Sample Examination III

Section I Part A

Directions: Solve each of the following problems, using available space for scratchwork. After examining the form of the choices, decide which is the best of the choices given. Do not spend too much time on any one problem. Calculators may NOT be used on this part of the exam.

In this test: Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. The area of the region between the graph of $y = 3x^2 + 2x$ and the x-axis from $x = 1$ to $x = 3$ is

(A) 36

$\boxed{\text{(B)}}$ 34

(C) 31

(D) 26

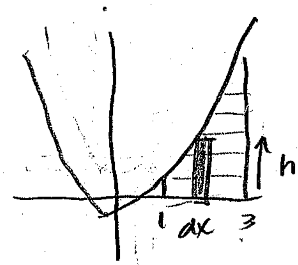
(E) 12

Intersection

$3x^2+2x=0$

$x(3x+2)=0$

$x=0 \quad x=-\frac{2}{3}$



$A=\int h\,dx$

$=\int_1^3 (3x^2+2x)\,dx$

$\left[x^3+x^2\right]_1^3$

$\left[3^3+3^2\right]-\left[1^3+1^2\right]$

$27+9-2$

$27+7$

34

2. $\lim_{x \to 3} \frac{(3-x)^2}{(x-3)}$ is $= \frac{0}{0}$ Indeterminant form, use L'Hopital's Rule!

$= \lim_{x \to 3} \frac{2(3-x)^1(-1)}{(1)} = \frac{0}{1} = 0$

(A) -2

(B) -1

(C) 0

(D) 1

(E) nonexistent

3. For a car traveling at a speed of s miles per hour, the fuel consumption of the car, $C(s)$, is measured in gallons per mile. What are the units of $\int_a^b C(s)\, ds$?

$\frac{gals}{mi}$ $\frac{mi}{hr}$

$\frac{gal}{hr}$

(A) gallons

(B) hours per gallon

(C) gallons per hour

(D) miles per hour per gallon

(E) gallons per miles per hour

4. If the radius of a sphere is increasing at the rate of 2 inches per second, how fast, in cubic inches per second, is the volume increasing when the radius is 10 inches?

(A) 40π

(B) 80π

(C) 800

$\boxed{\text{(D)}}$ 800π

(E) 3200π

$V_{sphere} = \frac{4}{3}\pi r^3$

Implicit differentiation WRT t:

$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$

$1 \cdot \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi(10)^2(2) = 800\pi$

5. A particle moves along a straight line so that its velocity is given by $v(t) = t^2$. How far does the particle travel between $t = 1$ and $t = 3$?

(A) $\frac{1}{3}$ $\boxed{\text{(B)}\ \frac{26}{3}}$ (C) 8 (D) 26 (E) 27

$s = \int v\,dt = \int_1^3 t^2\,dt = \left[\frac{1}{3}t^3\right]_1^3$

$= \frac{1}{3}3^3 - \frac{1}{3}1^3$

$= 9 - \frac{1}{3}$

$= \frac{26}{3}$

I find it easiest to multiply out first:

6. The derivative of $(4x)^3 \cdot (2x)^6$ is

$f = 4^3x^3 \cdot 2^6x^6 = 2^6 \cdot 2^6x^9 = 2^{12}x^9$

(A) $72x^8$

$f' = 2^{12}(9)x^8$ it is C, D, or E, try to match

(B) $124x^{17}$

(C) $30x(4x)^2(2x)^5$

(D) $\boxed{72x(4x)^2(2x)^5}$

(E) $144(4x)^2(2x)^5$

$x(4x)^2(2x)^5$

$x4^2x^2 2^5x^5$

$4^2 \cdot 2^5 x^8$

$2^4 2^5 x^8$

$2^9 x^8$

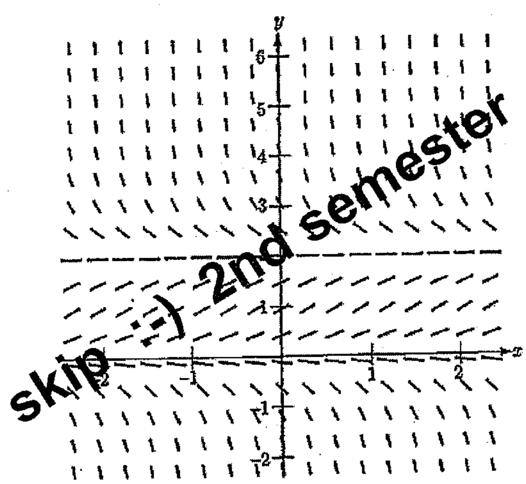
so $f' = 2^{12}(9)x^8 = Ax(4x)^2(2x)^5$

$= A2^9x^8$

$A = \frac{2^{12}(9)}{2^9} = 2^3 \cdot 9 = 72$

$f' = 72x(4x)^2(2x)^5$

7.



The slope field for a differential equation $\frac{dy}{dx} = f(y)$ is shown in the figure above.

Which statement is true about $y(x)$?

I. If $y(0) > 2$, then $\lim_{x\to\infty} y(x) \approx 2$

II. If $0 < y(0) < 2$, then $\lim_{x\to\infty} y(x) \approx 2$

III. If $y(0) < 0$, then $\lim_{x\to\infty} y(x) \approx 2$

(A) I only (B) II only (C) III only (D) I and II only (E) I, II, and III

8. $\int_1^3 \frac{x}{x^2+1}\,dx =$

(A) $\ln 5$ (B) $\ln 10$ (C) $2\ln 2$ $\boxed{\text{(D)}}$ $\frac{1}{2}\ln 5$ (E) $\ln\left(\frac{5}{2}\right)$

u-sub: $u=x^2+1$ $du=2x\,dx$ $x\,dx=\frac{1}{2}du$ $x=1 \to u=2$

$x=3 \to u=10$

$\frac{1}{2}\int_2^{10} \frac{1}{u}\,du = \frac{1}{2}\left[\ln|u|\right]_2^{10} = \frac{1}{2}(\ln 10 - \ln 2)$

$= \frac{1}{2}\ln\left(\frac{10}{2}\right)$

$= \frac{1}{2}\ln 5$

9. $\frac{d}{dx}\ln\left(\frac{1}{x^2-1}\right) =$ $\frac{1}{\left(\frac{1}{x^2-1}\right)}\frac{d}{dx}\left[\frac{1}{x^2-1}\right]$

quotent rule

(A) $\frac{1}{x^2-1}$ $(x^2-1)\left[\frac{\cancel{(x^2-1)(0)} - (1)(2x)}{(x^2-1)^2}\right]$

$\boxed{\text{(B)}}$ $-\frac{2x}{x^2-1}$ $\frac{\cancel{(x^2-1)}(-2x)}{\cancel{(x^2-1)}(x^2-1)}$

(C) $\frac{2x}{x^2-1}$ $\frac{-2x}{x^2-1}$

(D) $2x^3-2x$

(E) $2x-2x^3$

10. For $0 \leq x < \frac{\pi}{2}$, an antiderivative of $2\tan x$ is

(A) $\ln(\sec 2x)$

(B) $2\sec^2 x$

$\boxed{\text{(C)}}$ $\ln(\sec^2 x)$

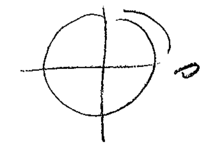
(D) $2\ln(\cos x)$

(E) $\ln(2\sec x)$

$2\int \tan x\,dx$

$2\ln|\sec(x)|$

$\pi/2$



$\cos x (+)$

$\sec x = \frac{1}{\cos x}$ (+) and defined

so ln defined

$= 2\ln(\sec x)$

log property:

$2\ln(\sec x)$

$\ln(\sec^2 x)$

11. If the substitution $u = x^2 + 1$ is made, the integral $\int_0^2 \frac{x^3}{x^2+1}\,dx =$

$\boxed{\text{(A)}}$ $\int_1^5 \frac{u-1}{2u}\,du$

(B) $\int_0^2 \frac{u-1}{2u}\,du$

(C) $\int_1^5 \frac{u+1}{2u}\,du$

(D) $\int_1^5 \frac{2(u-1)}{u}\,du$

(E) $\int_0^2 \frac{2(u-1)}{u}\,du$

$u = x^2 + 1$, $x^2 = u - 1$

$du = 2x\,dx$, $x\,dx = \frac{1}{2}du$

$x = 0 \Rightarrow u = 1$

$x = 2 \Rightarrow u = 5$

$\int_1^5 \frac{1}{u}(u-1)\frac{1}{2}du$

$\int_1^5 \frac{u-1}{2u}\,du$

12. The equation of the line tangent to the curve $y = \dfrac{kx+8}{k+x}$ at $x = -2$ is $y = x + 4$. What is the value of k?

quotient rule:

$$y' = \frac{(k+x)(k) - (kx+8)(1)}{(k+x)^2} = \frac{k^2 + kx - kx - 8}{(k+x)^2}$$

(A) -3

(B) -1

(C) 1

(D) 3

(E) 4

$$y' = \frac{k^2-8}{(k+x)^2} = m = 1 \quad \text{at } x = -2$$

$$k^2 - 8 = (k+x)^2$$

$$k^2 - 8 = (k-2)^2$$

$$k^2 - 8 = k^2 - 4k + 4$$

$$4k = 12$$

$$k = 3$$

13.

x	5	6	9	11	12
$f(x)$	10	7	11	12	8

The function f is continuous on the closed interval $[5, 12]$ and differentiable on the open interval $(5, 12)$ and f has the values given in the table above. Using the subintervals $[5, 6]$, $[6, 9]$, $[9, 11]$, and $[11, 12]$, what is the right-hand Riemann sum approximation to $\int_5^{12} f(x)\,dx$?

interval	x	f(x) · width	= area
[5,6]	6	7 · 1	= 7
[6,9]	9	11 · 3	= 33
[9,11]	11	12 · 2	= 24
[11,12]	12	8 · 1	= 8
			72

(A) 64

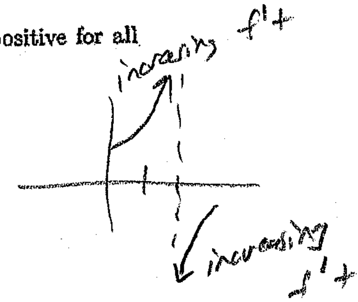
(B) 65

(C) 66

(D) 68.5

(E) 72

14. A function $f(x)$ has a vertical asymptote at $x = 2$. The derivative of $f(x)$ is positive for all $x \neq 2$. Which of the following statements are true?



I. $\lim_{x \to 2} f(x) = +\infty$ FALSE (DNE)

II. $\lim_{x \to 2^+} f(x) = +\infty$ FALSE ($-\infty$)

III. $\lim_{x \to 2^-} f(x) = +\infty$ True

(A) I only

(B) II only

(C) III only

(D) I and II only

(E) I, II, and III

15. If $y = 5^{(x^3-2)}$, then $\frac{dy}{dx} =$

$\frac{d}{dx}[a^x] = (\ln a)a^x$, then Chain Rule:

so $y' = (\ln 5)\, 5^{(x^3-2)} \cdot (3x^2)$

$= 3x^2(\ln 5) \cdot 5^{(x^3-2)}$

(A) $(x^3-2)5^{(x^3-2)}$

(B) $3x^2(\ln 5)5^{(x^3-2)}$

(C) $(3x^2)5^{(x^3-2)}$

(D) $(\ln 5)5^{(x^3-2)}$

(E) $x^3(\ln 5)5^{(x^3-2)}$

$\Delta x \quad x=0 \qquad f(x)=\sin(\pi x) \quad x=1$

16. If n is a positive integer, then $\lim\limits_{n\to+\infty} \frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{n\pi}{n}\right)$ is

Riemann Sum for $\Delta x = \frac{1}{n}$

(A) 0

(B) $\frac{2}{\pi}$

$x_i = i \sin\frac{\pi}{n}$

equivalent to $\int_0^1 \sin(\pi x)\,dx$

(C) $\frac{\pi}{2}$

(D) 2

$u=\pi x \quad du=\pi dx$

$dx=\frac{1}{\pi}du$

(E) 2π

$\frac{1}{\pi}\int_0^{\pi} \sin u\,du$

$\frac{1}{\pi}\left[-\cos u\right]_0^{\pi}$

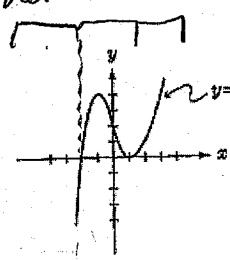
$\frac{1}{\pi}\left[-\cos\pi+\cos 0\right]$

$\frac{1}{\pi}\left[-(-1)+1\right]$

$\frac{2}{\pi}$

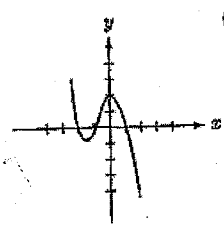
f: decr. incr increasing

17.



The graph of the derivative of f is shown in the figure above. Which of the following could be the graph of f?

(A)



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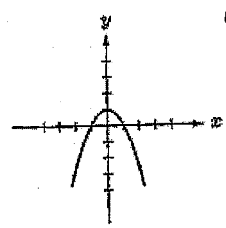
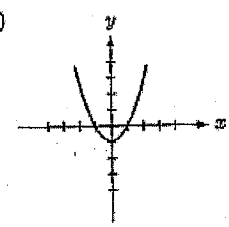
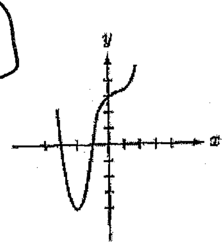


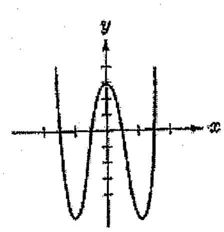
(C)



(D)



(E)



18. $\frac{d}{dx}\int_x^0 \frac{du}{1+u^2} = \frac{d}{dx}\left[-\int_0^x \frac{du}{1+u^2}\right] = -\frac{1}{1+x^2}$

(A) $\frac{1}{x^2+1}$ (B) $\boxed{\frac{-1}{x^2+1}}$ (C) x^2+1

(D) $-x^2+1$ (E) Arctan x

19. Let $f(x)$ be a differentiable function defined only on the interval $-2 \le x \le 10$. The table below gives the value of $f(x)$ and its derivative $f'(x)$ at several points of the domain.

x	-2	0	2	4	6	8	10
$f(x)$	26	27	26	23	18	11	2
$f'(x)$	1	0	-1	-2	-3	-4	-5

The line tangent to the graph of $f(x)$ and parallel to the segment between the endpoints intersects the y-axis at the point

Same slope as: $\frac{2-26}{10-(-2)} = \frac{-24}{12} = -2$

(A) $(0, 27)$

(B) $(0, 28)$

(C) $\boxed{(0, 31)}$

(D) $(0, 36)$

(E) $(0, 43)$

This slope occurs at $x = 4$

So this is the point of tangency, and the tangent line passes through $(4, 23)$

So tan equation is

$(y-23) = -2(x-4)$

$y - 23 = -2x + 8$

$y = -2x + 31$

& yint = $(0, 31)$

20. At each point (x, y) on a certain curve, the slope of the curve is $4xy$. If the curve contains the point $(0, 4)$, then its equation is

(A) $y = e^{2x^2} + 4$

(B) $y = e^{2x^2} + 3$

(C) $y = 4e^{2x^2}$

(D) $y^2 = 2x^2 + 4$

(E) $y = 2x^2 + 4$

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21. The equation of the tangent line to the curve $y = x^3 - 6x^2$ at its point of inflection is

(A) $y = -12x + 8$

(B) $y = -12x + 40$

(C) $y = 12x - 8$

(D) $y = -12x + 12$

(E) $y = 12x - 40$

$y' = 3x^2 - 12x$

$y'' = 6x - 12 = 0$ at inflection point

at $6x = 12$

$x = 2$

$y'\big|_{x=2} = 3(2)^2 - 12(2) = 12 - 24 = -12$

$y\big|_{x=2} = (2)^3 - 6(2)^2 = 8 - 24 = -16$

tan line: $(y - (-16)) = -12(x - 2)$

$y + 16 = -12x + 24$

$y = -12x + 8$

$24 - 8 = 16$

$24 - 16$

22. If $0 < k < \pi$, then $\int_0^k \cos(2x)\,dx = \frac{1}{2}$ when $k =$

(A) $\frac{\pi}{12}$

$\boxed{\text{(B)}}$ $\frac{\pi}{4}$

(C) $\frac{5\pi}{12}$

(D) $\frac{\pi}{2}$

(E) $\frac{3\pi}{4}$

$u = 2x$

$du = 2dx$

$\frac{1}{2}\int \cos u\,du = \frac{1}{2}\left[\sin(2x)\right]_0^k$

$\frac{1}{2}(\sin(2k) - \sin \cancel{0})$

$\frac{1}{2}\sin(2k) = \frac{1}{2}$

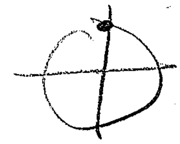
when $\sin(2k) = 1$

$\sin(\theta) = 1$

$\theta = \frac{\pi}{2}$

$2k = \frac{\pi}{2}$

$k = \frac{\pi}{4}$



23. Which of the following properties of the definite integral are true?

I. $\int_a^b kf(x)\,dx = k\int_a^b f(x)$ dx, k is a constant true

II. $\int_a^b xf(x)\,dx = x\int_a^b f(x)\,dx$ FALSE (NOT unless X is a constant)

III. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ true

(A) I only

(B) I and II only

$\boxed{\text{(C)}}$ I and III only

(D) II and III only

(E) I, II, and III

24. If $f(x) = \sqrt{e^{2x}+1}$, then $f'(0) =$

(A) $-\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{2}}{4}$ $\boxed{\text{(C)}}$ $\frac{\sqrt{2}}{2}$ (D) 1 (E) $\sqrt{2}$

$f = (e^{2x}+1)^{1/2}$ ← chain Rule

$f' = \frac{1}{2}(e^{2x}+1)^{-1/2}(e^{2x})(2) = \frac{e^{2x}}{\sqrt{e^{2x}+1}}$

$f'(0) = \frac{e^0}{\sqrt{e^0+1}} = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

25. If $x^2y + yx^2 = 6$, then $\frac{d^2y}{dx^2}$ at the point $(1,3)$ is

(A) -18

(B) -6

(C) 6

(D) 12

$\boxed{\text{(E)}}$ 18

implicit differentiation w/product rule:

$\frac{dy}{dx}$: $x^2\frac{d}{dx}[y] + y\frac{d}{dx}[x^2] + y\frac{d}{dx}[x^2] + x^2\frac{d}{dx}[y] = \frac{d}{dx}(6)$

$x^2\frac{dy}{dx} + y(2x) + y(2x) + x^2\frac{dy}{dx} = 0$

$2x^2\frac{dy}{dx} + 4xy = 0$, $\frac{dy}{dx} = \frac{-4xy}{2x^2} = \frac{-4y}{2x} = \frac{-2y}{x}$

$\frac{d^2y}{dx^2}$: quotient rule:

$\frac{d^2y}{dx^2} = \frac{x\frac{d}{dx}[-2y] - (-2y)\frac{d}{dx}[x]}{x^2}$

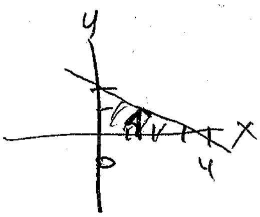
$= \frac{x(-2\frac{dy}{dx}) + 2y(1)}{x^2} = \frac{-2x\frac{dy}{dx} + 2y}{x^2}$

$= \frac{-2x\left(\frac{-2y}{x}\right) + 2y}{x^2} = \frac{4y+2y}{x^2} = \boxed{\frac{6y}{x^2}}$

now $\overset{x\ y}{(1,3)}$ $\frac{d^2y}{dx^2} = \frac{6(3)}{(1)^2} = 18$

26. The base of a solid is the region in the first quadrant bounded by the line $x + 2y = 4$ and the coordinate axes. What is the volume of the solid if every cross section perpendicular to the x-axis is a semicircle?

$\boxed{\text{(A)}}$ $\frac{2\pi}{3}$ (B) $\frac{4\pi}{3}$ (C) $\frac{8\pi}{3}$ (D) $\frac{32\pi}{3}$ (E) $\frac{64\pi}{3}$



$$V = \int_0^4 \frac{1}{2}\pi\left(-\frac{1}{4}x+1\right)^2 dx$$

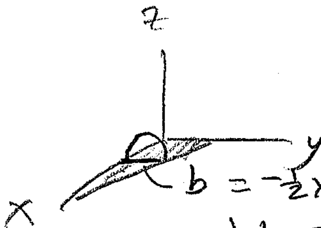
$$\frac{1}{2}\pi \int_0^4 \left(\frac{1}{16}x^2 - \frac{1}{2}x + 1\right) dx$$

$$\frac{\pi}{2}\left[\frac{1}{48}x^3 - \frac{1}{4}x^2 + x\right]_0^4$$

$x + 2y = 4$

$2y = -x + 4$

$y = -\frac{1}{2}x + 2$



$b = -\frac{1}{2}x + 2$

$r = \frac{1}{2}b = -\frac{1}{4}x + 1$

$A_{cross} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(-\frac{1}{4}x+1\right)^2$ $\quad \frac{\pi}{2}\left[\left(\frac{4}{3}\right) - 0\right]$

$\frac{4\pi}{6} = \boxed{\frac{2\pi}{3}}$

27. A particle moves along the x-axis so that at any time t its position is given by $x(t) = (t + 1)(t - 3)^3$. For what values of t is the velocity of the particle increasing?

(A) $t > 3$ only

(B) $0 < t < 3$ only

(C) $1 < t < 3$ only

$\boxed{\text{(D)}}$ $t < 1$ or $t > 3$

(E) $0 < t < 3$ or $t > 3$

product rule:

$v(t) = x'(t) = (t+1)\,3(t-3)^2(1) + (t-3)^3(1)$

$= 3(t+1)(t-3)^2 + (t-3)^3$

$= (t-3)^2[3(t+1) + (t-3)]$

$= (t-3)^2[3t+3+t-3]$

$= (t-3)^2\, 4t$

$a(t) = v'(t) = (t-3)^2(4) + 4t(2(t-3)(1))$

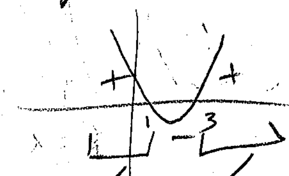
$= 4(t-3)^2 + 8t(t-3)$

$= (t-3)(4(t-3) + 8t)$

$= (t-3)(12t - 12)$

$12(t-3)(t-1) = 0$

graph this eq



$t < 1$ or $t > 3$

28. At any time $t \geq 0$, in days, the rate of growth of a bacteria population is given by $y' = ky$, where y is the number of bacteria present and k is a constant. The initial population is 1,500 and the population is quadrupled during the first 2 days. By what factor will the population have increased during the first 3 days?

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(A) 4

(B) 5

(C) 6

(D) 8

(E) 10

29. The average value of a continuous function $f(x)$ on the closed interval $[3, 7]$ is 12. What is the value of $\int_3^7 f(x)\,dx$?

$$AV = \frac{1}{7-3}\int_3^7 f(x)dx = 12$$

$$\frac{1}{4}\int_3^7 f(x)dx = 12$$

$$\int_3^7 f(x)dx = 48$$

(A) 3

(B) 4

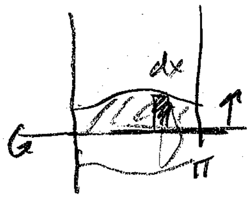
(C) 12

(D) 36

(E) 48

30. The region in the first quadrant enclosed by the x-axis, the line $x = \pi$, and the curve $y = \cos(\cos x)$ is rotated about the x-axis. What is the volume of the solid generated?

(A) 1.921 (B) 3.782 $\boxed{\text{(C)}}$ 6.040 (D) 8.130 (E) 23.781



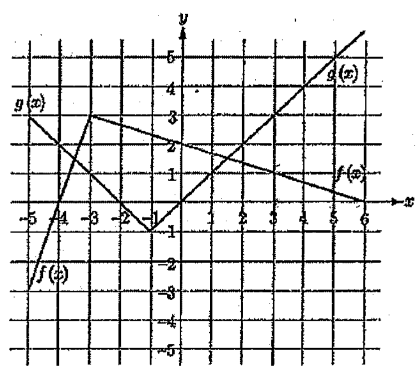
disc method: $V = \int \pi r^2 dx$

$r = \cos(\cos x)$

$= \int_0^{\pi} \pi (\cos(\cos x))^2 dx$ math 9

$= \boxed{6.0346589}$

31.



The functions f and g are piecewise linear functions whose graphs are shown above. If $h(x) = \frac{f(x)}{g(x)}$, then $h'(3) =$

$\boxed{\text{(A)}}$ $-\frac{2}{9}$

(B) $-\frac{1}{3}$

(C) 0

(D) $\frac{1}{3}$

(E) $\frac{8}{27}$

quotient rule

$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$h'(3) = \frac{g(3)f'(3) - f(3)g'(3)}{[g(3)]^2}$

$= \frac{(3)(-\frac{1}{3}) - (1)(1)}{(3)^2}$

$= \frac{-1-1}{9} = -\frac{2}{9}$

32. The region bounded by the x-axis and the part of the graph of $y = \cos x$ between $x = 0$ and $x = \pi/2$ is divided into two regions by the line $x = c$. If the area of the region for $0 \le x \le c$ is equal to the area of the region for $c \le x \le \pi/2$, then c must be

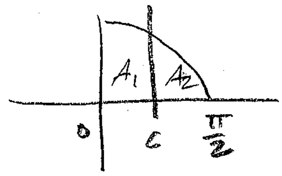
(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{6}$

(C) $\frac{\pi}{3}$

(D) $\frac{2\pi}{9}$

(E) $\frac{5\pi}{18}$



$A_1 = A_2$

$\int_0^c \cos x\,dx = \int_c^{\frac{\pi}{2}} \cos x\,dx$

$[\sin x]_0^c = [\sin x]_c^{\pi/2}$

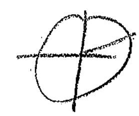
$\sin c - \sin 0 = \sin \frac{\pi}{2} - \sin c$

$\sin c - 0 = 1 - \sin c$

$2\sin(c) = 1$

$\sin(c) = \frac{1}{2}$

$c = \frac{\pi}{6}$



33. For what values of x is the function $f(x) = 5 + 15x + 6x^2 - x^3$ decreasing? when $f' < 0$

(A) $-1 < x < 5$

(B) $-5 < x < 1$

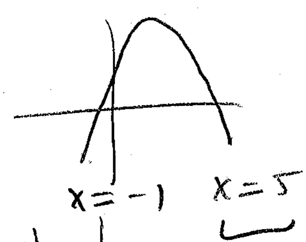
(C) $x < -5$ or $x > 1$

(D) $x < -1$ or $x > 5$

(E) All real numbers

$f'(x) = 15 + 12x - 3x^2$

graph this:



$x < -1$ or $x > 5$

34. If f is an antiderivative of $\dfrac{\tan^2 x}{x^2+1}$ such that $f(1) = \dfrac{1}{2}$, then $f(0) =$

(A) 0

(B) 0.155

(C) 0.345

(D) 0.845

(E) 1

$\int_0^1 \frac{\tan^2 x}{x^2+1}\,dx = f(1) - f(0)$

math 9

$0.3446399 = \frac{1}{2} - f(0)$

so $f(0) = \frac{1}{2} - 0.3446399 = .15536$

35. The second derivative of a function is given by $f''(x) = 0.5 + \cos x - e^{-x}$. How many points of inflection does the function $f(x)$ have on the interval $0 \le x \le 20$?

graph this

inflection point when $f'' = 0$

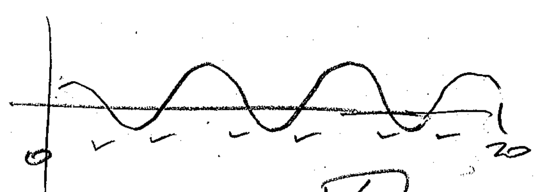
(A) None

(B) Three

(C) Six

(D) Seven

(E) Ten



$\boxed{6}$

36. The local linear approximation of a function f will always be greater than or equal to the function's value if, for all x in an interval containing the point of tangency,

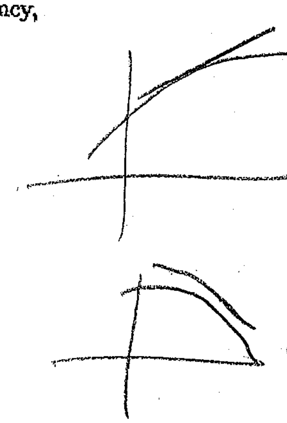
(A) $f' < 0$

(B) $f' > 0$

(C) $f'' < 0$

(D) $f'' > 0$

(E) $f' = f'' = 0$



$f' > 0$

$f'' < 0$

$f' < 0$

$f'' < 0$

37. Let f be the function given by $f(x) = 5 + 5.8\sin\left(\frac{\pi x}{4}\right) - 15.7\cos\left(\frac{\pi x}{3}\right)$. For $0 \le x \le 12$, f is increasing most rapidly when $x =$

(A) 1.328 $f'+$

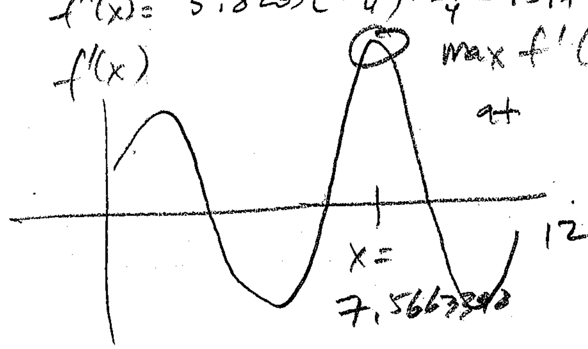
(B) 4.434

(C) 6.000

(D) 7.566

(E) 10.672

$f'(x) = 5.8\cos\left(\frac{\pi x}{4}\right)\cdot\frac{\pi}{4} - 15.7\left(-\sin\left(\frac{\pi x}{3}\right)\cdot\frac{\pi}{3}\right)$



38. If $\lim_{x\to 3} \frac{g(3)-g(x)}{3-x} = -0.628$, then near the point where $x = 3$, the graph of $g(x)$

↖ is $f'(3)$

(A) is decreasing

(B) is increasing

(C) is concave upwards

(D) is concave downwards

(E) has a point of inflection

AB Sample Exam III (Semester I Reveiw)

39. How many zeros does the function $y = \sin(\ln x)$ have for $0 < x \leq 1$?

(A) One

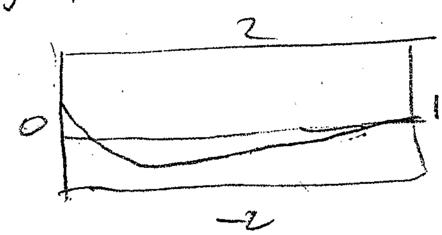
(B) Two

(C) Three

(D) Four

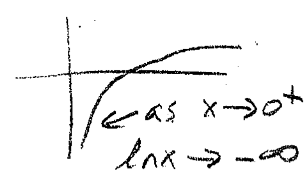
(E) More than four

graph in calculator

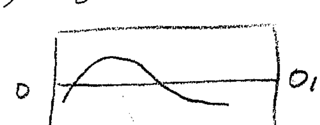


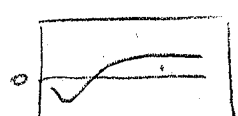
two?

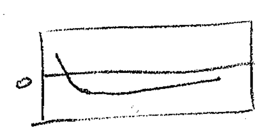
...but $\ln x$ graph is



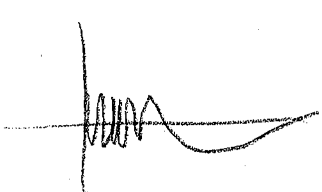
...so zoom in close to zero...







what's really happening:



40. Let $f(x) = x^3 - 7x^2 + 25x - 39$ and let g be the inverse function of f. What is the value of $g'(0)$?

(A) $-\frac{1}{25}$ (B) $\frac{1}{25}$ $\boxed{\text{(C)}}$ $\frac{1}{10}$ (D) 10 (E) 25

$g'(x) = \frac{1}{f'(g(x))}$ $f'(x) = 3x^2 - 14x + 25$

$f(3) = (3)^3 - 7(3)^2 + 25(3) - 39 = 0$

$g'(0) = \frac{1}{f'(g(0))}$ so $g(0) = 3$ (inverses)

$g'(0) = \frac{1}{f'(3)} = \frac{1}{3(3)^2 - 14(3) + 25} = \boxed{\frac{1}{10}}$

41. For all $x > 0$, if $f(\ln x) = x^2$, then $f(x) =$

trial and error w/provided answers.

(A) $\sqrt{e^x}$

(B) $2 \ln x$

(C) $e^{\sqrt{x}}$

(D) $\sqrt{\ln x}$

$\boxed{\text{(E)}\ e^{2x}}$

A) $\sqrt{e^{\ln x}} = \sqrt{x}$ NO

B) $2\ln(\ln x)$ NO

C) $e^{\sqrt{\ln x}}$ NO

D) $\sqrt{\ln(\ln x)}$ NO

E) $e^{2(\ln x)} = e^{\ln(x^2)} = x^2$ YES

42. A rectangle is to be inscribed in a semicircle of radius 8, with one side lying on the diameter of the circle. What is the maximum possible area of the rectangle?

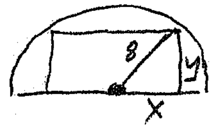
(A) $4\sqrt{2}$

(B) $8\sqrt{2}$

(C) 32

(D) $32\sqrt{2}$

(E) $\boxed{64}$



objective function

$A = 2xy$

$A = 2x\left(\sqrt{64-x^2}\right)$

(product rule) $A = 2x\left(64-x^2\right)^{1/2}$

constraint

$x^2+y^2=8^2$

$y^2=64-x^2$

$y=\pm\sqrt{64-x^2}$

always positive here

$y=\sqrt{64-x^2}$

$A' = 2x\left(\frac{1}{2}(64-x^2)^{-1/2}(-2x)\right) + (64-x^2)^{1/2}(2)$

$A' = \frac{-2x^2}{\sqrt{64-x^2}} + 2\sqrt{64-x^2} = 0$ (DNE at $x=8$ not possible)

$\frac{2x^2}{\sqrt{64-x^2}} = 2\sqrt{64-x^2}$

$x^2 = 64-x^2$

$2x^2 = 64$

$x^2 = 32$

$x = \pm\sqrt{32} = \sqrt{32}$

(negative not possible)

$x=\sqrt{32}$

$y=\sqrt{64-(\sqrt{32})^2}=\sqrt{64-32}=\sqrt{32}$

$A_{max} = 2(\sqrt{32})(\sqrt{32}) = 64$

---

43. If $f(x) = \begin{cases} e^{-x}+2, & \text{for } x<0 \\ ax+b, & \text{for } x \ge 0 \end{cases}$ is differentiable at $x=0$, then $a+b=$

must be continuous at $x=0$

(A) 0

(B) 1

(C) $\boxed{2}$

(D) 3

(E) 4

so $\lim_{x\to 0^-} e^{-x}+2 \overset{\text{must}}{=} \lim_{x\to 0^+} ax+b$

$e^{-0}+2$

$3 = b$ so $b=3$

derivative must also be continuous:

$f'(x) = \begin{cases} -e^{-x} & x<0 \\ a & x\ge 0 \end{cases}$

so $\lim_{x\to 0^-} -e^{-x} \overset{\text{must}}{=} \lim_{x\to 0^+} a$

$-1 = a$

So $a+b = (-1)+3 = \boxed{2}$

44. At how many points do the graphs of the functions $f(x) = 5 + x$[illegible] and $g(x) = 5e$[illegible] intersect?

(A) 0

(B) 1

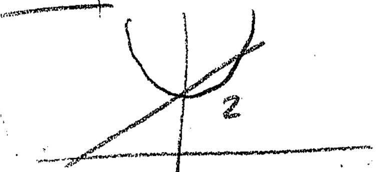
(C) 2

$\boxed{\text{(D)}}$ 3

(E) 4

graph in calculator

close



but eventually an exponential function rises more quickly than a power function

so zoom out ... set y from 0 to 10000000

AB Sample Exam III (Semester I Reveiw)

45. The line $y = mx + b$ with $b \geq 2$ is tangent to the graph of $f(x) = -2(x-2)^2 + 2$ at a point in the first quadrant. What are all possible values of b?

(A) $b = 2$ only

(B) $2 \leq b < 10$

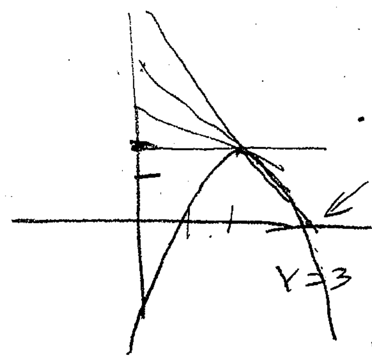
$\boxed{\text{(C)}}$ $2 \leq b < 12$

(D) $2 \leq b < 14$

(E) $2 \leq b < 20$

$f'(x) = -2(2(x-2)) = -4(x-2) = -4x+8$

graphing $f(x)$:



last tangent in I is when tangent at this zero

$f'(x) = -4(3)+8 = -4 = m$

through $(3,0)$

$y = -4x + b$

$(0) = -4(3) + b$

$b = 12$

$2 \leq b \leq 12$

29. The average value of a continuous function $f(x)$ on the closed interval $[3, 7]$ is 12. What is the value of $\int_3^7 f(x)\,dx$?

(A) 3

(B) 4

$AV = \frac{1}{7-3}\int_3^7 f(x)dx = 12$

FRQs (calculator allowed)

① $v(t) = (t-4)^4 \sin(t-4)$

(a) velocity $= v(3) = (3-4)^4 \sin(3-4) = \boxed{\sin(-1) = -0.84147}$

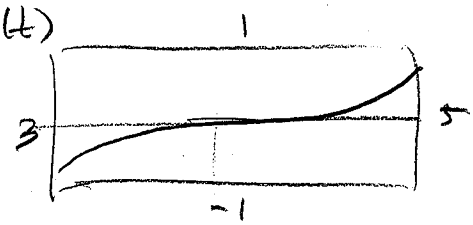
speed $= |\text{velocity}| = \boxed{0.84147}$

$a(t) = v'(t) = (t-4)^4 \cos(t-4) + \sin(t-4)\, 4(t-4)^3$

$a(3) = (3-4)^4 \cos(3-4) + \sin(3-4)\, 4\, (3-4)^3$

$= \boxed{\cos(-1) - 4\sin(-1) = 3.906186}$

(b) $v(t)$



(c) the graph shows that velocity increasing from $3 \leq t \leq 5$.

(d) distance $= -\int_3^4 (t-4)^4 \sin(t-4)\,dt + \int_4^5 (t-4)^4 \sin(t-4)\,dt$

$-(-.1466503276) + .1466503276$

$= \boxed{.2933}$

② (a) math 9

$$\int_0^9 \frac{5000}{t^3+50}\,dt = \boxed{415{,}421 \text{ cm}^3}$$

(b) $$S(6)-R(6) = \frac{5000}{(6)^3+50} - 23{,}967\sqrt{6} = \boxed{-39{,}9099 \text{ cm}^3/\text{hr}}$$

in _ out

This is the net rate of change of amount of sand, and it is negative because more sand is being removed by R than is been put in by S, so this is the current rate of decrease of volume of sand.

(c) Integrating a rate of change of a quantity gives the accumulated amount of that quantity, so net accumulation at time t would be:

$$A(t) = \int_0^t \frac{5000}{x^3+50}\,dx - \int_1^t 23{,}967\sqrt{x}\,dx$$

$$\text{So } A'(t) = \frac{5000}{x^3+50} - 23{,}967\sqrt{x} = 0 \quad \text{at extrema}$$

So maximum would occur when

$$\frac{5000}{x^3+50} = 23{,}967\sqrt{x} \quad \text{or} \quad S(t) = R(t)$$

(d) total accumulation would be

$$\int_0^9 \frac{5000}{x^3+50}\,dx - \int_1^9 23{,}967\sqrt{x}\,dx$$

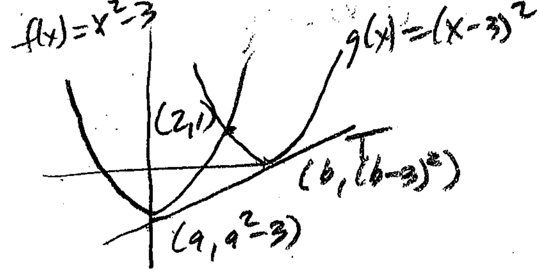
math 9

$$415{,}42091 - 415{,}428 = \boxed{-{,}00709 \text{ cm}^3}$$

(approximately empty)

(#3 is 2nd semester topic)

(4)



(a) slope of T must equal $f'(a)$ and $g'(b)$:

$f'(x) = 2x \qquad g'(x) = 2(x-3)(1)$

$f'(a) = 2a \qquad g'(b) = 2(b-3)$

so $\quad 2a = 2(b-3)$

$a = b-3$ ✓

(b) slope of T is given by $\dfrac{g(b)-f(a)}{b-a} = \dfrac{(b-3)^2-(a^2-3)}{b-a}$

and this slope must equal $f'(a)$ & $g'(b)$

using $f'(a) = 2a$: $\dfrac{(b-3)^2-(a^2-3)}{b-a} = 2a$

$(b-3)^2-(a^2-3) = 2a(b-a)$

$b^2-6b+9-a^2+3 = 2ab-2a^2$

also $a=b-3$, substituting ↑

$b^2-6b+9-(b-3)^2+3 = 2(b-3)b-2(b-3)^2$

$b^2-6b+9-(b^2-6b+9)+3 = 2b^2-6b-2(b^2-6b+9)$

$b^2-6b+9-b^2+6b-9+3 = 2b^2-6b-2b^2+12b-18$

$3 = 6b-18$

$21 = 6b$ so $b = \frac{21}{6} = \frac{3(7)}{3(2)} = \boxed{\frac{7}{2}}$

$a = b-3$

$a = \frac{7}{2}-\frac{6}{2} = \boxed{\frac{1}{2}}$

(c) slope $= g'(b) = g'(\frac{7}{2}) = 2(\frac{7}{2}-3) = 7-6 = 1$

$\frac{7}{2}-3$

$\frac{7}{2}-\frac{6}{2}$

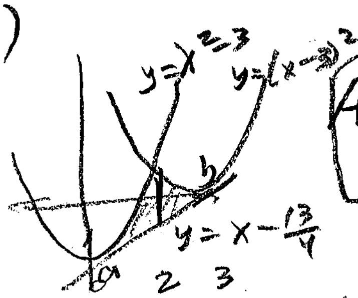
$\frac{1}{2}$

point $(b,(b-3)^2) = (7/2, (7/2-3)^2) = (7/2, \frac{1}{4})$

$\boxed{(y-\tfrac{1}{4}) = (1)(x-7/2)}$ or $y = x - 7/2 + 1/4$

$y = x - \frac{13}{4}$

(d)



$$\boxed{A = \int_{\frac{1}{2}}^{2} \left[(x^2-3)-(x-\tfrac{13}{4})\right]dx + \int_{2}^{7/2} \left[(x-3)^2-(x-\tfrac{13}{4})\right]dx}$$

(5) $f(x) = K + 12x + 3x^2 - 2x^3$

(a) $f(x)$ is increasing when $f'(x) > 0$

$f'(x) = 12 + 6x - 6x^2$

$-6x^2 + 6x + 12 = 0$

$-6(x^2 - x - 2) = 0$

$-6(x+1)(x-2) = 0$

$x = -1 \quad x = 2$

rel min ↓ at -1, rel max ↓ at 2

$-\infty$ ——— -1 ——— 2 ——— ∞

test (-2)	test (0)	test (3)
$f'(-2)$	$f'(0)$	$f'(3)$
$(-)(-)(-)$	$(-)(+)(-)$	$(-)(+)(+)$
$-$	$+$	$-$

$f(x)$ is increasing on $\boxed{(-1, 2)}$

(b) from (a), rel max occurs at $x = 2$

so $f(2) = 4$

$K + 12(2) + 3(2)^2 - 2(2)^3 = 4$

$K + 24 + 12 - 16 = 4$

$K + 20 = 4$

$\boxed{K = -16}$

(c) $f(x)$ is concave up when $f''(x) > 0$

$f''(x) = 6 - 12x$

$6 - 12x = 0$

$12x = 6$

$x = \frac{1}{2}$

$-\infty$ ——— $\frac{1}{2}$ ——— ∞

test (0)	test (1)
$f''(0)$	$f''(1)$
$6 - 12(0) = 6$	$6 - 12(1) = -6$
$+$	$-$

$f(x)$ is concave up on $\boxed{(-\infty, \frac{1}{2})}$

(d) from (a), rel min occurs at $x = -1$ & $K = -16$ (part b)

$f(-1) = -16 + 12(-1) + 3(-1)^2 - 2(-1)^3$

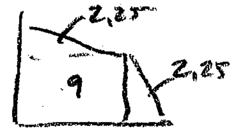
$= -16 - 12 + 3 + 2$

$= -28 + 5$

$= \boxed{-23}$

(6) $g(x) = \int_0^x f(t)\,dt$

(a) $g(4.5) = \int_0^{4.5} f(t)\,dt$ = area under curve



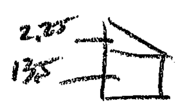
$= \boxed{13.5}$

$g'(4.5) = \frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = f(x) = f(4.5) = \boxed{0}$

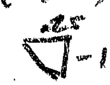
$g''(4.5) = f'(4.5) =$ "slope" of $f(x)$ at $x = 4.5 = \boxed{-2}$

(b) $AV = \frac{1}{5-(-3)} \int_{-3}^{5} f(x)\,dx = \frac{1}{8}\left[\int_{-3}^{0} f(x)\,dx + \int_{-3}^{4.5} f(x)\,dx + \int_{4.5}^{5} f(x)\,dx\right]$

$= \frac{1}{8}\left[15.75 + 13.5 + (-.25)\right] = \frac{1}{8}(29) = \boxed{\frac{29}{8}}$



(part a)



(c) points of inflection of $\underline{g(x)}$ occur when $g''(x) = 0$, which is where $g'(x)$ has a relative max or min. Since $g(x)$ is the antiderivative of $f(x)$, $f(x)$ is the derivative of $g(x)$ so we are looking for points when $f(x)$ has a relative max or min. This occurs only at $\boxed{x = 5}$.

(d) Maximum points of g may occur at endpoints of an interval or at relative maxima. $g(x)$ represents accumulated area from $x = 0$ for the $f(x)$ curve. Area would start at zero at $x = 0$ and increase as x increases until $x = 4.5$, to an area of $g(4.5) = 13.5$ (part a)

Between $x = 4.5$ and $x = 7$, the area is negative, so $g(x)$ would decrease until down to $g(7) = 13.5 - 1.25 = 12.25$. Then the area would increase again from $x = 7$ to $x = 9$, ending at an area of $g(9) = 12.25 + 1 = 13.25$.

Therefore, there is an absolute maximum at $x = 4.5$: $g(4.5) = 13.5$ and the endpoint maximum would be $g(9) = 13.25$.