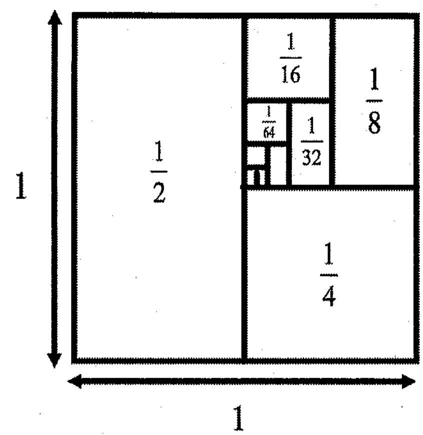
$$S_2=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

$$S_3=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$$

$$S_4=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}$$

$$S_n=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^n}=\frac{2^n-1}{2^n}$$

$$\lim_{n\to\infty}\frac{2^n-1}{2^n}=1$$



Example: Determine if the series converges and if so, to what sum.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

unique behavior - cancelation between terms (this is called a 'telescoping series')

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)\ \ldots$$

...so the 3rd partial sum is: $S_3=\frac{1}{1}-\frac{1}{4}$

The nth partial sum would therefore be: $S_n=1-\frac{1}{n+1}$

So we can determine convergence using the limit: $\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1-\frac{1}{\infty}=1$

This series also converges and happens to have a sum of 1 as well.

Example: Determine if the series converges and if so, to what sum.

$$\sum_{n=1}^{\infty}1$$

$$\sum_{n=1}^{\infty}1=1+1+1+1\ \ldots$$

The nth partial sum would be: $S_n=n$

So we can determine convergence using the limit: $\lim_{n\to\infty}(n)=\infty$

So this series **diverges**.

Determining convergence of infinite series when it is difficult to find S_n, Properties

In the examples we've seen so far, we have been able to determine convergence by writing out some terms of the series, and then finding a formula for the nth partial sum, then taking the limit of this partial sum expression as n approaches infinity.

But sometimes it is difficult to determine a formula for the nth partial sum, so we need to resort to other techniques to determine convergence and/or to find the sum:

- **Recognizing Known Series**: Sometimes, we can recognize that a given series is a particular known type which has conditions which determine whether or not it converges and what sum it converges to.
- **Theorems and Tests**: Sometimes, we can employ certain theorems or 'tests' to determine whether a series converges or diverges

For series which converge, the following properties also apply:

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series, and let *A*, *B*, and *c* be real numbers.
If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then the following series converge to the indicated sums:

$$\sum_{n=1}^{\infty} ca_n = cA$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$$

Telescoping series

Telescoping series are series which expand to the form: $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \dots$

so that cancellation occurs between terms...

leaving only the first and last term left: $b_1 - b_{n+1}$

For telescoping series, the nth partial sum is: $S_n = b_1 - b_{n+1}$

Since the first term is just a number, telescoping series will converge if the last term here converges: $\lim_{n\to\infty}(b_{n+1})$

...and will converge to the sum: $S = b_1 - \lim_{n\to\infty}(b_{n+1})$

We usually need to do some manipulation of the expression to see that a series is telescoping and also need to be very careful with how the cancellation actually occurs. This is best seen by looking at a more complicated example.

Determine if the series converges and if so to what sum: $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

This a_n has a factored denominator, which suggests we can expand using Partial Fractions:

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$
$$A(n+2) + B(n) = 1$$
$$(A+B)n + (2A) = (0)n + 1$$

$$\begin{cases} A+B=0 \\ 2A=1 \end{cases}$$
$$A = \frac{1}{2},\ B = -\frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\frac{1}{n} - \frac{1}{2}\frac{1}{n+2}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

two terms with subtraction = telescoping series

Now, write out some terms to see how the cancellation works:

$$\frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\dots\right]$$

cancellation happens, but one term is skipped

$$\frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\dots\right]$$

this leave two terms at beginning and end

so the expression for the nth partial sum is: $S_n = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$ (OK)

and the limit is: $\lim_{n\to\infty}\frac{1}{2}\left[1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right] = \frac{1}{2}\left[1+\frac{1}{2}-\frac{1}{\infty}-\frac{1}{\infty}\right] = \frac{1}{2}\left[1+\frac{1}{2}-0-0\right] = \frac{3}{4}$ **converges**

Geometric Series

Another infinite series pattern we can recognize is the **Geometric Series**:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + ... + ar^4 + ... \quad (for\ a \neq 0)$$

We were introduced to this back in honors algebra 2, but can now prove some things about geometric series.

The expression for the nth partial sum would be: $S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$

if we multiply this by r and then subtract this result from S_n:

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$
$$-rS_n = \quad ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n$$
$$(1-r)S_n = a - ar^n$$
$$(1-r)S_n = a(1-r^n)$$
$$S_n = \frac{a}{1-r}(1-r^n)$$

Taking the limit: $\lim_{n\to\infty} \frac{a}{1-r}(1-r^n)$ since we are subtracting the term r^n as n increases, this will go to zero only if $|r| < 1$

Therefore, a geometric series will converge if $|r| < 1$ and will converge to the sum $\frac{a}{1-r}$

Examples: Determine if the series converges and if so to what sum.

$$\sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n$$

$$a = 3\left(\frac{1}{4}\right)^0 = 3$$

$$r = \frac{1}{4} \quad converges$$

$$to: \frac{a}{1-r} = \frac{3}{1-\frac{1}{4}} = \frac{3}{\left(\frac{3}{4}\right)} = 4$$

$$\sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^n$$

$$a = 3\left(\frac{1}{4}\right)^1 = \frac{3}{4}$$

$$r = \frac{1}{4} \quad converges$$

$$to: \frac{a}{1-r} = \frac{\left(\frac{3}{4}\right)}{1-\frac{1}{4}} = \frac{\left(\frac{3}{4}\right)}{\left(\frac{3}{4}\right)} = 1$$

$$\sum_{n=1}^{\infty} 3(1.1)^n$$

$$a = 3(1.1)^1 = 3.3$$

$$r = 1.1 \quad diverges$$

You can use geometric series to investigate repeating decimals...

Example: Write the repeating decimal as a geometric series and as a ratio of two integers: $0.\overline{36}$

$$0.\overline{36} = 0.36363636363636...$$
$$= 0.36 + 0.0036 + 0.000036 + 0.00000036 + ...$$
$$= \sum_{n=0}^{\infty} 0.36\left(\frac{1}{100}\right)^n$$
$$a = 0.36$$
$$r = \frac{1}{100} \quad converges$$
$$to: \frac{a}{1-r} = \frac{0.36}{1-\frac{1}{100}} = \frac{0.36}{\frac{99}{100}} = \frac{100(0.36)}{99} = \frac{36}{99}$$

The nth-term test for divergence

If we can't find an expression for the nth partial sum, and the series isn't telescopic or geometric, we can at least determine if the series diverges using the following theorem and test.

Assume that $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L$ (even though we can't determine the S_n)

It would be true that $S_n = S_{n-1} + a_n$

If we take $L = \lim_{n\to\infty} S_n$

$$= \lim_{n\to\infty}(S_{n-1} + a_n)$$

$$= \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n$$

We know $\lim_{n\to\infty} S_n = L$

but since n is approaching infinity it is also true that $\lim_{n\to\infty} S_{n-1} = L$

so it must be true that $L = L + \lim_{n\to\infty} a_n$

which implies that the sequence $\{a_n\}$ converges to zero

Therefore: **If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$**

This is a theorem

This theorem isn't super useful directly:

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

But remember from geometry that you can either negate the sides of a statement or invert them to form the inverse, converse, and contrapositive? One of these, the contrapositive is always true if the statement is true, so the contrapositive of this theorem is also always true:

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

This is called the '**nth-term test for Divergence**'

The contrapositive of a true statement is always true, but the inverse or converse are not necessarily true or false.

We can therefore only use this test to demonstrate that a series diverges.

1) Compute the limit of the sequence for the series $\lim_{n\to\infty} a_n$

2) If the limit is non-zero: you can state that the infinite series diverges.
   If the limit is zero, you can make no conclusions: the infinite series might converge or might not converge (and if it does converge, you do not know that sum to which it converges).

Examples

Determine if the series converges and if so to what sum.

$$\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$$

$\lim_{n\to\infty} \frac{n+1}{2n-1}$ $\frac{\infty}{\infty}$

(L'Hopital)

$= \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$

$\frac{1}{2} \neq 0$

so the series **diverges**

$$\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$$

$\lim_{n\to\infty} \frac{4}{n(n+4)}$ $\frac{4}{\infty} = 0$

so we can't say yet if converges... investigate further:

partial fractions:

$\frac{4}{n(n+4)} = \frac{A}{n} + \frac{B}{n+4}$ $A(n+4) + Bn = 4$

$(A+B)n + (4A) = (0)n + (4)$

$\begin{cases} A+B=0 \\ 4A=4 \end{cases}$ $A=1$ $B=-1$

$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+4}\right)$ telescoping series

$\left(1-\frac{1}{5}\right)+\left(\frac{1}{2}-\frac{1}{6}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{4}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{6}-\frac{1}{10}\right)+\ldots$

$\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) - \left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}\right)$

converges if $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}\right)$ converges

$= \frac{1}{\infty}+\frac{1}{\infty}+\frac{1}{\infty}+\frac{1}{\infty} = 0$

so **converges** to sum $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} - (0)$

$= \boxed{\frac{25}{12}}$

Examples

Determine if the series converges and if so to what sum.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$\lim_{n\to\infty} \frac{1}{n} = 0$

no conclusion yet...

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

not geometric, not telescopic

partial sums?

$S_1 = 1$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$

$S_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}$

no obvious formula for S_n

So ... stuck (for now)

$$-2 + \frac{5}{2} - \frac{25}{8} + \frac{125}{32} - \cdots$$

$\times -\frac{5}{4}$, $\times -\frac{5}{4}$

geometric with $a = -2$, $r = -\frac{5}{4}$

$|r| = \frac{5}{4} > 1$

so **diverges**

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+5}\right)$$

$\lim_{n\to\infty} \ln\left(\frac{n}{2n+5}\right)$

by earlier theorem about limits...

$= \ln\left(\lim_{n\to\infty} \frac{n}{2n+5}\right)$ $\frac{\infty}{\infty}$

L'Hopital: $\lim_{n\to\infty} \frac{1}{2}$

$\ln\left(\frac{1}{2}\right) \neq 0$

so **diverges**

Determine if the series converges and if so to what sum.

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{1+n^2}}$$

$\lim_{n\to\infty} \frac{n}{\sqrt{1+n^2}}$ $\frac{\infty}{\infty}$ $\left(\frac{n}{(1+n^2)^{1/2}}\right)$

L'Hopital's:

$= \lim_{n\to\infty} \frac{1}{\frac{1}{2}(1+n^2)^{-1/2}(2n)}$

$\lim_{n\to\infty} \frac{1}{\frac{n}{\sqrt{1+n^2}}} = \lim_{n\to\infty} \frac{\sqrt{1+n^2}}{n}$ $\frac{\infty}{\infty}$

L'Hopital's:

$= \lim_{n\to\infty} \frac{\frac{1}{2}(1+n^2)^{-1/2}(2n)}{1} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{1+n^2}}}{1}$

(not working ☹) Okay then...

$\lim_{n\to\infty} \frac{n}{\sqrt{1+n^2}}$ if n is large $1+n^2 \approx n^2$

$\approx \lim_{n\to\infty} \frac{n}{\sqrt{n^2}} = \lim_{n\to\infty} \frac{n}{n} = 1$

$1 \neq 0$, so **diverges**

Find the values of x for which the series converges, and find the sum of the series for those values of x.

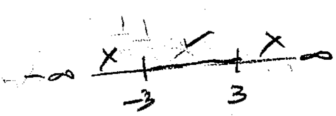
$$\sum_{n=1}^{\infty} \frac{x^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}x\right)^n$$

geometric, w/ $a = \frac{1}{3}x$

$r = \frac{1}{3}x$

converges if $|r| < 1$

$\left|\frac{1}{3}x\right| < 1$

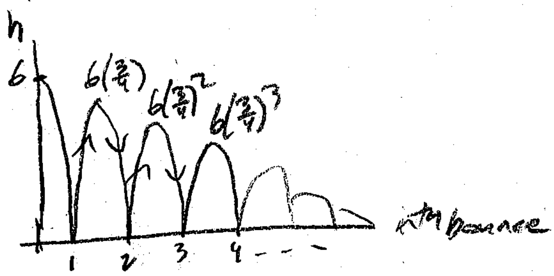


converges if $-3 < x < 3$ to

$\text{Sum} = \frac{a}{1-r} = \frac{\frac{1}{3}x}{1 - \frac{1}{3}x} = \boxed{\frac{x}{3-x}}$

Examples

A ball is dropped from a height of 6 feet and begins bouncing. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.



total distance $= 6 + (2)6(\frac{3}{4}) + (2)6(\frac{3}{4})^2 + (2)6(\frac{3}{4})^3 + \cdots$

↑ doubled for up & down (except 1st drop)

this part is a geometric series

w/ $a = 12(\frac{3}{4})$

$r = \frac{3}{4}$

converges because $|\frac{3}{4}| < 1$

to sum of $\frac{a}{1-r} = \frac{12(\frac{3}{4})}{1-\frac{3}{4}} = \frac{9}{1/4} = 36$

total distance $= 6 + 36 = \boxed{42 \text{ ft}}$

Unit 10-3: The Integral Test and p-series
Larsen: 8.3

The Integral Test

Much of this unit is about whether or not a series converges and if so to what sum. We've seen that sometimes we can recognize a given series as a particular form for which there are specific rules (like telescoping or geometric series) and we can sometimes use theorems (like the n-th term test) for at least determining convergence. In this section, we expand our list of recognizable series and theorems about convergence with **<u>The Integral Test</u>**:

> If f is positive, continuous, and <u>decreasing</u> for $x \ge 1$ and $a_n = f(n)$, then
>
> $$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$
>
> either both converge or both diverge.

This theorem applies when you have a series whose terms are always positive and always decreasing...in this case, we can use an integral of the a_n expressed as a function and if the integral converges then the series converges.

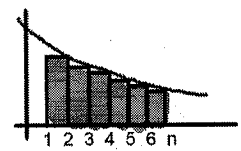
<u>Important note</u>: If the integral and series converge, the series **<u>does not</u>** (necessarily) converge to the same value as the integral. We are just testing to see <u>if</u> the series converges at all.

The Integral Test - proof

Much of this unit is about whether or not a series converges and if so to what sum. We've seen that sometimes we can recognize a given series as a particular form for which there are specific rules (like telescoping or geometric series) and we can sometimes use theorems (like the n-th term test) for at least determining convergence. In this section, we expand our list of recognizable series and theorems about convergence with **<u>The Integral Test</u>**:

If you have a function decreasing from x = 1 onward and represent the area between x = 1 and x = n, we could set the rectangle height using the RHS, but would mean starting at i = 2 and we would obtain the *inscribed area*:

$$\textit{inscribed area} = \sum_{n=2}^{n} f(i)$$



Using the LHS for rectangle height we would start at i = 1 and obtain the *circumscribed area*:

$$\textit{circumscribed area} = \sum_{n=1}^{n-1} f(i)$$

The exact area is between these two values: $\sum_{n=2}^{n} f(i) \le \int_1^n f(x)\,dx \le \sum_{n=1}^{n-1} f(i)$

Using the nth partial sum: $S_n = f(1) + f(2) + f(3) + \ldots + f(n)$

the previous area statement can be written as: $S_n - f(1) \le \int_1^n f(x)\,dx \le S_{n-1}$

Assuming that the integral converges to a sum which we will call L, then: $S_n - f(1) \le L$

$S_n \le L + f(1)$

Therefore $\{a_n\}$ is bounded and monotonic, and by previous theorem it therefore converges, so $\sum a_n$ converges.

Examples using the Integral Test

Use the Integral Test to determine if the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

✓ • positive for $n \ge 1$

✓ • $f(x) = \frac{x}{x^2+1}$ continuous for $x \ge 1$

✓ • $f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} < 0$ for $x \ge 1$

(f decreasing)

So integral test applies!

$\int_1^{\infty} \frac{x}{x^2+1}\,dx \qquad u = x^2+1, \quad du = 2x\,dx$

$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|x^2+1|$

$\lim_{b\to\infty}\left[\frac{1}{2}\ln|x^2+1|\right]_1^b$

$\lim_{b\to\infty} \frac{1}{2}\ln|b^2+1| - \frac{1}{2}\ln|(1)^2+1|$

$\infty - \frac{1}{2}\ln 2$

∞

diverges

so the series diverges

Examples using the Integral Test

Use the Integral Test to determine if the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

✓ • positive for $n \geq 1$

✓ • $f(x) = \frac{1}{x^2+1}$ continuous for $x \geq 1$

✓ • $f'(x) = \frac{(x^2+1)(0) - (1)(2x)}{(x^2+1)^2}$

$= \frac{-2x}{(x^2+1)^2} < 0$ for $x \geq 1$

(f is decreasing)

so integral test applies

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx$$

$$\lim_{b\to\infty} \left[\arctan x\right]_1^b$$

$$\lim_{b\to\infty} \arctan(b) - \arctan(1)$$

$$\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$ converges...

so the series converges

(but not to $\frac{\pi}{4}$)

The series doesn't have to start at n=1

Use the Integral Test to determine if the series converges or diverges:

$$\sum_{n=3}^{\infty} \frac{1}{n^2+1} = \sum_{n=1}^{\infty} \frac{1}{n^2+1} - \underbrace{\frac{1}{(1)^2+1}}_{n=1} - \underbrace{\frac{1}{(2^2)+1}}_{n=2}$$

$$- \frac{1}{2} - \frac{1}{5}$$

can start at $n=1$ even though the given series starts at a different value

by integral test (last problem), the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

therefore $\sum_{n=3}^{\infty} \frac{1}{n^2+1} = \left(\text{some value to which } \sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ converges to}\right) - \frac{1}{2} - \frac{1}{5}$ also converges

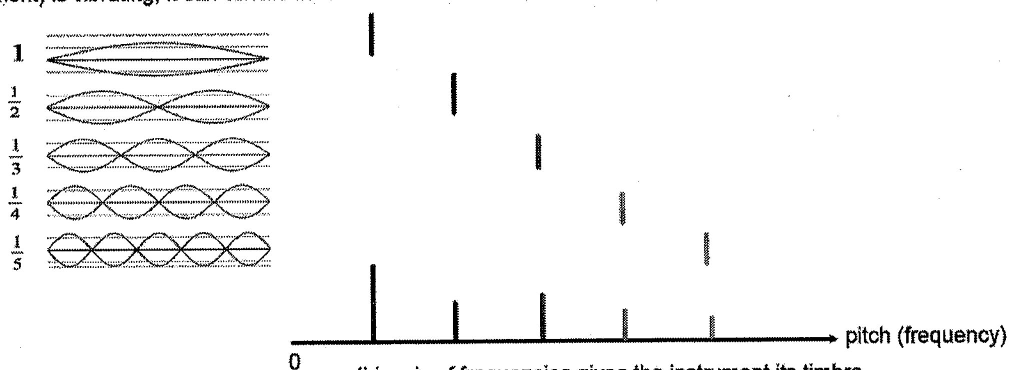
p-Series and Harmonic Series

Here are some more recognizable forms of infinite series:

$$p\text{-Series:} \quad \sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots$$

$$\text{"The Harmonic Series":} \quad \sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$

"The Harmonic Series" is called that because it relates to music theory. If a string (or column of air in a wind or brass instrument) is vibrating, it can vibrate in different modes:



this mix of frequencies gives the instrument its timbre (unique sound quality)

Convergence of *p*-Series

There is a theorem for determining convergence of *p*-series:

> The *p*-series $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots$ converges for $p>1$, and diverges for $0<p\le 1$.

Proof: Using the Integral Test, $f(x)=\frac{1}{x^p}$ and $\int_1^{\infty}\frac{1}{x^p}\,dx=\int_1^{\infty}x^{-p}\,dx$

$$=\left[\frac{x^{-p+1}}{-p+1}\right]_1^{\infty}$$

$$=\left[\lim_{b\to\infty}\frac{b^{-p+1}}{-p+1}\right]-\left[\frac{(1)^{-p+1}}{-p+1}\right]$$

$$=\left[\lim_{b\to\infty}\frac{b^{-p+1}}{-p+1}\right]+\frac{1}{p}-1$$

If the exponent is positive, $-p+1>0$, then b^{-p+1} will increase without bound and the integral and series diverge. The integral (and series) will only converge if the exponent is negative, because this moves the b^{-p+1} term to the bottom of the fraction. Therefore, the *p*-series will converge when $-p+1<0$

which is when $p>1$

Examples using the convergence of *p*-series theorem

Determine if the series converges or diverges:

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

p-series w/ $p=1$

converges if $p>1$

so **diverges**

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

p-series w/ $p=2$

converges if $p>1$

so **converges**

Determine if the series converges or diverges:

$$\sum_{n=2}^{\infty}\frac{1}{n\ln n}$$

integral test

✓ positive for $n\geq 1$

✓ $f(x)=\frac{1}{x\ln x}$ continuous for $x\geq 1$

✓ $f'(x)=\frac{x\ln x(0)-(1)[x\frac{1}{x}+\ln x]}{(x\ln x)^2}$

$=\frac{-1-\ln x}{(x\ln x)^2}<0$ when $-1-\ln x<0$, $\ln x>-1$, so for $x\geq 1$

integral test applies

$\int_1^{\infty}\frac{1}{x\ln x}dx$ $\quad u=\ln x$, $du=\frac{1}{x}dx$

$\int\frac{1}{u}du$

$[\ln|\ln x|]_1^{\infty}$

$\lim_{b\to\infty}\ln|\ln b|-\ln(\ln 1)$ $\quad \ln(0)$ also bad

$\lim_{b\to\infty}\ln|\ln b|-\lim_{d\to 0^+}\ln(d)$

$\infty-(-\infty)=\infty$

diverges

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

integral test

✓ positive for $n\geq 1$

✓ $f(x)=\frac{1}{\sqrt{x}}$ continuous for $x\geq 1$

✓ $f'(x)=-\frac{1}{2}x^{-3/2}=-\frac{1}{2\sqrt{x^3}}<0$ for $x\geq 1$

integral test applies

$\int_1^{\infty}\frac{1}{\sqrt{x}}dx=\int_1^{\infty}x^{-1/2}dx$

$2x^{1/2}=[2\sqrt{x}]_1^{\infty}$

$2\sqrt{\infty}-2\sqrt{1}$

$\lim_{b\to\infty}2\sqrt{b}-2$

$\infty-2$

$=\infty$

diverges

$$\sum_{n=1}^{\infty}\frac{1}{3n+1}$$

integral test

✓ positive for $n\geq 1$

✓ $f(x)=\frac{1}{3x+1}$ continuous for $x\geq 1$

✓ $f'(x)=\frac{(3x+1)(0)-1(3)}{(3x+1)^2}$

$=\frac{-3}{(3x+1)^2}<0$ for $x\geq 1$

integral test applies

$\int_1^{\infty}\frac{1}{3x+1}dx$ $\quad u=3x+1$, $du=3dx$

$\frac{1}{3}\int\frac{1}{u}dx$

$[\frac{1}{3}\ln|3x+1|]_1^{\infty}$

$\frac{1}{3}\ln\infty-\frac{1}{3}\ln|4|$

∞

diverges

More examples

Determine if the series converges or diverges using both the integral test and the p-series theorem: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

integral test

✓ • positive for $n \geq 1$

✓ • $f(x) = \frac{1}{x^2+1}$ continuous for $x \geq 1$

✓ • $f'(x) = \frac{(x^2+1)(0) - (1)(2x)}{(x^2+1)^2}$

$= \frac{-2x}{(x^2+1)^2} < 0$ for $x \geq 1$

integral test applies

$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \left[\arctan x\right]_1^{\infty}$

$\arctan\infty - \arctan 1$

$\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}$

converges

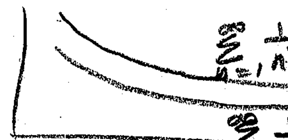
p-series

$\sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

p-series w/ $p = 2$

converges if $p > 1$

so converges



$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is always less

then $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges

Find the values of *p* for which the series converges: $\sum_{n=1}^{\infty} n\left(1+n^2\right)^p$

integral test?

✓ • positive for $n \geq 1$

✓ • $f(x) = x(1+x^2)^p$ continuous for $x \geq 1$

• $f'(x) = x\left(p(1+x^2)^{p-1}(2x)\right) + (1+x^2)^p(1)$

$= 2x^2p(1+x^2)^{p-1} + (1+x^2)^p < 0$

$= 2x^2p(1+x^2)^{p-1} + (1+x^2)(1+x^2)^{p-1}$

$= (1+x^2)^{p-1}\left[2x^2p - (1+x^2)\right]$

$= (1+x^2)^{p-1}\left[2x^2p - x^2 - 1\right]$

$= (1+x^2)^{p-1}\left[(2p-1)x^2 - 1\right] = 0$

$f'(x) < 0$ (decreasing)

stop here, assume met...

if $p < -1$, then

$2p - 1 < 0$, so

so $(2p-1)x^2 - 1 < 0$

so $f'(x)$ decreasing

(applied)

if integral test applies:

$\int_1^{\infty} x(1+x^2)^p\,dx$

$u = 1+x^2$

$du = 2x\,dx$

$dx = \frac{1}{2}x\,dx$

$\frac{1}{2}\int u^p\,du$

$\frac{1}{2}\frac{u^{p+1}}{p+1} = \left[\frac{1}{2}\frac{(1+x^2)^{p+1}}{p+1}\right]_1^{\infty}$

$\frac{1}{2}\frac{(1+(\infty)^2)^{p+1}}{p+1} - \frac{1}{2}\frac{2^{p+1}}{p+1}$

converges if 1st term exponent $p+1$ is negative (to move to bottom)

$p+1 < 0$

$p < -1$

go back and check integral test applied

Unit 10-4: Comparison of Series
Larsen: 8.4

Comparison Tests

We've seen a number of ways to check whether a series converges or diverges which involve either recognizing the form of the series or performing some calculation using the series itself or a function representing the series. But some series are difficult to analyze in this way.

So instead, we sometimes determine if a series converges or diverges by comparing it in some way to another series for which it is easier to determine divergence/converge. There are called comparison tests and in this section we will example two of these:

The Direct Comparison Test: Used when we know that one term is always higher than another, term by term.

The Limit Comparison Test: Used when the ratio of two series is always positive as n approaches infinity (meaning that, eventually for large n, one series converges to a higher value than another, even though we can't say every term in one series is higher than the corresponding term in the other series).

The Direct Comparison Test

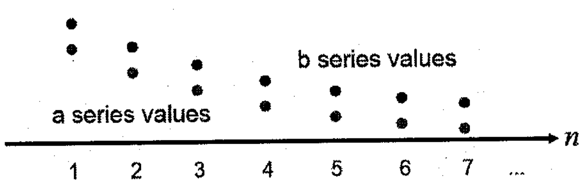
Here is the formal statement:

Let $0 < a_n \le b_n$ for all n.

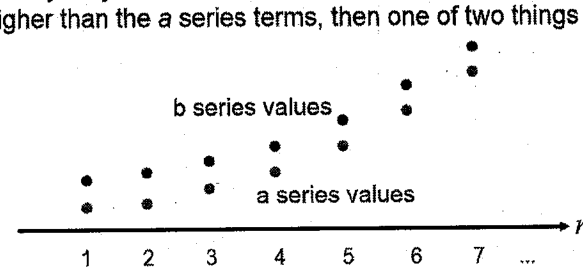
If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Instead of a proof, let's understand what this is saying intuitively. If you have two series, and comparing each term, term-by-term, one series (the *b* series) terms are always higher than the *a* series terms, then one of two things can happen:



If the b series is converging to a value, it 'squeezes' the a between this value and zero, so a must also be converging (although not to any particular value)



If the a series diverges, then it 'pushes' the b values up too, so the b series must also diverge

The way we use direct comparison is for our given series, we need to identify a different series which we can show is always above or below our series (term by term) but for which it is easier to determine convergence.

Example: Determine if the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

This series is similar in form to $\sum_{n=1}^{\infty} \frac{1}{3^n}$ and we can write out a few terms of each series...

n	1	2	3	4	
$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$	$\frac{1}{5}$	$\frac{1}{11}$	$\frac{1}{29}$	$\frac{1}{83}$	$\sum_{n=1}^{\infty} \frac{1}{2+3^n} = \sum_{n=1}^{\infty} a_n$
$\sum_{n=1}^{\infty} \frac{1}{3^n}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$	$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} b_n$

...to convince ourselves that $0 < a_n \le b_n$

Our new series is a geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} 1\left(\frac{1}{3}\right)^n$ with $r = \frac{1}{3}$ and since $|r| < 1$ $\sum_{n=1}^{\infty} \frac{1}{3^n}$ *converges.*

Since this series is always higher than our original series, $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ *converges.*

The Direct Comparison Test

Here is an example that shows the divergent case:

Example: Determine if the following series converges or diverges: $\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$

This series is similar in form to $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ and because the denominator of our original series is always 2 higher, this new series is always higher then the original series: $\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}=\sum_{n=1}^{\infty}a_n$ $\quad\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}=\sum_{n=1}^{\infty}b_n$ $\quad 0<a_n\le b_n$

$\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ is a *p*-Series with $p=\frac{1}{2}$ which diverges (p must be > 1 for convergence).

Since this new series is higher series is above our original series, it diverging doesn't tell us anything about our original series - turns out this was a bad choice for the other series.

We will need to pick a different series to compare to, and maybe one which is instead lower than our original series.

To stay under our original series, we need the denominator to grow more rapidly with n, so let's try the series $\sum_{n=1}^{\infty}\frac{1}{n}$

n	1	2	3	4	5	6	7
$\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$	0.33	0.29	0.27	0.25	0.24	0.22	0.215
$\sum_{n=1}^{\infty}\frac{1}{n}$	1	0.5	0.33	0.25	0.20	0.17	0.143

(using calculator table function to check)

This new series starts with terms higher than the original, but for n=5 and higher, it remains below our original series. It doesn't really matter what happens with the first terms - this is all about convergence/divergence as $n\to\infty$

So we will now call the new series *a* and the original series *b* so we have:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\sum_{n=1}^{\infty}a_n \qquad \sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}=\sum_{n=1}^{\infty}b_n \qquad 0<a_n\le b_n \quad [\text{for } n\ge 4]$$

This new series is a *p*-Series with p = 1, so it diverges.

Since this series is always lower than our original series, $\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$ *diverges.*
(for $n>4$)

Example: Determine if the following series converges or diverges: $\sum_{n=1}^{\infty}\frac{3^n}{2^n-1}=\sum_{n=1}^{\infty}b_n$

$\frac{3^n}{2^n}<\frac{3^n}{2^n-1}$ (smaller denom = bigger)

$\sum_{n=1}^{\infty}\frac{3^n}{2^n}=\sum_{n=1}^{\infty}a_n$

$0<a_n\le b_n$

$\sum_{n=1}^{\infty}\frac{3^n}{2^n}=\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$ is geometric w/ $r=\frac{3}{2}$ so $|r|>1$
this diverges

$0<a_n\le b_n$
(a_n diverges) ↑ so this diverges too

$\sum_{n=1}^{\infty}\frac{3^n}{2^n-1}$ **diverges**

The Limit Comparison Test

Sometimes a series closely resembles another series, but it is difficult to establish, term-by-term, that even after some value of *n* one series is always higher than the other. In this case, we can use a different test:

> If $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and L is *finite* and *positive*, then
>
> $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either *both converge*, or *both diverge*.

(See the textbook if you want formal proof of this or the Direct Comparison Test.)

Let's consider a few examples to see how it works...

Example: Determine if the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$

For a similar form series, consider ignoring the constants in the denominator: $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Both series have positive terms, so now we need to evaluate the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{3n^2 - 4n + 5}\right)}{\left(\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{n^2}{3n^2 - 4n + 5} = \frac{\infty}{\infty}$$

by L'Hopital:

$$= \lim_{n\to\infty} \frac{2n}{6n - 4} = \frac{\infty}{\infty}$$

by L'Hopital:

$$= \lim_{n\to\infty} \frac{2}{6} = \frac{1}{3} \quad \textit{converges}$$

This limit (*L=1/3*) is *finite* and *positive* so these two series will either both diverge or both converge. Evaluating the easier series:

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a *p*-Series with *p* = 2 so it converges, therefore

$\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$ also converges.

Example: Determine if the following series converges or diverges: $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Let's try this series: $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\sin\left(\frac{1}{n}\right)\right)}{\left(\frac{1}{n}\right)} = \frac{\sin(0)}{0} = \frac{0}{0}$$

by L'Hopital:

$$= \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right) \bullet \left(-n^{-2}\right)}{-n^{-2}} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$$

So L = 1 is finite and positive, both series converge or diverge together.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a *p*-Series with *p* = *1* so it diverges, therefore

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ also diverges.

The Limit Comparison Test will often work in place of the Direct Comparison Test

Early, we used the Direct Comparison Test to do this example...

Example: Determine if the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

...and determined that the series diverges. It was difficult to find a series that was term-by-term larger (and we had to select one which only became larger after a certain number of terms). We could have instead just used the Limit Comparison Test:

Select another series, perhaps by ignoring the constant in the denominator: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Evaluate the limit of the ratio of terms: $\sum_{n=1}^{\infty} \frac{a_n}{b_n} = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2+\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{2+\sqrt{n}} \left(\frac{\infty}{\infty}\right)$ (use L'Hopital)

$\sum_{n=1}^{\infty} \frac{a_n}{b_n} = \frac{\frac{1}{2}n^{-\frac{1}{2}}}{\frac{1}{2}n^{-\frac{1}{2}}} = 1$ which is finite and positive so the two series are 'locked together'

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a p-series with $p \le 1$ so it diverges, therefore the original series also diverges.

Summary of Tests so far...

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series ($r \neq 0$)	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \dfrac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

More examples... Determine if the following series converge or diverge

$\sum_{n=1}^{\infty} \frac{1}{n^3-1}$ compare to $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$\frac{1}{n^3} < \frac{1}{n^3-1}$

direct comparison

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is p-series w/ $p=3$ (converges for $p>1$)

doesn't help w/ orig. series (wrong side)

try limit comparison

$\lim_{n\to\infty} \frac{\left(\frac{1}{n^3-1}\right)}{\left(\frac{1}{n^3}\right)} = \lim_{n\to\infty} \frac{n^3}{n^3-1} \quad \frac{\infty}{\infty}$

L'Hop $= \lim_{n\to\infty} \frac{3n^2}{3n^2} = 1$

finite, positive so series are "linked"

since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges,

$\sum_{n=1}^{\infty} \frac{1}{n^3-1}$ also **converges**

$\sum_{n=1}^{\infty} \frac{5}{2+3^n}$ compare to $\sum_{n=1}^{\infty} \frac{5}{3^n}$

$\frac{5}{2+3^n} < \frac{5}{3^n}$

direct comparison

$\sum_{n=1}^{\infty} \frac{5}{3^n} = \sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n$ is geometric w/ $r=1/3$ (converges)

$\frac{5}{2+3^n} < \frac{5}{3^n}$ (converges)

so this pushes down original series

and $\sum_{n=1}^{\infty} \frac{5}{2+3^n}$ also **converges**

$\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ compare to $\sum_{n=1}^{\infty} \frac{3^n}{2^n}$

$\frac{3^n}{2^n} < \frac{4+3^n}{2^n}$

direct comparison

$\sum_{n=1}^{\infty} \frac{3^n}{2^n} = \sum_{n=1}^{\infty} 1\left(\frac{3}{2}\right)^n$ is geometric w/ $r=3/2>1$ diverges

$\frac{3^n}{2^n} < \frac{4+3^n}{2^n}$

(diverges)

so this pushes up the original series

$\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ also **diverges**

$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+4}}$ compare to $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2} \cdot n^{-1}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$\frac{n}{\sqrt{n^5+4}} < \frac{n}{\sqrt{n^5}} \left(= \frac{1}{n^{3/2}}\right)$

direct comparison

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is p-series w/ $p=3/2>1$ converges

so it pushes down the orig. series

$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+4}}$ also **converges**

Unit 10-5: Alternating Series
Larsen: 8.5

The Alternating Series Test

An **Alternating Series** is a series in which the signs of the terms alternative from term to term:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\ldots$$

We can test whether or not an alternating series converges by using the **Alternating Series Test**:

> Let $a_n > 0$. The alternating series
>
> $$\sum_{n=1}^{\infty}(-1)^n a_n \quad and \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$
>
> converge when the following are both true:
>
> 1) $\lim_{n\to\infty} a_n = 0$
>
> 2) $a_{n+1} \le a_n$ *for all n*

Example: $\sum_{n=1}^{\infty}(-1)^n \frac{n}{2^{n-1}}$

$\lim_{n\to\infty} \frac{n}{2^{n-1}}$ $\frac{\infty}{\infty}$

L'Hopital:

$= \lim_{n\to\infty} \frac{1}{\ln(2)2^{n-1}}$ $\frac{1}{\infty} = 0$ ✓

so series **converges**

$a_{n+1} \le a_n$?

$\frac{n+1}{2^n} \le \frac{n}{2^{n-1}}$

$\frac{n+1}{2 \cdot 2^{n-1}} \le \frac{n}{2^{n-1}}$

$\frac{\left(\frac{n+1}{2}\right)}{2^{n-1}} \le \frac{n}{2^{n-1}}$ ✓ true $\left(\frac{n+1}{2} < n\right)$

Alternating Series Test can only show convergence

If the two conditions apply for the Alternating Series Test, then the series converges, but technically if these do not apply, then we can't use the Alternating Series Test and must choose another test to verify that the series diverges.

Example: $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2}{n^2+4}$

$\lim_{n\to\infty} \frac{n^2}{n^2+4}$ $\frac{\infty}{\infty}$

L'Hopital:

$= \lim_{n\to\infty} \frac{2n}{2n} = 1$

$\neq 0$

so alternating series test can't be used

...instead: test for divergence

if $\lim_{n\to\infty} a_n \neq 0$ diverges

$\lim_{n\to\infty} \frac{n^2}{n^2+4} = 1 \neq 0$

so series **diverges**

More examples to try: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln(n)}{n}$

$\lim_{n\to\infty}\frac{\ln n}{n}\;\frac{\infty}{\infty}$

L'Hopital: $=\lim_{n\to\infty}\frac{\frac{1}{n}}{1}=\frac{1}{\infty}=0$ ✓

$a_{n+1}\le a_n$? not obvious

$\frac{\ln(n+1)}{n+1}\le\frac{\ln(n)}{n}$? check with derivative

$f(x)=\frac{\ln x}{x}$ $f'(x)=\frac{x(\frac{1}{x})-\ln x(1)}{x^2}=\frac{1-\ln x}{x^2}$

for larger x $\ln x>1$ so decreasing

so $a_{n+1}<a_n$ ✓ **converges**

$\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\ldots = \sum_{n=1}^{\infty}(-1)^n\frac{n}{n+2}$

$\lim_{n\to\infty}\frac{n}{n+2}\;\frac{\infty}{\infty}$ alternating series test can't be used.

L'Hopital $=\lim_{n\to\infty}\frac{1}{1}=1\neq 0$

but test for divergence: if $\lim_{n\to\infty}a_n\neq 0$ diverges

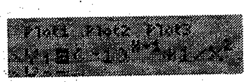
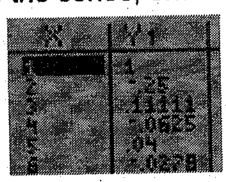
So this series **diverges**

Alternating Series Remainder

For a convergent alternating series, the partial sum S_n can be a useful approximation to the series sum S.
You can use a calculator Series function to either write out the terms of a series, or to find partial sums:

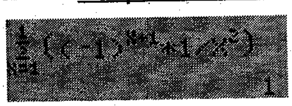
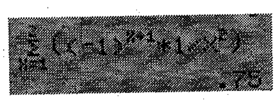
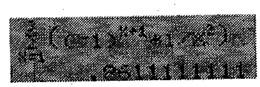
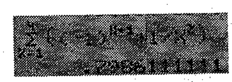
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$$

To write out the terms of the series, enter a_n as if you are going to graph and use a table:

a_1	a_2	a_3	a_4	...
1	−.25	.11111	−.0625	...

To write out the partial sums of the series, use the Summation function (under MATH):

S_1	S_2	S_3	S_4	...
1	.75	.861111	.798611	...

...or you can just write out the terms, them sum them up manually: $S_3=1-.25+.111111=.861111$

Alternating Series Remainder

If you go far enough out in the terms (and the alternating series is converging) you can get as close as you wish to the actual sum...you get closer to the actual sum the more terms you including in the partial sum. The difference between the actual sum and your approximation is called the **Remainder** and represents how much error there is in your approximation:

$$remainder,\ R_N=S-S_N \quad \textit{for an N term partial sum}$$

The following theorem is helpful for approximating an alternating series sum:

> If a convergent alternating series satisfies the condition $a_{n+1}\le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than or equal to the first neglected term:
>
> $$|S-S_N|=|R_N|\le a_{N+1}$$

Alternating Series Remainder

Example: Approximate the sum of the series by its first six terms: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}$

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}+\frac{1}{120}-\frac{1}{720}=\frac{91}{144}\approx .6319444$$

next term is the error $\frac{1}{5040}$

so approximation of S is

$$\frac{91}{144}-\frac{1}{5040}<S<\frac{91}{144}+\frac{1}{5040}$$

$$\boxed{0.631746<S<0.632143}$$

Finding N for a given allowable error

We can also determine the minimum number of terms a partial sum must have to approximate the series sum of an alternating series to a given maximum error:

Example: Determine the number of terms required to approximate the sum of the series with an error of less than 0.001: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$

error is N+1 term

so $\frac{1}{(N+1)^4}<.001$

$$(N+1)^4>1000$$

$$N+1>\sqrt[4]{1000}$$

$$N>\sqrt[4]{1000}-1$$

$$N>4.6234 \uparrow$$

$$\boxed{N=5}$$

Absolute Convergence

Sometimes, a series may have both positive and negative terms, but not be an alternating series, for example:

$$\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2}$$
$$=.84147+.22732+.01588-.0473-.0384-.0078+.01341\ldots$$

One way to learn things about the convergence of such a series is to investigate the series with absolute value of a_n:

$$\sum_{n=1}^{\infty}\left|\frac{\sin(n)}{n^2}\right|$$ Using the Direct Comparison Test, this series can be compared to $\sum_{n=1}^{\infty}\frac{1}{n^2}$

$$\left|\frac{\sin(n)}{n^2}\right|\le\frac{1}{n^2}$$ ...and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a *p*-Series with p = 2 which converges, so this 'squeezes' the left series to also converge.

$0<a_n\le b_n$ Therefore the absolute value of our original series, $\sum_{n=1}^{\infty}\left|\frac{\sin(n)}{n^2}\right|$ converges.

How does this help us evaluate the convergence of $\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2}$? We need another theorem...

If a series $\sum\lvert a_n\rvert$ converges, then the series $\sum a_n$ also converges.

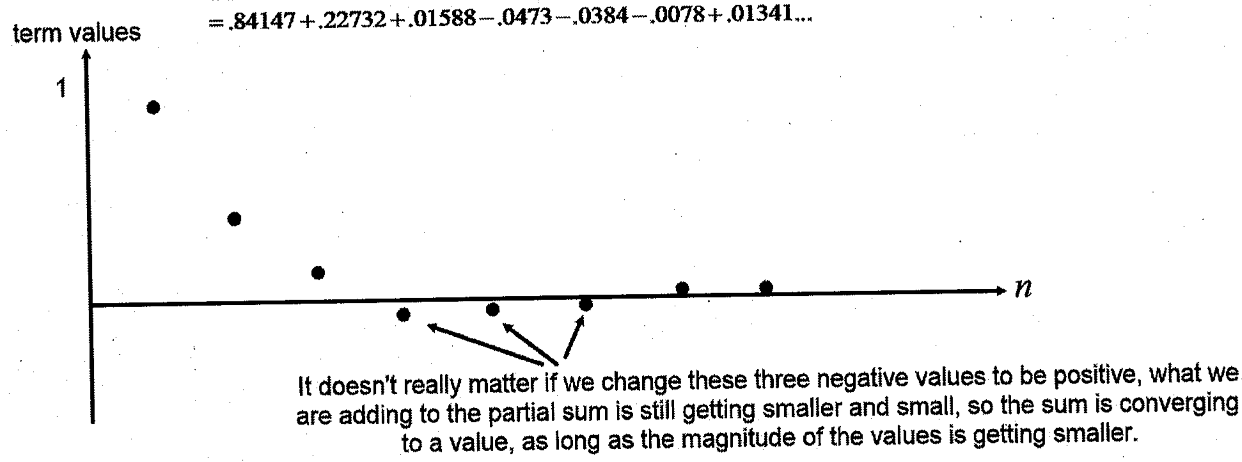
Therefore $\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2}$ also converges.

Absolute Convergence

If a series $\sum\lvert a_n\rvert$ converges, then the series $\sum a_n$ also converges.

The formal proof of this theorem is in our textbook, but consider this intuitively with our example:

$$\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2}$$
$$=.84147+.22732+.01588-.0473-.0384-.0078+.01341\ldots$$



But the converse is not true...

Okay, so if the absolute value of a series converges, then the original series also converges. But the converse would be, if a series converges, then the absolute value of the series also converges, and this is not necessarily true. Consider this example of an alternating harmonic series:

By the alternating series test...

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots$$

$$\lim_{n\to\infty}\frac{1}{n}=0 \qquad a_{n+1}\le a_n \qquad \frac{1}{n+1}\le\frac{1}{n}$$

this series converges.

But if we took the absolute value: $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n}\right|$

This would be equivalent to this series: $\sum_{n=1}^{\infty}\frac{1}{n}$ which is a *p*-Series with p = 1, which diverges.

Absolute vs. Conditional Convergence

There are specific terms for cases related to this, defined as follows:

> A series is called **Absolutely Convergent** if $\sum_{n=1}^{\infty}|a_n|$ converges (and therefore $\sum_{n=1}^{\infty}a_n$ also converges.)
>
> A series is called **Conditionally Convergent** if $\sum_{n=1}^{\infty}a_n$ converges, but $\sum_{n=1}^{\infty}|a_n|$ diverges.

Examples: Determine whether the series coverges absolutely or conditionally, or diverges.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n^2}\right|$

$= \sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series w/ $p=2$

so converges

therefore $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ converges absolutely

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+3}$$

$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n+3}\right|$

$= \sum_{n=1}^{\infty}\frac{1}{n+3}$

use limit comparison test

$b_n = \frac{1}{n}$

$\lim_{n\to\infty}\frac{a_n}{b_n}$

$= \lim_{n\to\infty}\frac{\frac{1}{n+3}}{\frac{1}{n}}$

$= \lim_{n\to\infty}\frac{n}{n+3}$ $\frac{\infty}{\infty}$

L'Hopital

$= \lim_{n\to\infty}\frac{1}{1} = 1$ finite, positive

So since $\sum_{n=1}\frac{1}{n}$ is p-series w/ $p=1$ (diverges)

then $\sum_{n=1}^{\infty}\frac{1}{n+3}$ diverges

original series: $\sum_{n=1}(-1)^{n+1}\frac{1}{n+3}$

use alternating series test:

$\lim_{n\to\infty}\frac{1}{n+3} = 0$ ✓

$a_{n+1} \le a_n$?

$\frac{1}{n+4} \le \frac{1}{n+3}$ ✓

So $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n+3}$ converges

therefore, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+3}$ is conditionally convergent

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n+3)}{n+10}$$

try alternating series test first:

$\lim_{n\to\infty}\frac{2n+3}{n+10}$ $\frac{\infty}{\infty}$

L'Hopitals

$= \lim_{n\to\infty}\frac{2}{1} = 2 \ne 0$

so alternating series test doesn't apply

but divergence test (nth term)

$\lim_{n\to\infty}a_n = 2 \ne 0$

mean this series diverges

limit compare w/ $\frac{2n}{n}$?

(diverges)

yes diverges

$\lim_{n\to\infty}a_n \ne 0$

Last point: Rearrangement of Series

Later in this unit we may encounter series where it is advantageous to rearrange the order of the terms, and determine the sum of the series. Counter-intuitively, when you rearrange the terms of an infinite series it can actually change the sum to which a series (if convergent) converges to. We can use the fact that the series is absolutely or conditional convergent to predict things about the effect of term rearrangement.

This is not an issue with a finite series: $1+3-2+5-4 \quad S=3$

$1+3+5-2-4 \quad S=3$

But consider this series: $$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\frac{1}{13}-\frac{1}{14}\cdots$$

here are the partial sums adding 1, then 2, then 3, etc. terms:

$$1,\ .5,\ .83,\ .58,\ .78,\ .62,\ .76,\ .63,\ .75,\ .6,\ .74,\ .6532,\ .7301,\ .6587 \rightarrow \ln 2$$

However, if you rearrange the terms like this:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=\left(\frac{1}{1}-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)\cdots$$

now you get these partial sums: $.5,\ .25,\ 417,\ .292,\ .392,\ .3083,\ .4869 \rightarrow \frac{1}{2}\ln 2$

...the partial sums are converging on a different sum for the series, just be regrouping the terms!

This is counter-intuitive but is a real effect, one of the many strange things that happen as we consider things involving infinity.

The reason this is important to us in this course, is that later we may want to rearrange terms while finding a sum, but we don't want to do that if rearrangement would affect the series sum. Fortunately, this effect doesn't happen for all series, and we can predict when it will happen:

> If a series is **absolute convergent**, then its terms can be rearranged in any order without changing the sum of the series.
>
> If a series is **conditionally convergent**, then rearranging its terms will change the sum of the series.

More examples

For what values of p is the series convergent?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$$

alt. series test:

$\lim_{n\to\infty}\frac{1}{n^p}=\frac{1}{\infty}=0$ ✓

$a_{n+1}\le a_n$?

$\frac{1}{(n+1)^p}\le\frac{1}{n^p}$ ✓

as long as $\boxed{p>1}$

$(p=0)$: $\frac{1}{1}\le\frac{1}{1}$

$(p<1)$: $\frac{(n+1)^{(-p)}}{1}\le\frac{(n)^{(-p)}}{1}$

(moves to numerator)

How many terms of the series do we need to approximate the sum to an error of < 0.002?

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}$$

N+1 term: $\frac{N+1}{4^{N+1}}<.002$

no good way to solve, just try some values

$n=1 \quad \frac{1}{4^1}=\frac{1}{4}=.25$

$n=2 \quad \frac{2}{4^2}=\frac{2}{16}=.125$

$n=3 \quad \frac{3}{4^3}=\frac{3}{64}=.046875$

$n=4 \quad \frac{4}{4^4}=\frac{4}{256}=.015625$

$n=5 \quad \frac{5}{4^5}=\frac{5}{1024}=.00488$

$\boxed{n=6} \quad \frac{6}{4^6}=\frac{6}{4096}=.00146$

Unit 10-6: The Ratio and Root Tests
Larsen: 8.6

The Ratio Test

> Let $\sum a_n$ be a series with nonzero terms:
>
> 1) The series $\sum a_n$ converges absolutely when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$
>
> 2) The series $\sum a_n$ diverges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ *or* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$
>
> 3) The Ratio Test is inconclusive when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$

** The Ratio Test is especially good for factorials and for series expressed as a function of the previous term.

Examples: $\sum_{n=1}^{\infty}\frac{2^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{\left(\frac{2^{n+1}}{(n+1)!}\right)}{\left(\frac{2^n}{n!}\right)}\right|$$

$$=\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{2\cdot 2^n\, n!}{(n+1)n!\, 2^n}\right|$$

$$=\lim_{n\to\infty}\frac{2}{n+1}$$

$$=0$$

$$<1$$

converges absolutely

$a_1=2,\quad a_{n+1}=\frac{5n+1}{4n+3}a_n$

2, 2, 2.13333333, 2.357894....

$$\frac{a_{n+1}}{a_n}=\frac{5n+1}{4n+3}$$

$$\lim_{n\to\infty}\frac{5n+1}{4n+3}=\frac{\infty}{\infty}$$

(L'Hôpital)

$$=\lim_{n\to\infty}\frac{5}{4}$$

$$=\frac{5}{4}$$

$$>1$$

diverges

$\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$

$$\lim_{n\to\infty}\frac{\sqrt{n+1}}{n+2}\cdot\frac{n+1}{\sqrt{n}}$$

$$=\lim_{n\to\infty}\sqrt{\frac{n+1}{n}}\cdot\lim_{n\to\infty}\frac{n+1}{n+2}$$

$$=\sqrt{\lim_{n\to\infty}\frac{n+1}{n}}\cdot\lim_{n\to\infty}\frac{n+1}{n+2}\qquad\frac{\infty}{\infty}\quad\frac{\infty}{\infty}$$

(L'Hôpital)

$$=\sqrt{\lim_{n\to\infty}\frac{1}{1}}\cdot\lim_{n\to\infty}\frac{1}{1}$$

$$\sqrt{1}\cdot 1=1$$

(inconclusive)

The Root Test

For a series $\sum a_n$:

1) The series $\sum a_n$ converges absolutely when $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

2) The series $\sum a_n$ diverges when $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ *or* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$

3) The Root Test is inconclusive when $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

** The Root Test is especially good for series involving nth powers.

Examples: $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

$\lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}}$

$= \lim_{n\to\infty} \frac{(e^{2n})^{1/n}}{(n^n)^{1/n}}$

$= \lim_{n\to\infty} \frac{e^2}{n}$

$= 0$

< 1

converges absolutely

$\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^n}$

$= \lim_{n\to\infty} \frac{2n}{n+1} = \frac{\infty}{\infty}$

(L'Hopital)

$= \lim_{n\to\infty} \frac{2}{1}$

$= 2$

> 1

diverges

If these tests are inconclusive, you must use another method to evaluate convergence...

Earlier example was inconclusive for the Ratio Test $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+1}$ $\left(\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1\right)$

alternating series test...

1) $\lim_{n\to\infty} a_n = 0$?

$\lim_{n\to\infty} \frac{\sqrt{n}}{n+1}$ $\frac{\infty}{\infty}$

(L'Hopital)

$= \lim_{n\to\infty} \frac{\frac{1}{2}n^{-1/2}}{1}$

$= \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$

✓

2) $a_{n+1} \le a_n$? check with derivative...

$f(x) = \frac{\sqrt{x}}{x+1} = \frac{x^{1/2}}{x+1}$

$f'(x) = \frac{(x+1)(\frac{1}{2}x^{-1/2}) - (x^{1/2})(1)}{(x+1)^2} = \frac{\frac{x+1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2}$

will be < 0 (decreasing) if

$\frac{x+1}{2\sqrt{x}} - \sqrt{x} < 0$

$\frac{x+1}{2\sqrt{x}} - \frac{\sqrt{x}(2\sqrt{x})}{2\sqrt{x}} < 0$

$\frac{x+1-2x}{2\sqrt{x}} < 0$, $\frac{1-x}{2\sqrt{x}} < 0$ decreasing for $n > 1$

so $a_{n+1} \le a_n$ ✓

So $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+1}$ **converges**

...to check if converges absolutely, must also check $\sum_{n=1}^{\infty}\left|(-1)^n \frac{\sqrt{n}}{n+1}\right|$

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ can evaluate using limit comparison w/ $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$

check $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt{n}}{n+1}\right)}{\left(\frac{\sqrt{n}}{n}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} \frac{n}{\sqrt{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \frac{\infty}{\infty}$

(L'Hopital)

$= \lim_{n\to\infty} \frac{1}{1} = 1$ finite, positive

so $\sum a_n$ & $\sum b_n$ converge or diverge together

so now evaluate $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

is a p-series w/ $p = 1/2 < 1$

so diverges

therefore $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1} = \sum_{n=1}^{\infty}\left|(-1)^n \frac{\sqrt{n}}{n+1}\right|$ diverges

therefore the original series, $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+1}$ is only **conditionally convergent**

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series ($r \neq 0$)	$\sum_{n=0}^{\infty} ar^n$	$\|r\| < 1$	$\|r\| \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Alternating Series ($a_n > 0$)	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\|R_N\| \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} < 1$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} > 1$ or $= \infty$	Test is inconclusive when $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| > 1$ or $= \infty$	Test is inconclusive when $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Unit 10: Strategies for testing Series

Some hints from various textbooks on how to determine the best test for series convergence...

Try these, in order:
1) Does the *n*th term approach 0? If not, the series diverges.
2) Is the series one of the special types - geometric, *p*-Series, telescoping, or alternating?
3) Can the Integral Test, the Root Test, or the Ratio Test be applied?
4) Can the series be compared favorably to one of the special types?

It is not wise to apply a list of tests in a specific order until one finally works...uses up too much time. Instead, classify the series according to its *form*:

1) $\sum \frac{1}{p_n}$ *is p – Series, convergent if* $p > 1$, *divergent if* $p \leq 1$

2) $\sum ar^{n-1}$ *or* $\sum ar^{n}$ *is geometric, convergent if* $|r| < 1$, *divergent if* $|r| \geq 1$

3) Form similar to *p*-Series or geometric? Try comparison test. In particular, if a_n is a rational function or an algebraic function of *n* (involving roots of polynomials), then the series should be compared with a *p*-Series. The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.

4) If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$; then the Test for Divergence should be used.

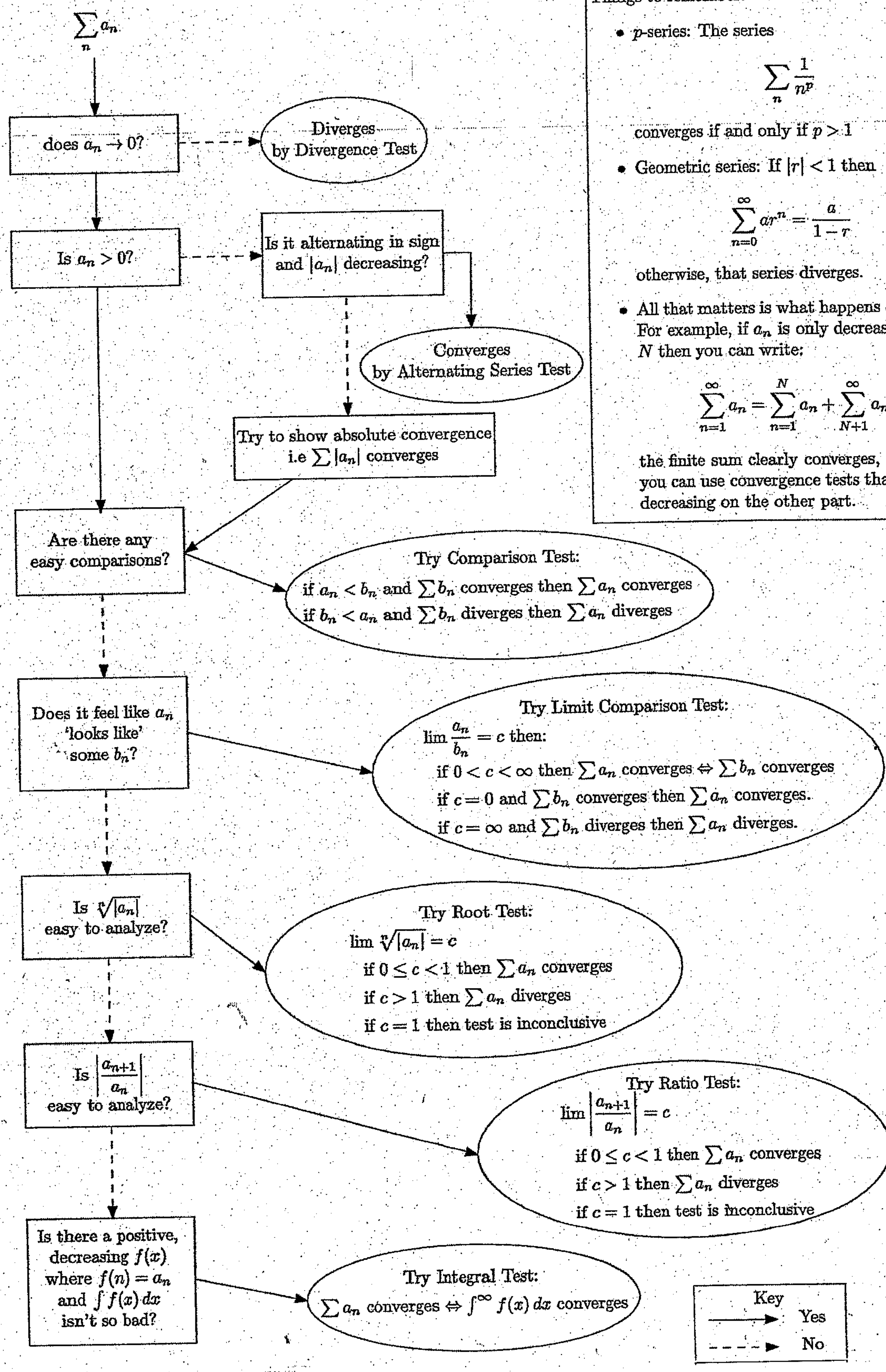
5) If the series is of the form $\sum (-1)^n a_n$ *or* $\sum (-1)^{n+1} a_n$ try the Alternating Series Test.

6) For series that involve factorials or other products (including a constant raised to the *n*th power) the Ratio Test is often a good choice. But Ratio Test is not good for p-Series and all rational or algebraic functions of *n*.

7) If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8) If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (if conditions are satisfied).

Series Convergence Flowchart



Things to remember:

- p-series: The series

$$\sum_n \frac{1}{n^p}$$

converges if and only if $p > 1$

- Geometric series: If $|r| < 1$ then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

otherwise, that series diverges.

- All that matters is what happens on a tail. For example, if a_n is only decreasing after N then you can write:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{N+1}^{\infty} a_n$$

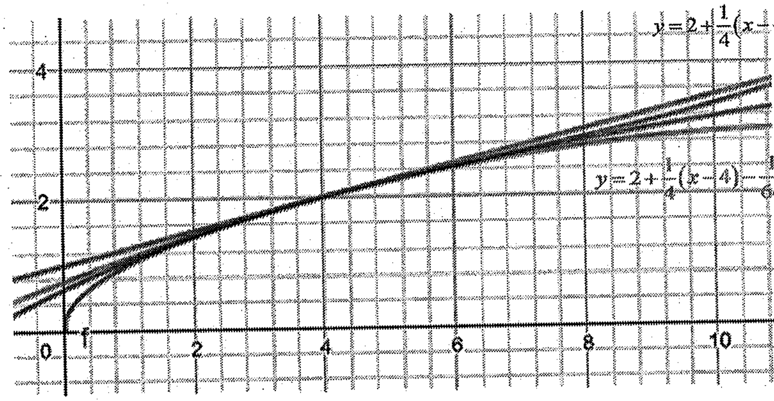
the finite sum clearly converges, and then you can use convergence tests that require decreasing on the other part.

Unit 10-7: Taylor Polynomials and Approximations
Larsen: 8.7

Polynomials can approximate a function over a small region

There are times when it is advantageous to use a polynomial to approximate a function. The approximation may match the function's y-value exactly at one x-value, and then approximately match y over a small region around this x-value:

(approximating around x=4)



The higher-degree polynomial you use, the better the approximation is over a given region of x-values.

The way we form such a polynomial is to first require that the y-values of the original function, $f(x)$, and the polynomial approximation function, $P(x)$, match at the desired x-value, which we label *c*:

$$P(c)=f(c)$$

To add terms in order to get a higher-degree polynomial approximation, we then also require one or more derivatives to be equal between the original function and the polynomial function. Wherever we decide to stop, that polynomial is called the 'nth order polynomial approximation', labeled $P_n(c)$ where *c* is the x-value we are approximating around.

For example, to create the fourth-degree approximating polynomial $P_4(x)$ we need:

$$P(c)=f(c)$$
$$P'(c)=f'(c)$$
$$P''(c)=f''(c)$$
$$P'''(c)=f'''(c)$$
$$P^{(4)}(c)=f^{(4)}(c)$$

This is easiest to understand by looking at a specific example...

Polynomials can approximate a function over a small region

Let's find a 3rd-order polynomial to approximate $f(x)=e^x$ around x=0.

We can first postulate a 3rd-degree polynomial with unknown coefficients:

$$P_3(x)=a_0+a_1(x-0)+a_2(x-0)^2+a_3(x-0)^3$$
$$P_3(x)=a_0+a_1x+a_2x^2+a_3x^3$$

We can then take 3 derivatives of both the original function and the polynomial approximation:

original function	*polynomial approximation function*
$f(x)=e^x$	$P_3(x)=a_0+a_1x+a_2x^2+a_3x^3$
$f'(x)=e^x$	$P_3'(x)=a_1+2a_2x+3a_3x^2$
$f''(x)=e^x$	$P_3''(x)=2a_2+6a_3x$
$f'''(x)=e^x$	$P_3'''(x)=6a_3$

Now we can replace all *x* values with the particular *c* value (in this case 0) and set the original function and polynomial function equal for each derivative (and y expressions). This will form a system of equations where the values to solve for are the unknown coefficients:

original function	*polynomial approximation function*	
$f(0)=e^0=1$	$P_3(0)=a_0+a_1(0)+a_2(0)^2+a_3(0)^3=a_0$	$a_0=1$
$f'(0)=e^0=1$	$P_3'(0)=a_1+2a_2(0)+3a_3(0)^2=a_1$	$a_1=1$
$f''(0)=e^0=1$	$P_3''(0)=2a_2+6a_3(0)=2a_2$	$2a_2=1$
$f'''(0)=e^0=1$	$P_3'''(0)=6a_3$	$6a_3=1$

system:

$$\begin{cases} a_0=1 \\ a_1=1 \\ 2a_2=1 \\ 6a_3=1 \end{cases}$$

solution to the system: $a_0=1,\ a_1=1,\ a_2=\frac{1}{2},\ a_3=\frac{1}{6}$

There is a pattern to these coefficients: $\left(a_2=\frac{1}{2!},\ a_3=\frac{1}{3!}\right)$

That means $$\boxed{f(x)=e^x\approx P_3(x)=1+1x+\frac{1}{2}x^2+\frac{1}{6}x^3 \quad (\textit{in the vicinity of } x=0)}$$

Taylor and Maclaurin Polynomials

Defining this process more generally provides the definition of **nth Taylor and Maclaurin Polynomial Approximations**:

If f has n derivatives at c, then the polynomial

$$P_n(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+....+\frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the ***nth Taylor polynomial for f at*** c.

If $c=0$, then the polynomial

$$P_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+....+\frac{f^{(n)}(0)}{n!}x^n$$

is called the ***nth Maclaurin polynomial for f at*** c.

(the full derivation of this more general definition is shown in our textbook if you are interested in seeing it)

1) Find the n=3 Taylor polynomial for $f(x)=\sqrt{x}$ at c = 4

2) Find the nth Maclaurin polynomial for $f(x)=e^x$

3) Find the n=4 Taylor polynomial for $f(x)=\ln x$ at c = 1

① $f=\sqrt{x}=x^{1/2}$

$f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

$f''(x)=-\frac{1}{4}x^{-3/2}$

$f'''(x)=\frac{3}{8}x^{-5/2}$

$P_3(4)=f(4)+f'(4)(x-4)+\frac{f''(4)}{2!}(x-4)^2+\frac{f'''(4)}{3!}(x-4)^3$

$=\boxed{2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3}$

② $f(x)=e^x$

$f'(x)=e^x$

$f''(x)=e^x$

$\vdots$

$f^{(n)}(x)=e^x$

$P_n(0)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3+\dots+\frac{f^{(n)}(0)}{n!}(x-0)^n$

$=\boxed{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\dots\frac{1}{n!}x^n}$

③ $f(x)=\ln x$

$f'(x)=\frac{1}{x}=x^{-1}$

$f''(x)=-x^{-2}=\frac{-1}{x^2}$

$f'''(x)=2x^{-3}=\frac{2}{x^3}$

$f^{(4)}(x)=-6x^{-4}=\frac{-6}{x^4}$

$P_4(1)=f(1)+f'(1)(x-1)^2+\frac{f''(1)}{2!}(x-2)^2+\frac{f'''(1)}{3!}(x-2)^3+\frac{f^{(4)}(1)}{4!}(x-2)^4$

$=\boxed{0+(x-1)^2-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4}$

Error and the Remainder of a Taylor Polynomial

Consider our Taylor polynomial approximation to $f(x)=\ln x$

$$P_4(x)=0+(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4$$

it will be exactly correct at the x we approximated around:

$$f(1)=\ln(1)=0 \qquad P_4(1)=0+((1)-1)-\frac{1}{2}((1)-1)^2+\frac{1}{3}((1)-1)^3-\frac{1}{4}((1)-1)^4=0$$

But if evaluate at a different x, such as x=1.5, a Taylor polynomial approximation will not be exact, and the more terms we include in the approximation, the lower the difference between the exact value and the approximation will be:

$$f(1.5)=\ln(1.5)=\underline{0.4054651081}$$

$$P_1(1)=0+((1.5)-1)=0.5 \quad \textit{difference}=\underline{0.094535}$$

$$P_2(1)=0+((1.5)-1)-\frac{1}{2}((1.5)-1)^2=0.375 \quad \textit{difference}=\underline{-0.030465}$$

$$P_3(1)=0+((1.5)-1)-\frac{1}{2}((1.5)-1)^2+\frac{1}{3}((1.5)-1)^3=0.416666667 \quad \textit{difference}=\underline{0.01120}$$

$$P_4(1)=0+((1.5)-1)-\frac{1}{2}((1.5)-1)^2+\frac{1}{3}((1.5)-1)^3-\frac{1}{4}((1.5)-1)^4=0.4010416667 \quad \textit{difference}=\underline{-0.004423}$$

This difference between the value of the actual function and the Taylor polynomial, is called the **remainder**:

$$f(x)=P_n(x)+R_n(x)$$

exact value — approx. value — remainder

The absolute value of the remainder is called the **error**:

$$\boxed{Error=|R_n(x)|=|f(x)-P_n(x)|}$$

Taylor's Theorem states:

> **If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that:**
>
> $$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\ldots+\frac{f^{(n)}(c)}{n!}(x-c)^n+R_n(x)$$
>
> **where**
>
> $$R_n(x)=\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

This is saying that the maximum error will always be less than or equal to the first term in the approximation which is not included (what would be the next term if you continued the polynomial to higher degree), and this error will always be less than something called the Lagrange Error Bound for a Taylor Polynomial:

$$|R_n(x)|\le\frac{|x-c|^{n+1}}{(n+1)!}\max\left|f^{(n+1)}(z)\right|$$

The z is whichever value in the interval between x (the x-value we are approximating) and c (the value we've centered the Taylor polynomial at) which produces the maximum value of the derivative of the first neglected term.

Error and the Remainder of a Taylor Polynomial

We can use this either to determine the accuracy of a Taylor polynomial with a known number of terms or we can use this to determine how many terms (the degree of) the Taylor polynomial which would be required to guarantee an error below some specified value.

Examples: 1) Find the n=3 Maclaurin polynomial approximation for $f(x) = \sin x$ then use the polynomial to approximate $f(0.1)$ and determine the accuracy of the approximation.

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \quad (c=0)$$

$f(x) = \sin x$ $\quad f(0) = \sin(0) = 0$

$f'(x) = \cos x$ $\quad f'(0) = \cos(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = -\sin(0) = 0$

$f'''(x) = -\cos x$ $\quad f'''(0) = -\cos(0) = -1$

$$P_3(x) = (0) + (1)(x-0) + \frac{0}{2}(x-0)^2 + \frac{(-1)}{6}(x-0)^3$$

$$\boxed{P_3(x) = x - \frac{1}{6}x^3}$$

error $\leq$ 1st rejected term

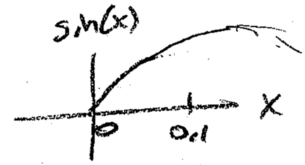
$$\text{error} \leq \frac{f^{(n)}(z)}{4!}(x-0)^4 \qquad f^{(4)}(x) = \sin x$$

$$\text{error} \leq \frac{\sin(z)}{4!}(x-0)^4$$

by theorem: we should find z in the interval and use that value.

f(0.1): c=0, x=0.1 so the interval is 0 — 0.1

and sine increases for larger x in this interval:



so z=0.1, therefore:

$$\text{error} \leq \frac{\sin(0.1)}{4!}(0.1-0)^4 = \boxed{4.2\ 10^{-7}}$$

and $P(0.1) = 0.1 - \frac{1}{6}(0.1)^3 = 0.099833\overline{3}$ so adding and subtracting this error:

$$\boxed{0.0998329133 < \sin(0.1) < 0.0998337533}$$

but in practice, many textbooks, including ours, don't bother to find z and just state that the max for sin(x) = 1, so they use this value for sin(z):

$$\text{error} \leq \frac{(1)}{4!}(0.1-0)^4 = \boxed{4.2\ 10^{-6}}$$

which means:

$$\boxed{0.0998291333 < \sin(0.1) < 0.0998375333}$$

2) Determine the degree of the Taylor polynomial expanded about $c = 1$ needed to approximate $\ln(1.2)$ to within 0.001 of the correct value.

we previous found $\ln(x) \approx 0 + (x-1)^2 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots$ centered at $c=1$

so for $\ln(1.2)$ $\quad c=1,\ x=1.2$

we could take derivatives to see if there is a pattern:

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -x^{-2} = \frac{-1}{x^2}$

$f'''(x) = 2x^{-3} = \frac{2}{x^3}$

$f^{(4)}(x) = -6x^{-4} = \frac{-6}{x^4}$

unlike e^x, or $\sin x$ which has an easily determined $\max|f^{(n+1)}(z)|$ ← this would be difficult to know without knowing n

so... switch to just brute force... try terms until one drops below error allowed.

$P_1 = (x-1)$, next term is $\frac{1}{2}(x-1)^2$ (max at $x=1.2$), error $\leq \frac{1}{2}(1.2-1)^2 = .02$ (not good enough)

$P_2 = (x-1) - \frac{1}{2}(x-1)^2$, next term is $\frac{1}{3}(x-1)^3$ (max at $x=1.2$), error $\leq \frac{1}{3}(1.2-1)^3 = .0026667$ (not good enough)

$P_3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$, next term is $\frac{1}{4}(x-1)^4$ (max at $x=1.2$), error $\leq \frac{1}{4}(1.2-1)^4 = .0004$ ✓

$\boxed{n=3}$

Unit 10-8: Power Series
Larsen: 8.8

Taylor Polynomials approximate a given function, Power Series provides an exact representation

The Taylor/Maclaurin polynomials we've learned provide an approximation for a given function in the vicinity of the x-value where we center the approximation, and the higher-degree polynomial we use, the better the approximation.

In this section, we will study **Power Series** which are infinite series which provide an exact equivalent representation of a given function. However, this representation may only converge to the representative value over some portion of the x-value domain.

> If x is a variable, then an infinite series of the form:
>
> $$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$
>
> is called a **Power Series**. More generally, an infinite series of the form:
>
> $$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots + a_n(x-c)^n + \ldots$$
>
> is called a **Power Series centered at *c***, where c is a constant.

Examples of Power Series

A Power Series centered at 0: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

A Power Series centered at -1: $\displaystyle\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 - (x+1)^3 + \ldots$

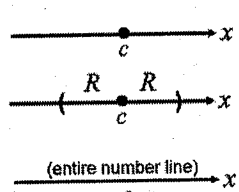
A Power Series centered at 5: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}(x-5)^n = (x-5) + \frac{1}{2}(x-5)^2 + \frac{1}{3}(x-5)^3 + \ldots$

Radius and Interval of Convergence

A Power Series can be thought of as a function of x: $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

where the domain of f is the set of all x for which the power series converges.

The constant c always lies in the domain of f, and there are 3 possibilities for the domain:



1) A single point: *series converges only at* $x = c$

2) an interval with radius R: *series converges absolutely for* $|x-c| < R$, *diverges for* $|x-c| > R$

3) all x: *series converges absolutely for all* x

R = **Radius of convergence** of the power series:
1) A single point: $R = 0$
2) an interval: $R = R$
3) all x: $R = \infty$

The set of all x for which the series converges is called the **interval of convergence**.

Determining the Radius and Interval of Convergence

Although we may use any method to determine convergence of a power series, our main tool is the Ratio Test.

Examples: Find the radius of convergence for the given series:

$$\sum_{n=0}^{\infty} 3(x-2)^n$$

ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$\lim_{n\to\infty}\left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(x-2)(x-2)^n}{(x-2)^n}\right|$

$\lim_{n\to\infty}|x-2|$

$|x-2|<1$ (centered at 2)

$1<x<3$

$R=2$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

ratio test:

$\lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$

$\lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$

$\lim_{n\to\infty}\left|\frac{x^2 x^{2n+1}}{(2n+3)(2n+2)(2n+1)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$

$\lim_{n\to\infty}\left|\frac{1}{(2n+3)(2n+2)}\cdot\frac{x^2}{1}\right|$

$\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)}|x^2|$

$(0)|x^2|$

$0<1$

converges for all x

$-\infty<x<\infty$

$R=\infty$

$$\sum_{n=0}^{\infty} n!x^n$$

ratio test:

$\lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$

$\lim_{n\to\infty}\left|\frac{(n+1)n!\,x\,x^n}{n!\,x^n}\right|$

$\lim_{n\to\infty}|(n+1)x|$

$\lim_{n\to\infty}(n+1)|x|<1$

$|x|<\lim_{n\to\infty}\frac{1}{n}$

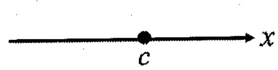
$|x|<0$

only when $x=0$

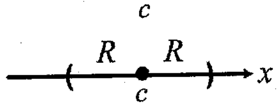
$R=0$

Endpoint Convergence

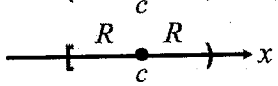
The Ratio Test is inconclusive if the limit = 1, so that means that we must test for convergence at $x - R$, and $x + R$ by hand using others tests of convergence. Each endpoint may converge or diverge independently of the other, which means there are really 6 possible cases for the interval of convergence:

1) A single point:


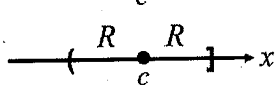
series converges only at $x=c$

2a)


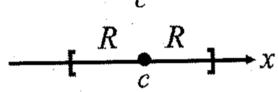
series converges for $(x-R,\ x+R)$

2b)


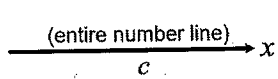
series converges for $[x-R,\ x+R)$

2c)


series converges for $(x-R,\ x+R]$

2d)


series converges for $[x-R,\ x+R]$

3) all x:


series converges for $(-\infty,\ \infty)$

Endpoint Convergence

Examples: Find the interval of convergence for the given series:

$$\sum_{n=1}^{\infty}\frac{x^n}{n}$$

ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$ *converges*

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\frac{n}{x^n}\right|$$

$$\lim_{n\to\infty}\frac{n}{n+1}|x|$$

$(1)|x|$ *converges when:*

$|x|<1$ *so* $-1<x<1$

Now test each endpoint...

when x = -1:

$$\sum_{n=1}^{\infty}\frac{x^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

is an alternating series,
so use alternating series test:

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{n}=0$$

$$a_{n+1}\overset{?}{<}a_n \quad \frac{1}{n+1}<\frac{1}{n}$$

converges for $x=-1$

when x = 1:

$$\sum_{n=1}^{\infty}\frac{x^n}{n}=\sum_{n=1}^{\infty}\frac{(1)^n}{n}=\sum_{n=1}^{\infty}\frac{1}{n}$$

is a p-Series with p = 1,
so it diverges, and the
power series diverges at $x=1$

interval of convergence = [−1,1)

$$\sum_{n=0}^{\infty}\frac{(-1)^n(x+1)^n}{2^n}$$

ratio test:

$$\lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(x+1)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x+1)(x+1)^n}{2\cdot 2^n}\frac{2^n}{(x+1)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{x+1}{2}\right|$$

$$\left|\frac{x+1}{2}\right|<1$$

$$|x+1|<2$$

$$R=2$$

(centered at $x=-1$)

so

$$-3<x<1$$

now test $x=-3$:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-3+1)^n}{2^n}=\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{2^n}$$

$$=\sum_{n=0}^{\infty}\frac{(2)^2}{2^n}=\sum_{n=0}^{\infty}(1) \text{ diverges}$$

test $x=1$:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(1+1)^n}{2^n}=\sum_{n=0}^{\infty}\frac{(-1)^n(2)^n}{2^n}$$

$$=\sum_{n=0}^{\infty}(-1)^n \text{ diverges}$$

so interval of convergence is $(-3,1)$

$$\sum_{n=1}^{\infty}\frac{x^n}{n^2}$$

ratio test:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\cdot\frac{x\cdot x^n}{x^n}\right|$$

$$\lim_{n\to\infty}\frac{n^2}{(n+1)^2}|x|$$

$\frac{\infty}{\infty}$ by L'Hopital's rule:

$$=\lim_{n\to\infty}\frac{2n}{2(n+1)} \quad \frac{\infty}{\infty}$$

(L'Hopital)

$$=\lim_{n\to\infty}\frac{2}{2}=1$$

$1|x|<1 \quad -1<x<1$

now test $x=-1$:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$$

alternating series test:

$\lim_{n\to\infty}\frac{1}{n^2}=0$ ✓

$a_{n+1}<a_n$?

$\frac{1}{(n+1)^2}<\frac{1}{n^2}$ ✓

converges at $x=-1$

test $x=1$:

$$\sum_{n=1}^{\infty}\frac{(1)^n}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

is a p-series w/ $n=2$ (>1)
so converges at $x=1$

so interval of convergence is $[-1,1]$

Derivatives and Integrals of Power Series

Since a power series can be thought of as equivalent to a function, you can also take derivatives and integrals of power series.

If the function $f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c)+a_2(x-c)^2+...a_3(x-c)^3+...$

has a radius of convergence of $R>0$, then, on the interval $(c-R,\ c+R)$

f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are:

$$f'(x)=\sum_{n=0}^{\infty} na_n(x-c)^{n-1} = a_1+2a_2(x-c)+3a_3(x-c)^2+...$$

$$\int f(x)\,dx = C+\sum_{n=0}^{\infty} a_n\frac{(x-c)^{n+1}}{n+1} = C+a_0(x-c)+a_1\frac{(x-c)^2}{2}+a_2\frac{(x-c)^3}{4}...$$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The interval of convergence, however, may differ as a result of behavior at the endpoints.

Derivatives and Integrals of Power Series

Example: Find the intervals of convergence of $f(x)$, $f'(x)$, and $\int f(x)\,dx$ for $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$$

ratio test: $\lim_{n\to\infty}\left|\frac{(x-1)^{n+2}}{(n+2)}\frac{(n+1)}{(x-1)^{n+1}}\right|$

$\lim_{n\to\infty}\left|\frac{(n+1)}{(n+2)}\frac{(x-1)(x-1)^{n+1}}{(x-1)^{n+1}}\right|$

$\lim_{n\to\infty}\frac{n+1}{n+2}|(x-1)|$

$(1)|(x-1)|<1$

$0<x<2$

test $x=0$...

$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(0-1)^{n+1}}{n+1}$

$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(-1)^{n+1}}{n+1}$

$\sum_{n=0}^{\infty}\frac{(1)^{n+1}}{n+1}=\sum_{n=0}^{\infty}\frac{1}{n+1}$

positive, decreasing

integral test:

$\int_1^{\infty}\frac{1}{x+1}dx=\left[\ln|x+1|\right]_1^{\infty}$

$\lim_{b\to\infty}\ln|b+1|-\ln|1|$

$\infty-0$ *diverges at* $x=0$

test $x=2$...

$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2-1)^{n+1}}{n+1}$

$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$

alternating series test

$\lim_{n\to\infty}\frac{1}{n+1}=0$

$a_{n+1}\overset{?}{<}a_n\quad\frac{1}{n+2}<\frac{1}{n+1}$

converges at $x=2$

$$\textit{interval of convergence}=(0,2]$$

$$f'(x)=\sum_{n=0}^{\infty}\frac{(n+1)(-1)^{n+1}(x-1)^{n}}{n+1}=\sum_{n=0}^{\infty}(-1)^{n+1}(x-1)^{n}$$

same R, so

$0<x<2$, *recheck endpoints*

test $x=0$...

$\sum_{n=0}^{\infty}(-1)^{n+1}(0-1)^{n}$

$\sum_{n=0}^{\infty}(-1)(-1)^{n}(-1)^{n}$

$\sum_{n=0}^{\infty}(-1)(1)^{n}$

$\sum_{n=0}^{\infty}(-1)$ *diverges at* $x=0$

test $x=2$...

$\sum_{n=0}^{\infty}(-1)^{n+1}(2-1)^{n}$

$\sum_{n=0}^{\infty}(-1)(-1)^{n}(1)^{n}$

$\sum_{n=0}^{\infty}(-1)(-1)^{n}$ *diverges at* $x=2$

$$\textit{interval of convergence}=(0,2)$$

$$\int f(x)\,dx=C+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}$$

same R, so

$0<x<2$, *recheck endpoints*

test $x=0$...

$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(0-1)^{n+2}}{(n+1)(n+2)}$

$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(-1)(-1)^{n+1}}{(n+1)(n+2)}$

$\sum_{n=0}^{\infty}\frac{(-1)(1)^{n+1}}{(n+1)(n+2)}$

$\sum_{n=0}^{\infty}\frac{(-1)}{(n+1)(n+2)}$

nth term test:

$\lim_{n\to\infty}\frac{(-1)}{(n+1)(n+2)}=0$

converges at $x=0$

test $x=2$...

$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2-1)^{n+2}}{(n+1)(n+2)}$

$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(1)^{n+2}}{(n+1)(n+2)}$

$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)(n+2)}$

alternating series test:

$\lim_{n\to\infty}\frac{1}{(n+1)(n+2)}=0$

$a_{n+1}\overset{?}{>}a_n\quad\frac{1}{(n+2)(n+3)}<\frac{1}{(n+1)(n+2)}$

converges at $x=2$

$$\textit{interval of convergence}=[0,2]$$

Geometric Power Series

It is sometimes helpful to represent a given function by a power series. We often do this by trying to match the given function to the sum of a geometric series.

If we have a function of the form $f(x)=\frac{1}{1-x}$ this closely resembles the sum of a geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r},\ |r|<1$

So if we match a = 1 and r = x, then

$$f(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty} ar^n=\sum_{n=0}^{\infty} 1(x)^n=1+x+x^2+x^3+....$$

over the interval of convergence for this series.

Therefore, a function may be represented by a power series (over its interval of convergence), although this representation only works at the x where the series is centered. Here, that would be $x = 0$.

If we wanted a power series to represent this function around $x = -1$, that would require a different series centered at -1.

That means in place of x we need (x+1). Here are the steps needed to convert from centering at 0 to -1:

$$\frac{1}{1-x}=\frac{1}{1-x+1-1}=\frac{1}{1+1-x-1}=\frac{1}{2-(x+1)}=\frac{\left(\frac{1}{2}\right)}{1-\left(\frac{x+1}{2}\right)}$$

Now we match to the form $\frac{a}{1-r}$ to find a and r: $a=\frac{1}{2},\quad r=\frac{x+1}{2}$

Then we build the corresponding geometric series: $\sum_{n=0}^{\infty} ar^n=\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{x+1}{2}\right)^n$

$$=\frac{1}{2}+\frac{1}{4}(x+1)+\frac{1}{8}(x+1)^2+\frac{1}{16}(x+1)^3+...$$

We need to also find the interval of convergence. Since this is a geometric series...

$$|r|<1$$
$$\left|\frac{x+1}{2}\right|<1$$
$$-1<\frac{x+1}{2}<1$$
$$-2<x+1<2$$
$$-3<x<1$$
$$(-3,1)$$

(We don't need to test endpoints because when a geometric series always diverges for r = 1)

Examples of converting functions to geometric form

We can sometimes also convert other function forms to geometric form to find corresponding geometric power series.

Example: $f(x)=\frac{4}{x+2}$ centered at 0

$$\frac{4}{2-(-x)}$$

$$\frac{2}{1-\left(-\frac{x}{2}\right)}$$

$$a=2, \quad r=-\frac{x}{2}$$

$$\sum_{n=0}^{\infty} ar^n=\sum_{n=0}^{\infty} 2\left(-\frac{x}{2}\right)^n$$

$$=2\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n$$

$$\left|\frac{x}{2}\right|<1$$

$$(-2,2)$$

Example: $f(x)=\frac{1}{x}$ centered at 1

$$\frac{1}{1-1+x}$$

$$\frac{1}{1-1-(-x)}$$

$$\frac{1}{1-(-x+1)}$$

$$a=1, \quad r=-x+1$$

$$\sum_{n=0}^{\infty} ar^n=\sum_{n=0}^{\infty} 1(-x+1)^n$$

$$=\sum_{n=0}^{\infty}(-1(x-1))^n$$

$$=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

$$|x-1|<1$$

$$-1<x-1<1$$

$$0<x<2$$

$$(0,2)$$

Using polynomial division to find power series

For some functions, we can use polynomial division to find the power series representation.

$f(x)=\frac{4}{x+2}$ Previous example, this is represented by:

$$2\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n=2-x+\frac{1}{2}x^2-\frac{1}{4}x^3+\ldots$$

$$\begin{array}{r} 2-x+\frac{1}{2}x-\frac{1}{4}x^3\ldots \\ 2+x\,\overline{)\,4\qquad\qquad\qquad} \\ -(4+2x) \\ \hline -2x \\ -(2x-x^2) \\ \hline x^2 \\ -\left(x^2+\frac{1}{2}x^3\right) \\ \hline -\frac{1}{2}x^3\ldots \end{array}$$

Operations with Power Series

Some properties related to operations on power series turn out to be helpful...

Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$

1. $f(kx)=\sum_{n=0}^{\infty} a_n k^n x^n$
2. $f(x^N)=\sum_{n=0}^{\infty} a_n x^{nN}$
3. $f(x)\pm g(x)=\sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

But operations of any kind may affect the interval of convergence. With these properties, if using more than one function, the interval of convergence is the intersection of the individual intervals:

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\left(1\pm\frac{1}{2^n}\right)x^n$$

$(-1,1)$ $(-2,2)$ *so* $(-1,1)$

An example where properties are helpful

$f(x)=\frac{3x-1}{x^2-1}$

centered at 0

First, this can be separated using Partial Fraction Expansion:

$$\frac{3x-1}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}$$

$$A(x+1)+B(x-1)=3x-1$$

$$(A+B)x+(A-B)=(3)x-1$$

$$\begin{cases} A+B=3 \\ A-B=-1 \end{cases} \quad A=1,\ B=2$$

$$\frac{3x-1}{x^2-1}=\frac{1}{x-1}+\frac{2}{x+1}$$

$$=\frac{-1}{1-x}+\frac{2}{1-(-x)}$$

$$\sum_{n=0}^{\infty}(-1)(x)^n + \sum_{n=0}^{\infty}(2)(-x)^n$$

$$\sum_{n=0}^{\infty}(-1)(x)^n + \sum_{n=0}^{\infty}(2)(-1)^n(x)^n$$

$$\sum_{n=0}^{\infty}\left(2(-1)^n-1\right)x^n$$

Using derivatives and integrals to find power series representations of other form function

So far, we've been restricted to rational function forms, but what if we wanted a power series representation for $f(x)=\ln(x)$?

Note that the derivative of this is: $f'(x)=\frac{1}{x}$ centered at 0

Which we previous saw was represented by $\sum_{n=0}^{\infty}(-1)^n(x-1)^n$ over $(0,2)$

Since $f(x)=\ln(x)=\int\frac{1}{x}\,dx$

$$f(x)=\int\sum_{n=0}^{\infty}(-1)^n(x-1)^n\,dx$$

$$=C+\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{n+1}}{n+1}$$

So $\boxed{\ln(x)=\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{n+1}}{n+1}}$

to establish C we use the x value where the series was centered here that was 1: $\ln(x)=C+\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{n+1}}{n+1}$

$$\ln(1)=C+\sum_{n=0}^{\infty}(-1)^n\frac{((1)-1)^{n+1}}{n+1}$$

$$0=C+\sum_{n=0}^{\infty}(-1)^n\frac{(0)^{n+1}}{n+1}$$

$$0=C+0,\quad C=0$$

When we use this procedure, we should recheck the endpoints. Sometimes, even if they were divergent before, now they may converge:

$$\ln(x)=\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{n+1}}{n+1}\qquad 0,2$$

$$(0,2]$$

$\underline{x=0}$

$$\sum_{n=0}^{\infty}(-1)^n\frac{((0)-1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{(-1)(-1)^n}{n+1}$$

$$\sum_{n=0}^{\infty}(-1)(1)^n\frac{1}{n+1}$$

$$-\sum_{n=0}^{\infty}\frac{1}{n+1}$$

$\sum_{n=0}^{\infty}\frac{1}{n+1}$ *is positive, decreasing*

use integral test:

$$\int_1^{\infty}\frac{1}{x+1}dx=\lim_{b\to\infty}\left[\ln|x+1|\right]_1^b$$

$\infty-0$ *diverges*

$\underline{x=2}$

$$\sum_{n=0}^{\infty}(-1)^n\frac{((2)-1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}$$

use alternating series test:

$$\lim_{n\to\infty}\frac{1}{n+1}=0$$

$$a_{n+1}\overset{?}{<}a_n$$

$$\frac{1}{n+2}<\frac{1}{n+1}$$

converges

Consider this result...

$$\ln(x)=\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{n+1}}{n+1}\quad\text{over }(0,2]$$

$$\ln(x)=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\ldots$$

how interesting and symmetrical this is about the point x value where lnx goes to zero :)

More examples using integrals

$f(x)=\ln(x+1)$ centered at 0

$$f'(x)=\frac{1}{x+1}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(1)(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n$$

$$so\ f(x)=\ln(x+1)=C+\int\sum_{n=0}^{\infty}(-1)^n x^n\,dx$$

$$=\ln(x+1)=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$

at x = 0: $\ln((0)+1)=C+\sum_{n=0}^{\infty}(-1)^n\frac{(0)^{n+1}}{n+1}$

$$0=C$$

$$so\ f(x)=\ln(x+1)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$

interval of convergence

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$

ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+2}}{(n+2)}\frac{(n+1)}{x^{n+1}}\right|=\lim_{n\to\infty}\frac{n+1}{n+2}\left|\frac{xx^{n+1}}{x^{n+1}}\right|=(1)|x|<1$$

$$-1<x<1$$

check endpoints:

$\underline{x=-1}$:

$$\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{n+1}}{n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{(-1)(-1)^n}{n+1}=\sum_{n=0}^{\infty}(-1)\frac{(1)^n}{n+1}=-\sum_{n=0}^{\infty}\frac{1}{n+1}$$

$\sum_{n=0}^{\infty}\frac{1}{n+1}$ *is positive, decreasing - integral test*

$$\int_1^{\infty}\frac{1}{x+1}\,dx=\lim_{b\to\infty}\left[\ln|x+1|\right]_1^b=\infty-0$$

diverges

$\underline{x=1}$:

$$\sum_{n=0}^{\infty}(-1)^n\frac{(1)^{n+1}}{n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}$$

alternating series test:

$$\lim_{n\to\infty}\frac{1}{n+1}=0$$

$$a_{n+1}\overset{?}{<}a_n$$

$$\frac{1}{n+2}<\frac{1}{n+1}$$ *converges*

so interval of convergence:

$$(-1,1]$$

You can sometimes find the sum of a convergent series by evaluating the function it represents

Example: Find the sum of the convergent series by using a well-known function.

$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{3^n n}$ In the last example we found: $\ln(x+1)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$

To match n in denominator in given series, we can rewrite the ln(x+1) series like this:

$$\ln(x+1)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

$$\ln(x+1)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

Now, try to turn the given series into the above form:

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{3^n n}=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\left(\frac{1}{3^n}\right)}{n}=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\left(\frac{1}{3}\right)^n}{n}$$

matches form with... $x=\frac{1}{3}$

so this series sum = $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{3^n n}=\ln\left(\left(\frac{1}{3}\right)+1\right)\approx 0.28768$

Another useful example which also includes a symbol substitution

$f(x)=\arctan x$ centered at 0

$f'(x)=\frac{1}{1+x^2}$ now use a symbol substitution $u=x^2$

$$f'(x)=\frac{1}{1+u}=\frac{1}{1-(-u)}=\sum_{n=0}^{\infty}(1)(-u)^n$$

re-substitute to return to the x variable...

$$f'(x)=\sum_{n=0}^{\infty}(1)(-x^2)^n=\sum_{n=0}^{\infty}(-1)^n(x^2)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

now integrate to find f(x)...

$$f(x)=\arctan x=C+\int\sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

and find c for x=0

$$\arctan(0)=C+\sum_{n=0}^{\infty}(-1)^n\frac{(0)^{2n+1}}{2n+1}$$

$$0=C+0,\ C=0$$

so $\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$

interval of convergence...

$$\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

ratio test:

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)}\frac{(2n+1)}{x^{2n+1}}\right|=\lim_{n\to\infty}\frac{2n+1}{2n+2}\left|\frac{xx^{2n+1}}{x^{2n+1}}\right|=(1)|x|<1$$

$$-1<x<1$$

test endpoints:

<u>$x=-1$:</u>

$$\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{(-1)(-1)^{2n}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{(-1)((-1)^2)^n}{2n+1}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{(-1)(1)^n}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{(-1)}{2n+1}=-\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$$

use alternating series test on $\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$

$$\lim_{n\to\infty}\frac{1}{2n+1}=0$$

$a_{n+1}\overset{?}{<}a_n,\ \frac{1}{2n+2}<\frac{1}{2n+1}$ *converges*

<u>$x=1$:</u>

$$\sum_{n=0}^{\infty}(-1)^n\frac{(1)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$$

use alternating series test on $\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$

$$\lim_{n\to\infty}\frac{1}{2n+1}=0$$

$a_{n+1}\overset{?}{<}a_n,\ \frac{1}{2n+2}<\frac{1}{2n+1}$ *converges*

so the interval of convergence is $[-1,1]$

Unit 10-10: Taylor and Maclaurin Series
Larsen: 8.10

Taylor and Maclaurin Series

In previous sections we've discovered that Taylor (or Maclaurin) polynomials can represent functions, and also power series can represent functions. In this last section we put these two ideas together to define **Taylor and Maclaurin Series**.

We defined a Taylor Polynomial for f at c as... $P_n(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\dots\frac{f^{(n)}(c)}{n!}(x-c)^n$

Consider a function which is represented by a power series: $f(x)=\sum_{n=0}^{\infty}a_n(x-c)^n$

We can write out the terms of the power series and then take derivatives...

$$f(x)=a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+\dots+a_n(x-c)^n+\dots$$
$$f'(x)=a_1+2a_2(x-c)+3a_3(x-c)^2+\dots+na_n(x-c)^{n-1}+\dots$$
$$f''(x)=2a_2+3(2)a_3(x-c)+\dots+n(n-1)a_n(x-c)^{n-2}+\dots$$
$$f'''(x)=3(2)a_3+\dots+n(n-1)(n-2)a_n(x-c)^{n-3}+\dots$$
...
$$f^{(n)}(x)=n!a_n$$

If these derivatives are evaluated at $x = c$, only the first terms remain:

$$f(c)=a_0$$
$$f'(c)=a_1$$
$$f''(c)=2a_2$$
$$f'''(c)=3(2)a_3$$
...
$$f^{(n)}(c)=n!a_n$$

...which means the coefficients for the terms of the power series are always of the form:

$$a_n=\frac{f^{(n)}(c)}{n!}$$

Taylor and Maclaurin Series

Substituting this into the power series, gives the definition of a **Taylor Series**:

> If a function f has derivatives of all orders at $x=c$, then the series
>
> $$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\dots\frac{f^{(n)}(c)}{n!}(x-c)^n+\dots$$
>
> is called the **Taylor Series for f at c**. If c = 0, then the series is called the **Maclaurin Series for f**.

Using the definition of Taylor Series to find a series represention of a function

Example: Find the Maclaurin Series for the function $f(x)=\sin x$

First, write out the function, and a few derivatives, and solving for the values at x = c (c = 0 for Maclaurin)...

$$\begin{array}{ll} f(x)=\sin x & f(0)=0 \\ f'(x)=\cos x & f'(0)=1 \\ f''(x)=-\sin x & f''(0)=0 \\ f'''(x)=-\cos x & f'''(0)=-1 \\ f^{(4)}(x)=\sin x & f^{(4)}(0)=0 \\ f^{(5)}(x)=\cos x & f^{(5)}(0)=1 \end{array}$$

Next, use the definition of Taylor Series to build some terms of the series...

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\ldots\frac{f^{(n)}(c)}{n!}(x-c)^n+\ldots$$

$$=0+1(x-0)+\frac{0}{2!}(x-0)^2+\frac{-1}{3!}(x-0)^3+\frac{0}{4!}(x-0)^4+\frac{1}{5!}(x-0)^5+\ldots$$

Then, eliminate any zero terms, and write the term number under each term...

$$=1(x-0)+\frac{-1}{3!}(x-0)^3+\frac{1}{5!}(x-0)^5+\ldots$$

$n=0 \qquad n=1 \qquad n=2$

Now we need to figure out an expression for the nth term...

$$\frac{(-1)^n}{(2n+1)!}(x-0)^{2n+1}$$

Finally write out the series:

$$\boxed{\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\ldots}$$

Using the definition of Taylor Series to find a series represention of a function

Example: Find the Taylor Series for the function $f(x)=\frac{1}{1-x}$ centered at c = 2.

First, write out the function. and a few derivatives. and solving for the values at x = c (c = 0 for Maclaurin)...

$$f(x)=\frac{1}{1-x}=(1-x)^{-1} \qquad f(2)=\frac{1}{1-2}=-1$$

$$f'(x)=-(1-x)^{-2}(-1)=(1-x)^{-2}=\frac{1}{(1-x)^2} \qquad f'(2)=1$$

$$f''(x)=-2(1-x)^{-3}(-1)=2(1-x)^{-3}=\frac{2}{(1-x)^3} \qquad f''(2)=-2$$

$$f'''(x)=2(-3)(1-x)^{-4}(-1)=6(1-x)^{-4}=\frac{6}{(1-x)^4} \qquad f'''(2)=6$$

$$f^{(4)}(x)=6(-4)(1-x)^{-5}(-1)=24(1-x)^{-5}=\frac{24}{(1-x)^5} \qquad f^{(4)}(2)=-24$$

Next, use the definition of Taylor Series to build some terms of the series...

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\ldots\frac{f^{(n)}(c)}{n!}(x-c)^n+\ldots$$

$$=-1+1(x-2)+\frac{-2}{2!}(x-2)^2+\frac{6}{3!}(x-2)^3+\frac{-24}{4!}(x-2)^4+\ldots$$

Then, eliminate any zero terms, and write the term number under each term and find an expression for the nth term...

$$= \underset{n=0}{-1} \quad \underset{n=1}{+(x-2)} \underset{n=2}{+\frac{-2}{2!}(x-2)^2} \underset{n=3}{+\frac{6}{3!}(x-2)^3} \underset{n=4}{+\frac{-24}{4!}(x-2)^4} +\ldots$$

$$\frac{(-1)^{n+1}n!}{n!}(x-2)^n=(-1)^{n+1}(x-2)^n$$

Finally write out the series:
$$\boxed{\frac{1}{1-x}=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n=-1+(x-2)-(x-2)^2+(x-2)^3+\ldots}$$

Binomial Series

If the function we are representing is of the form: $f(x)=(1+x)^k$

then the series is called a **Binomial Series**.

Example: Find the Maclaurin Series to represent $f(x)=(1+x)^k$

$$f(x)=(1+x)^k \qquad f(0)=1$$
$$f'(x)=k(1+x)^{k-1} \qquad f'(0)=k$$
$$f''(x)=k(k-1)(1+x)^{k-2} \qquad f''(0)=k(k-1)$$
$$f'''(x)=k(k-1)(k-2)(1+x)^{k-3} \qquad f'''(0)=k(k-1)(k-2)$$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\ldots\frac{f^{(n)}(c)}{n!}(x-c)^n+\ldots$$
$$=1+k(x-0)+\frac{k(k-1)}{2!}(x-0)^2+\frac{k(k-1)(k-2)}{3!}(x-0)^3\ldots$$

$$\boxed{=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3\ldots}$$

which can be shown to have an interval of convergence of (-1, 1)

Binomial Series example

Example: Find the power series for $f(x)=\sqrt[3]{1+x}$

Since $f(x)=\sqrt[3]{1+x}=(1+x)^{\frac{1}{3}}$ this will be a binomial series with k = 1/3, so instead of starting from scratch, we can just just use the more general result we found in the last example:

$$(1+x)^k=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\frac{k(k-1)(k-2)(k-3)}{4!}x^4\ldots$$
$$(1+x)^{\frac{1}{3}}=1+\frac{1}{3}x+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{4!}x^4\ldots$$
$$=1+\frac{1}{3}x+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}x^2+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}x^3+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!}x^4+\ldots$$

Now, we simplify things in each term (moving down some denominators) then try to find an nth term expression...

$$\boxed{(1+x)^{\frac{1}{3}}=1+\frac{1}{3}x+(-1)\frac{(2)}{3^2 2!}x^2+(1)\frac{(2)(5)}{3^3 3!}x^3+(-1)\frac{(2)(5)(8)}{3^4 4!}x^4+\ldots}$$

Sometimes, you can't find a good expression for the nth term, but at least we can write out a few representative terms.

Binomial Series example

Example: Find the power series for $f(x)=\frac{1}{(1+x)^4}=(1+x)^{-4}$

$$(1+x)^k=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\frac{k(k-1)(k-2)(k-3)}{4!}x^4\ldots$$

$$(1+x)^{-4}=1+-4x+\frac{-4(-4-1)}{2!}x^2+\frac{-4(-4-1)(-4-2)}{3!}x^3+\frac{-4(-4-1)(-4-2)(-4-3)}{4!}x^4\ldots$$

$$=1+(-1)4x+\frac{-4(-5)}{2!}x^2+\frac{-4(-5)(-6)}{3!}x^3+\frac{-4(-5)(-6)(-7)}{4!}x^4\ldots$$

$$=1+(-1)4x+(1)\frac{4(5)}{2!}x^2+(-1)\frac{4(5)(6)}{3!}x^3+(1)\frac{4(5)(6)(7)}{4!}x^4\ldots$$

Sometimes, you can use factorials with cancelation to represent strings of consecutive numbers...

$$=1+(-1)4x+(1)\frac{5(4)(3)(2)(1)}{(3)(2)(1)2!}x^2+(-1)\frac{6(5)(4)(3)(2)(1)}{(3)(2)(1)3!}x^3+(1)\frac{7(6)(5)(4)(3)(2)(1)}{(3)(2)(1)4!}x^4\ldots$$

$$=1+(-1)4x+(1)\frac{5!}{3!2!}x^2+(-1)\frac{6!}{3!3!}x^3+(1)\frac{7!}{3!4!}x^4\ldots$$

...which allows us to write an expression for the nth term and the series:

$$\boxed{\frac{1}{(1+x)^4}=\sum_{n=0}^{\infty}(-1)^n\frac{(n+3)!}{3!n!}x^n}$$

But more complicated function can be very difficult...

Example: Find the power series for $f(x)=\sin(x^2)$

Taking derivatives... $f(x)=\sin(x^2)$

$$f'(x)=\cos(x^2)(2x)$$

$$f''(x)=\cos(x^2)(2)+(2x)(-\sin(x^2)(2x))$$

...we can see that the derivatives quickly get very complicated, and are not of the same form, so it will not be possible in a case like this to use the definition of the Taylor Series to find the series.

In cases like this, we can employ a list of pre-calculated power series for elementary functions.

Power Series for Elementary Functions

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \leq x \leq 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \leq x \leq 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1^*$

*The convergence at $x = \pm 1$ depends on the value of k.

Using the basic list to find power series

Example: Find the power series for $f(x)=\cos(\sqrt{x})$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}+\ldots\frac{(-1)^n x^{2n}}{(2n)!}+\ldots$$

Just substitute the expression for the argument in place of the x...

$$\cos(\sqrt{x})=\sum_{n=0}^{\infty}\frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!}$$

$$\cos(\sqrt{x})=\sum_{n=0}^{\infty}\frac{(-1)^n\left(x^{\frac{1}{2}}\right)^{2n}}{(2n)!}$$

$$\boxed{\cos(\sqrt{x})=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n)!}=1-\frac{1}{2!}x+\frac{1}{4!}x^2-\frac{1}{6!}x^3\ldots}$$

Example: Find the power series for $f(x)=e^x\arctan x$

You can also combine basic list power series using arithmetic operation such as multiplication...

Note: you cannot just multiply the expressions for the nth term because of the term combination interactions.

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$$

$$e^x\arctan x=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right)$$

$$=1\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right)+x\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right)+\frac{x^2}{2!}\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right)+\frac{x^3}{3!}\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right)$$

$$=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+x^2-\frac{x^4}{3}+\frac{x^6}{5}-\frac{x^8}{7}+\frac{x^3}{2}-\frac{x^5}{6}+\frac{x^7}{10}-\frac{x^9}{14}+\frac{x^4}{6}-\frac{x^4}{18}+\frac{x^8}{30}-\frac{x^{10}}{42}+\ldots$$

$$\boxed{=x+x^2+\frac{1}{6}x^3-\frac{2}{9}x^4+\frac{1}{30}x^5+\frac{1}{5}x^6-\frac{3}{70}x^7-\frac{23}{210}x^8-\frac{1}{14}x^9-\frac{1}{42}x^{10}+\ldots}$$

Using the basic list to find power series

Example: Find the power series for $f(x) = \tan x$

You can use division to find a series for tanx...

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \ldots$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \ldots}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots}$$

using polynomial division...

$$\begin{array}{r} x + \frac{1}{3}x^3 + \frac{2}{15}x^5 \ldots \\ 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots \overline{\big) x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \ldots} \\ -\left(x - \frac{1}{2}x^3 + \frac{1}{24}x^5\right) \\ \hline \frac{1}{3}x^3 - \frac{1}{30}x^5 + \ldots \\ -\left(\frac{1}{3}x^3 - \frac{1}{6}x^5 + \frac{1}{72}x^7\right) \\ \hline \frac{2}{15}x^5 + \ldots \end{array}$$

$$\boxed{\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 \ldots}$$

Example: Find the power series for $f(x) = \sin^2 x$

You can use identities to convert things into forms in the basic list: $\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos(2x)$

$$\boxed{\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}}$$

We can just work with the nth term here because we are just multiplying every term by the same thing.

Using the basic list to find power series

Example: Use a power series to approximate $\int_0^1 e^{-x^2}\,dx$

First expand out terms using the basic list form:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots$$

$$e^{(-x^2)} = 1 + \left(-x^2\right) + \frac{1}{2}\left(-x^2\right)^2 + \frac{1}{6}\left(-x^2\right)^3 + \frac{1}{24}\left(-x^2\right)^4 + \frac{1}{120}\left(-x^2\right)^5 + \dots$$

$$e^{(-x^2)} = 1 + -x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10} + \dots$$

Then integrate term by term:

$$\int_0^1 e^{-x^2}dx = \int_0^1 \left(1 + -x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10} + \dots\right)dx$$

$$= \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 - \frac{1}{1320}x^{11} + \dots\right]_0^1$$

$$= \left[(1) - \frac{1}{3}(1)^3 + \frac{1}{10}(1)^5 - \frac{1}{42}(1)^7 + \frac{1}{216}(1)^9 - \frac{1}{1320}(1)^{11} + \dots\right] - \left[(0) - \frac{1}{3}(0)^3 + \frac{1}{10}(0)^5 - \frac{1}{42}(0)^7 + \frac{1}{216}(0)^9 - \frac{1}{1320}(0)^{11} + \dots\right]$$

$$\boxed{= (1) - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \dots} \approx 0.746729$$

Interval of convergence, Remainders, and Error

We've been focusing on finding a series to represent a given function, but not much about how well this series represents the function and over what interval. There are theorems addressing this. From earlier:

> If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that
>
> $$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^4 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$
>
> where $R_n(x) = \frac{f^{(n)}(z)}{(n+1)!}(x-c)^{n+1}$

This theorem told us that if you build a Taylor Polynomial (which is a Taylor Series truncated at some term), then the error between the series and the actual function is less than or equal to the value of the remainder expression.

$f^{(n)}(z)$ is the maximum value of the derivative in the interval of interest.

This section includes the following theorem:

> If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$
>
> where $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

This theorem is telling us that a power series formed with Taylor coefficients $a_n = \frac{f^{(n)}(c)}{n!}$

converges to a function from which it was derived at precisely those values for which the remainder approaches 0 as $n \to \infty$

Since the result is a series, you can always determine convergence and interval of convergence using the test from the first part of Unit 10. In practice, the bigger issue is that even if we have a Taylor series representation of a function we don't necessarily want to compute the infinite number of terms in the series to represent the original function.

Often, we are asked to use a Taylor series to estimate a given function to within a given amount of error.

Interval of convergence, Remainders, and Error

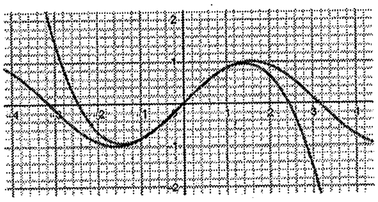
If we use the basic list to find a series representation for $f(x) = \sin x$ centered at $c = 0$

$$f(x) = \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \ldots$$

Let's see what happens if we truncate the series to just 2 terms or to just 4 terms:

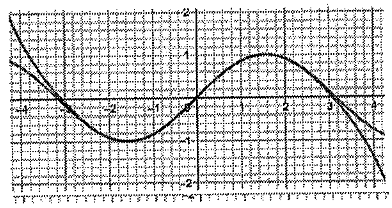
<u>2 terms:</u>

$$\sin x \approx x - \frac{1}{6}x^3$$



<u>4 terms:</u>

$$\sin x \approx x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$$



The theorems tell us that the error will be less than or equal to this value from the first dropped term: $R_n(x) = \frac{f^{(n)}(z)}{(n+1)!}(x-c)^{n+1}$

Here, since the function is sine, the nth derivative will be a sine or cosine, which has a maximum value of 1, so the error should be less than or equal to $error \le \frac{1}{(n+1)!}(x-c)^{n+1}$

For 2 terms, max error is next term: $\frac{1}{120}x^5$

For 4 terms, max error is next term: $\frac{1}{362880}x^9$

...but this error depends upon *x* (how far away we are from the centering value when we found the series).

Let's find the max error and actual error at x = 1 and at x = 2: $actual\ value: \sin(1) = 0.8414709848$

$actual\ value: \sin(2) = 0.9092974268$

<u>2 terms, error at x=1:</u>

max error

$\frac{1}{120}(1)^5 = 0.0083333$

approx value:

$\sin(1) = (1) - \frac{1}{6}(1)^3$

$= 0.83333333$

actual error:

0.833333333
<u>−0.841470948</u>
−0.00814

<u>2 terms, error at x=2:</u>

$\frac{1}{120}(2)^5 = 0.2666666$

approx value:

$\sin(2) = (2) - \frac{1}{6}(2)^3$

$= 0.666666666$

actual error:

0.6666666666
<u>−0.9092974268</u>
−0.24263

<u>4 terms, error at x=1:</u>

max error

$\frac{1}{362880}(1)^9 = 0.00000276$

approx value:

$\sin(1) = (1) - \frac{1}{6}(2)1^3 + \frac{1}{120}(1)^5 - \frac{1}{5040}(1)^7$

$= 0.841468254$

actual error:

0.841468254
<u>−0.841470948</u>
−0.00000269

<u>4 terms, error at x=2:</u>

$\frac{1}{362880}(2)^9 = 0.00141093$

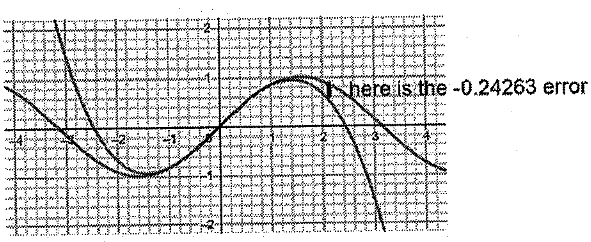
approx value:

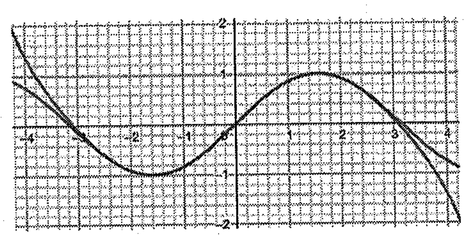
$\sin(2) = (2) - \frac{1}{6}(2)^3 + \frac{1}{120}(2)^5 - \frac{1}{5040}(2)^7$

$= 0.9080068593$

actual error:

0.9080068593
<u>−0.9092974268</u>
−0.00129

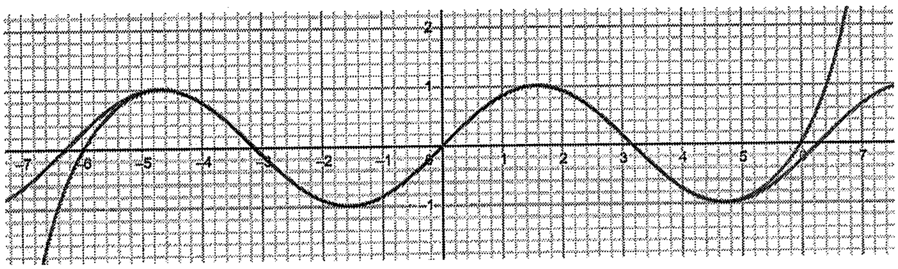




Interval of convergence, Remainders, and Error

These theorems also tell us that as long as we have a function which meets the requirements for creating a Taylor series representation, if we include all infinity of terms, then the series exactly matches the function for all x. This doesn't seem believable, but take a look at the Taylor series for sinx with more terms included:

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11} + \frac{1}{6227020800}x^{13} - ..$$



With just 7 terms, it is almost perfectly matching the sine curve over just under 2 full periods!

Examples of working with error

Use a Taylor series to approximate $\int_0^{0.5} x^2 e^{-x^2}\,dx$ with error < 0.001

$$\int_0^{0.5} x^2 e^{-x^2}\,dx$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + ...$$

$$e^{(-x^2)} = 1 + (-x^2) + \frac{1}{2}(-x^2)^2 + \frac{1}{6}(-x^2)^3 + \frac{1}{24}(-x^2)^4 + \frac{1}{120}(-x^2)^5 + \frac{1}{720}(-x^2)^6 + ...$$

$$e^{(-x^2)} = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10} + \frac{1}{720}x^{12} + ...$$

$$\int_0^{0.5} x^2 e^{-x^2}\,dx = \int_0^{0.5} x^2\left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10} + \frac{1}{720}x^{12} + ...\right)dx$$

$$= \int_0^{0.5}\left(x^2 - x^4 + \frac{1}{2}x^6 - \frac{1}{6}x^8 + \frac{1}{24}x^{10} - \frac{1}{120}x^{12} + \frac{1}{720}x^{14} + ...\right)dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{14}x^7 - \frac{1}{54}x^9 + \frac{1}{264}x^{11} - \frac{1}{1560}x^{13} + \frac{1}{10800}x^{15} + ...\right]_0^{0.5}$$

$$= \left[\frac{1}{3}(0.5)^3 - \frac{1}{5}(0.5)^5 + \frac{1}{14}(0.5)^7 - \frac{1}{54}(0.5)^9 + \frac{1}{264}(0.5)^{11} - \frac{1}{1560}(0.5)^{13} + \frac{1}{10800}(0.5)^{15} + ...\right] - 0$$

$$= .08333 - .00625 + \underline{.0000558}$$

error here is < 0.001 so only need the first 2 terms.

$$\boxed{\int_0^{0.5} x^2 e^{-x^2}\,dx \approx .08333 - .00625 = 0.0770833}$$