SECTION I PART A

NO CALCULATOR IS ALLOWED IN THIS SECTION.

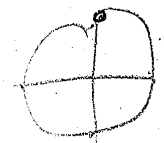
Directions: Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given. Do not spend too much time on any one problem.

In this exam:

(1) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(2) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$)

(0,1)



$$f(x) = \begin{cases} x^2 + bx & \text{for } x \le 5 \\ 5\sin\left(\frac{\pi}{2}x\right) & \text{for } x > 5 \end{cases}$$

1. Let f be the function defined above, where b is a constant. If f is a continuous function, what is the value of b ?

$5^2+5b = 5\sin(\frac{5\pi}{2})$

$25+5b = 5 \quad 5b = -20$

$b=-4$

(A) -5 (B) -4 (C) 4 (D) 5

2. The graph of $y = 3x^2 - x^3$ has a relative maximum at

$y' = 6x-3x^2=0 \quad y''=6-6x \quad f(2)=3(2)^2-8$

$3x(2-x)=0 \quad 12-8$

$x=0 \quad x=2 \quad (2,4) \quad 4$

$y''=6 \quad y''=-6$

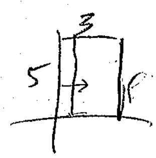
(A) (0,0) only

(B) (1,2) only

(C) (2,4) only

(D) (4, −16) only

3. A forest is in the shape of a rectangle 5 miles long and 3 miles wide. The density of trees at a distance of x miles from the 5-mile side is given by $\rho(x)$ trees per square mile. Which expression gives the number of trees in the forest?



derivative, deriv. per mile (in x)

$\rho(x) = \frac{\text{trees}}{\text{square mile}}$ is $5\rho(x)$

So accumulation $= 5\int_0^3 \rho(x)\,dx$

(A) $3\int_0^3 \rho(x)\,dx$

(B) $5\int_0^3 \rho(x)\,dx$

(C) $3\int_0^3 \rho'(x)\,dx$

(D) $5\int_0^3 \rho'(x)\,dx$

$f'(x) = e^{\sin x} \cdot \cos x = 0$

never zero $\quad \cos x = 0$

$x = \pi/2 \; \& \; 3\pi/2$

(0,1)

(0,−1)

4. If $f(x) = e^{\sin x}$, how many zeros does $f'(x)$ have on the closed interval $[0, 2\pi]$?

(A) None (B) One (C) Two (D) Three

5. $\lim\limits_{x \to 0} \frac{3x^2 - \sin(x)}{2x^2 + x} =$ $\frac{0}{0}$, L'Hopital: $= \lim\limits_{x\to 0} \frac{6x - \cos(x)}{4x+1} = \frac{-1}{1} = -1$

(A) $\frac{3}{2}$ (B) 1 (C) 0 (D) -1

$\frac{dy}{dx} = \frac{1}{3x+5}(3) = \frac{3}{3x+5}$

6. If $y = \ln(3x+5)$, then $\frac{d^2y}{dx^2} =$ $\quad \frac{d^2y}{dx^2} = \frac{(3x+5)(0)-(3)(3)}{(3x+5)^2} = \frac{-9}{(3x+5)^2}$

(A) $\frac{3}{(3x+5)^2}$ (B) $\frac{9}{(3x+5)^2}$ (C) $\frac{-3}{(3x+5)^2}$ (D) $\frac{-9}{(3x+5)^2}$

$(2-4\sin x)^{1/2}$

7. If $f(x) = \sqrt{2-4\sin x}$, then $f'(\pi) =$ $\quad \frac{1}{2}(2-4\sin x)^{-1/2}(-4\cos x)$ at $x=\pi$:

(A) $-\sqrt{2}$ (B) 0 (C) $\frac{\sqrt{2}}{2}$ (D) $\sqrt{2}$

$\frac{1}{2}\frac{1}{\sqrt{2-4\sin\pi}}(-4\cos\pi)$

$\frac{1}{2}\frac{1}{\sqrt{2-0}}(-4(-1)) = \frac{2}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

8. Let $R(x)$ be the radius of a round pipe that drains water from a dam, where x is measured in feet from the dam. Which choice best explains the meaning of $\pi\int_{10,000}^{30,000}(R(x))^2\,dx$?

x

$\pi r^2\,dx$ = Volume

(A) The amount of water in square feet that the pipe can hold in the section from 10,000 to 30,000 feet from the dam.

(B) The amount of water in cubic feet that the pipe can hold in the section from 10,000 to 30,000 feet from the dam.

(C) The amount of water in cubic feet flowing through the pipe from 10,000 to 30,000 feet.

(D) The amount of water in cubic feet in any 20,000-foot section of the pipe.

9. An equation of the line tangent to the curve $x^2 + y^2 = 169$ at the point $(5, -12)$ is

implicit diff:

$2x + 2y\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y} = \frac{-5}{-12} = \frac{5}{12}$

(A) $12x + 5y = 119$

(B) $5x - 12y = 119$

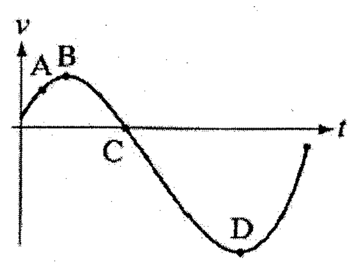
(C) $5x - 12y = 169$

(D) $12x - 5y = 169$

$(y+12) = \frac{5}{12}(x-5)$

$12y + 144 = 5x - 25$

$5x - 2y = 169$



10. For an object moving along a straight line, the graph above represents the velocity of the moving object as a function of time. At which of the marked points is the speed the greatest?

(A) A (B) B (C) C (D) D

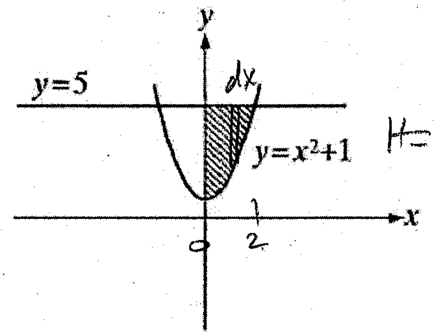
= |magnitude of velocity|

Intersection:

$\begin{cases} y=5 \\ y=x^2+1 \end{cases}$

$x^2+1=5 \quad x^2=4$

$x=\pm 2$



$H = 5-(x^2+1)$

$A = \int H\,dw$

$= \int_0^2 (5-(x^2+1))\,dx$

$\int_0^2 (5-x^2-1)\,dx$

$\int_0^2 (4-x^2)\,dx$

$= \left[4x-\frac{1}{3}x^3\right]_0^2$

$= 8-\frac{8}{3} = \frac{24-8}{3} = \frac{16}{13}$

11. For the figure above, the area of the shaded region is

(A) $\frac{14}{3}$ [✓ (B)] $\frac{16}{3}$ (C) $\frac{28}{3}$ (D) $\frac{32}{3}$

12. $\int \frac{1}{\sqrt{4-x^2}}\,dx =$ trig substitution

2, x, θ, $\sqrt{4-x^2}$

$\cos\theta = \frac{\sqrt{4-x^2}}{2}$

$\sqrt{4-x^2} = 2\cos\theta$

$\sin\theta = \frac{x}{2}$

$x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta\,d\theta$

[✓ (A)] $\sin^{-1}\left(\frac{x}{2}\right)+C$

(B) $2\sqrt{4-x^2}+C$

(C) $\sqrt{4-x^2}+C$

(D) $\frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right)+C$

$\int \frac{1}{2\cos\theta}\,2\cos\theta\,d\theta = \int 1\,d\theta = \theta + C$

$= \sin^{-1}\left(\frac{x}{2}\right)+C$

$\sin\theta = \frac{x}{2}$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$f(x) = 2x^2+kx^{-1}$

13. If the graph of $f(x) = 2x^2 + \frac{k}{x}$ has a point of inflection at $x=-1$, then the value of k is

(A) -2 (B) -1 (C) 1 [✓ (D)] 2

$f'(x) = 4x - kx^{-2}$

$f''(x) = 4 + 2kx = 0$ when $x=-1$

$4+2k(-1)=0 \quad 2k=4$

$k=2$

14. $\int \sin(3x+4)\,dx =$

$u = 3x+4$

$du = 3\,dx$

$\int \sin u \,\frac{1}{3}\,du$

$= \frac{1}{3}\int \sin u\,du$

$= -\frac{1}{3}\cos u + C$

$= -\frac{1}{3}\cos(3x+4)+C$

[✓ (A)] $-\frac{1}{3}\cos(3x+4)+C$

(B) $-\cos(3x+4)+C$

(C) $\cos(3x+4)+C$

(D) $\frac{1}{3}\cos(3x+4)+C$

15. For what values of x is the graph of $y = \frac{2}{4-x}$ concave downward?

(A) No values of x

[✓ (B)] $x>4$

(C) $x<4$

(D) All real numbers

$y' = \frac{(4-x)(0)-(2)(-1)}{(4-x)^2}$

$= \frac{2}{(4-x)^2}$

$y'' = \frac{(4-x)^2(0)-(2)(2(4-x)(-1))}{(4-x)^4}$

$= \frac{4(4-x)}{(4-x)^4} = \frac{4}{(4-x)^3}$

y'' DNE at $x=4$

$-\infty \quad (0) \quad + \quad 4 \quad (5) \quad - \quad \infty$

so $g(0) = -1$ $\qquad g'(x) = \frac{1}{f'(g(x))}$

16. $f(x) = x^3 + 3x^2 - 2$ and $f(-1) = 0$. If $g(x) = f^{-1}(x)$, the value of $g'(0)$ is $\qquad f'(x) = 3x^2 + 6x$

$g'(0) = \frac{1}{f'(g(0))} = \frac{1}{f'(-1)}$

(A) $-\frac{1}{3}$ $\qquad$ (B) $-\frac{1}{4}$ $\qquad$ (C) -3 $\qquad$ (D) nonexistent

$3(-1)^2 + 6(-1)$

$3 - 6 = -3 \qquad \frac{1}{-3}$

17. Determine the slope of the line tangent to the graph of $f(x) = \dfrac{e^x}{1+e^x}$ at $x = 1$.

quotient rule:

$f'(x) = \dfrac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2}$

(A) $\dfrac{e}{1+e^2}$ $\qquad$ (C) $\dfrac{-e}{1+e^2}$

(B) $\dfrac{e}{(1+e)^2}$ $\qquad$ (D) $\dfrac{-e}{(1+e)^2}$

$f'(1) = \dfrac{(1+e^1)e - e^2}{(1+e)^2} = \dfrac{e+e^2-e^2}{(1+e)^2} = \dfrac{e}{(1+e)^2}$

18. The rate of change of the surface area of a cube, S, with respect to time, t, is directly proportional to the square root of one-sixth of the surface area. Which of the following is a differential equation that best describes this relationship?

* SKIP (2nd semester topic)

(A) $\dfrac{dS}{dt} = \dfrac{k}{\sqrt{6S}}$ $\qquad$ (C) $\dfrac{dS}{dt} = k\sqrt{\dfrac{S}{6}}$ $\qquad$ $\dfrac{dS}{dt} = k\sqrt{\dfrac{S}{6}}$

(B) $\dfrac{dS}{dt} = k\sqrt{\dfrac{t}{6}}$ $\qquad$ (D) $S(t) = \dfrac{\sqrt{S}}{6}$

19. For $x > 2$, $\dfrac{d}{dx}\displaystyle\int_4^{x^2} \frac{dt}{1-\sqrt{t}} =$

$u = x^2$

$du = 2x\,dx$

$\displaystyle\int_4^{u} \frac{dt}{1-\sqrt{t}}\,dt = \frac{1}{1-\sqrt{u}}\,du$

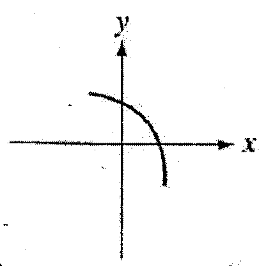
$\dfrac{1}{1-\sqrt{x^2}}\,2x = \dfrac{2x}{1-x}$

(A) $\dfrac{2x}{1-x}$ $\qquad$ (C) $\dfrac{1}{1-x} - 1$

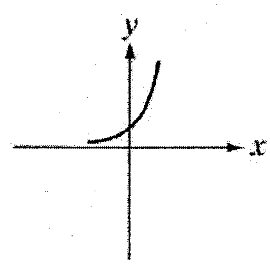
(B) $\dfrac{2x}{1-x} - 1$ $\qquad$ (D) $\dfrac{1}{1-x}$

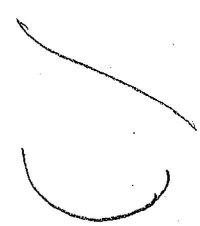
21. If, for all real numbers x, $f'(x) < 0$ and $f''(x) > 0$, which of the following curves could be part of the graph of f?

(A)

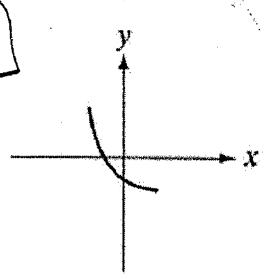


(B)

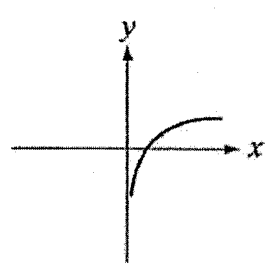




(C)



(D)



22. If f is a function such that $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = 0$, which of the following must be true?

(A) $\lim_{x \to a} f(x)$ does not exist.

(B) $f'(a) = 0$

(C) $f(a) = 0$

(D) $f(x)$ is continuous at $x = 0$.

x	2	8
$f(x)$	1	3
$f'(x)$	2	12
$g(x)$	2	−6
$g'(x)$	4	−12

23. The table above gives the values of f, f', g, and g' for selected values of x.

What is the value of $\int_2^8 \frac{f(x)\,g'(x) - g(x)\,f'(x)}{(f(x))^2}\,dx$?

$\int_2^8 \frac{d}{dx}\left[\frac{g(x)}{f(x)}\right] dx = \frac{g(8)}{f(8)} - \frac{g(2)}{f(2)}$

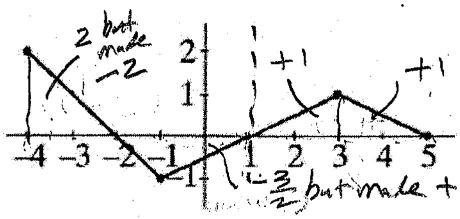
$= \frac{-6}{3} - \frac{2}{1}$

$= -2 - 2 = -4$

(A) $-\frac{1}{4}$ (B) $-\frac{1}{3}$ (C) -3 (D) -4

below $x = 1$

$\int_1^x f(t)\,dt = -\int_x^1 f(t)\,dt$



Graph of f

$g(-4) = -2 + 3/2 = -1/2$

$g(-2) = 3/2$

$g(1) = 0$

$g(5) = 1 + 1 = 2$

$g(-4) < g(1) < g(-2) < g(5)$

24. The graph of a piecewise linear function f is shown in the figure above. If $g(x) = \int_1^x f(t)\,dt$, which of the following is true?

(A) $g(-4) < g(1) < g(-2) < g(5)$

(B) $g(-4) < g(-2) < g(1) < g(5)$

(C) $g(-2) < g(-4) < g(1) < g(5)$

(D) $g(-2) < g(1) < g(-4) < g(5)$

25. The circumference of a circle is increasing at the rate of 0.5 meters/minute. What is the rate of change of the area of the circle when the radius is 4 meters?

(A) 2 m^2/min

(B) 4 m^2/min

(C) 4π m^2/min

(D) 8π m^2/min

$\frac{dC}{dt} = .5$ $C = 2\pi r$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$

$(.5) = 2\pi \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi(4)\left(\frac{.5}{2\pi}\right) = 2$

$\frac{dr}{dt} = \frac{.5}{2\pi}$

26. Let $f(x)$ be the function defined by $f(x) = \begin{cases} x & \text{for } x \le 0 \\ x+1 & \text{for } x > 0 \end{cases}$.

What is the value of $\int_{-2}^{1} x\, f(x)\, dx$?

(A) 0 (B) 3 $\boxed{\text{(C)}}$ $\frac{7}{2}$ (D) $\frac{11}{2}$

$\int_{-2}^{0} x(x)dx + \int_{0}^{1} (x+1)x\, dx$
$(x^2) \quad (x^2+x)$
$\left[\frac{1}{3}x^3\right]_{-2}^{0} + \left[\frac{1}{3}x^3+\frac{1}{2}x^2\right]_0^1$
$0 - \frac{1}{3}(-8) + \frac{1}{3} + \frac{1}{2} - 0$
$\frac{8}{3} + \frac{1}{3} + \frac{1}{2} = 3 + \frac{1}{2}$
$\frac{7}{2}$

27. The average value of the function $f(x) = \cos\left(\frac{1}{2}x\right)$ on the closed interval $[-4, 0]$ is

(A) $-\frac{1}{2}\sin(2)$ (C) $\frac{1}{4}\sin(2)$

(B) $-\frac{1}{4}\sin(2)$ $\boxed{\text{(D)}}$ $\frac{1}{2}\sin(2)$

$u = \frac{1}{2}x \quad du = \frac{1}{2}dx$
$dx = 2du$
$Av = \frac{1}{0-(-4)} 2\left[\sin(\tfrac{1}{2}x)\right]_{-4}^{0}$ $\int \cos\frac{1}{2}x\, dx = 2\int \cos u\, du$
$= 2\sin(\tfrac{1}{2}x)$
$= \frac{1}{2}(\sin 0 - \sin(-2))$
$= \frac{1}{2}(0 - \sin(-2)) = \frac{1}{2}(0 - (-\sin(2))) = \frac{1}{2}\sin 2$

28. Using the substitution $u = \sqrt{2x}$, $\int_{2}^{8} \frac{dx}{\sqrt{2x}+1}$ is equivalent to

(A) $\int_2^4 \frac{du}{u+1}$ (C) $\int_2^8 \frac{u\,du}{u+1}$

(B) $\int_2^8 \frac{du}{u+1}$ $\boxed{\text{(D)}}$ $\int_2^4 \frac{u\,du}{u+1}$

$u = (2x)^{1/2} \quad du = \frac{1}{2}(2x)^{-1/2}(2)\,dx$
$du = \frac{1}{\sqrt{2x}}dx$
$x = 2, u = 2$
$x = 8, u = 4$
$\int_2^4 \frac{1}{u+1}\sqrt{2x}\,du$
$\int_2^4 \frac{u}{u+1}du$

29. The volume of the solid formed by revolving the region bounded by the graphs of $y = 9$ and $y = (x-3)^2$ about the line $y = 9$ is given by which of the following integrals?

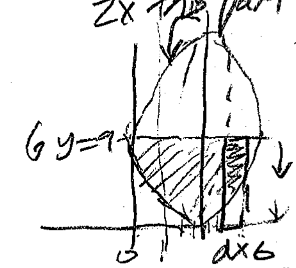
(A) $\pi\int_0^6 (9^2 - (x-3)^4)\, dx$

$\boxed{\text{(B)}}$ $2\pi\int_0^3 (9-(x-3)^2)^2 dx$

(C) $\pi\int_0^6 ((x-3)^4 - 9^2)\, dx$

(D) $2\pi\int_0^3 (9-(x-3)^2)\, dx$

2x this part



$r = 9-(x-3)^2$

$V = \int \pi r^2 dx$
$V = \int_0^6 \pi\,(9-(x-3)^2)^2\,dx$
$= 2\pi\int_0^3 (9-(x-3)^2)^2 dx$

intersection: $(x-3)^2 = 9$
$x-3 = \pm 3$
$x = 3+3 = 6$
$x = -3+3 = 0$

30. Suppose that $f(x)$ is an even function and let $\int_0^1 f(x)\,dx = 5$ and $\int_0^7 f(x)\,dx = 1$.

What is the value of $\int_{-7}^{-1} f(x)\,dx$? $= \int_{-7}^{0} f(x)dx - \int_{-1}^{0} f(x)dx$

but f(x) is even, so

(A) -5 $\boxed{\text{(B)}}$ -4 (C) 4 (D) 5

$= \int_0^7 f(x)dx - \int_0^1 f(x)dx$
$= 1 - 5$
$= -4$

SECTION I PART B

A GRAPHING CALCULATOR IS REQUIRED FOR SOME QUESTIONS IN THIS SECTION.

31. Let f be the function given by $f(x) = \tan x$ and let g be the function given by $g(x) = x^2$. At what value of x in the interval $0 \le x \le \pi$ do the graphs of f and g have parallel tangent lines? $f'(x) = g'(x)$

(A) 0.660 $\qquad$ $\boxed{\text{(B)}}$ 2.083 $\qquad$ (C) 2.194 $\qquad$ (D) 2.207

$\underline{f'(x)}$ $\sec^2(x)$ $\qquad$ $\underline{g'(x)}$ $2x$

$\frac{1}{(\cos x)^2} = 2x$

zero at 2.08287

graph $Y1 = \frac{1}{(\cos x)^2} - 2x$ and find zeros

32. Let $f(t) = \frac{1}{t}$ for $t > 0$. For what value of t is $f'(t)$ equal to the average rate of change of f on the closed interval $[a,b]$? Avg rate of change $= \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a}$ $\qquad$ $f'(t) = -\frac{1}{t^2}$

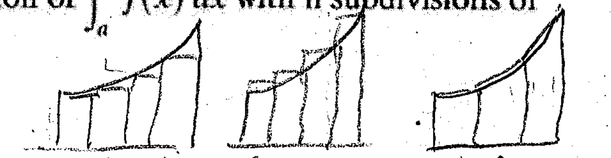
$\boxed{\text{(A)}}$ $\sqrt{ab}$ $\qquad$ (B) $\frac{1}{\sqrt{ab}}$ $\qquad$ (C) $-\frac{1}{\sqrt{ab}}$ $\qquad$ (D) $-\sqrt{ab}$

$-\frac{1}{t^2} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a}$

$t^2 = \frac{a-b}{\frac{1}{b}-\frac{1}{a}} = \frac{ab(a-b)}{a-b} = ab$

$t = \pm\sqrt{ab}$ $\qquad$ $a, b +$ so $+\sqrt{ab}$

33. On the interval $a \le x \le b$ the function f is positive, increasing and concave upwards. Let $A = \int_a^b f(x)\,dx$, $L =$ the left Riemann sum approximation of $\int_a^b f(x)\,dx$ with n subdivisions of equal length, $R =$ the right Riemann sum approximation of $\int_a^b f(x)\,dx$ with n subdivisions of equal length, and let $T =$ the trapezoidal sum approximation of $\int_a^b f(x)\,dx$ with n subdivisions of equal length. Which of the following inequalities is true?



$\boxed{\text{(A)}}$ $L < A < T < R$ $\qquad$ (C) $R < A < T < L$

(B) $L < T < A < R$ $\qquad$ (D) $R < T < A < L$

$L < A < T < R$

34. Let $R(t)$ represent the rate at which water is leaking out of a tank, where t is measured in hours. Which of the following expressions represents the total amount of water in gallons that leaks out in the first three hours?

(A) $\frac{1}{3}\int_0^3 R'(t)\,dt$ $\qquad$ (C) $\int_0^3 R'(t)\,dt$

(B) $\frac{1}{3}\int_0^3 R(t)\,dt$ $\qquad$ $\boxed{\text{(D)}}$ $\int_0^3 R(t)\,dt$

$\int_0^3 R(t)\,dt$

integrating = accumulation of the value the integrand is a derivative for

(like: $\int$ velocity = accumulated distance)

chain rule:
$h'(x) = f'(g(x)) \cdot g'(x)$
$h'(1) = f'(g(1)) \cdot g'(1)$
$= f'(3) \cdot (-3)$
$= (-5)(-3)$
$= 15$

35. Let f and g be differentiable functions such that

$$f(1) = 4,\ g(1) = 3,\ f'(3) = -5$$

$$f'(1) = -4,\ g'(1) = -3,\ g'(3) = 2$$

If $h(x) = f(g(x))$, then $h'(1) =$

$\boxed{\text{(A)}}$ 15 (B) -5 (C) -12 (D) -15

36. If f is differentiable and increasing on the interval $[0,b]$ and c is the number guaranteed by the Mean Value Theorem on this interval, then which statement must be true?

(A) $f'(c) = \dfrac{f(b)}{b}$

(B) $f'(c) = 0$

$\boxed{\text{(C)}}$ $f'(c) > 0$

(D) $f'(x)$ changes sign at $x = c$.

secant slope $= \dfrac{f(b) - f(0)}{b - 0}$ so $f'(c) = \dfrac{f(b) - f(0)}{b}$

If f is increasing $f(b) - f(0)$ is positive

interval is $[0, b]$ in order, so $b > 0$

$f'(c) = \dfrac{f(b) - f(0)}{b} = \dfrac{(+)}{(+)}$ so $f'(c) > 0$

37. The area of the region enclosed by the graphs of $y = e^{(x^2)} - 2$ and $y = \sqrt{4 - x^2}$ is

(A) 2.525 (B) 4.049 $\boxed{\text{(C)}}$ 5.050 (D) 6.289

$A = \int_{-1.137279}^{1.1372792} \left(\sqrt{4-x^2} - (e^{(x^2)} - 2)\right) dx$

MATH 9 = 5.049959

1.1372792

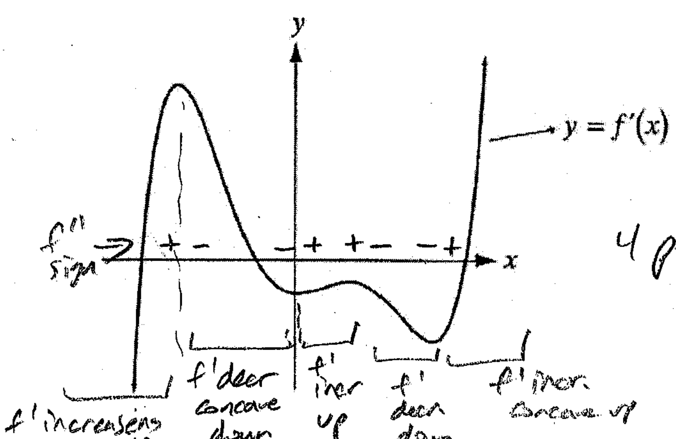
-1.137279

38. $\lim\limits_{x \to 1^-} \dfrac{x^3 - 1}{|x^3 - 1|}$ is

for 1^- $x^3 - 1 < 0$ so $|x^3 - 1| = -(x^3 - 1)$ $\lim\limits_{x \to 1^-} (-1) = -1$

(or could graph it)

(A) 1 (B) 0 $\boxed{\text{(C)}}$ -1 (D) undefined



4 points of inflection

39. The figure above shows the graph of $f'(x)$, the derivative of a polynomial function f. How many points of inflection does the graph of f have?

(A) One (B) Two (C) Three $\boxed{\text{(D)}}$ Four

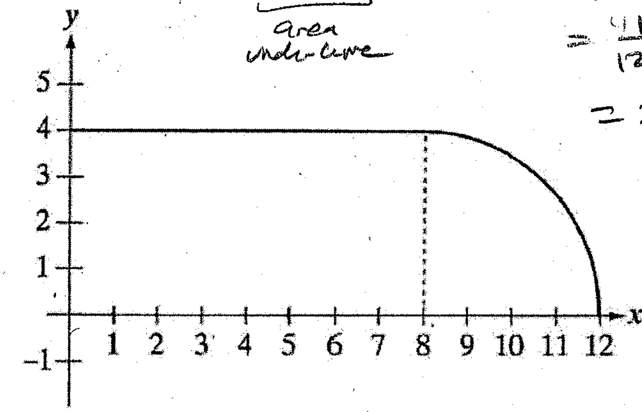
inflection when f'' changes sign
concave up when f' increasing
concave down when f' decreasing

$AV = \frac{1}{b-a}\underbrace{\int_a^b f(x)\,dx}_{\text{area under curve}}$

$AV = \frac{1}{12-0}\left[(4)(8) + \frac{1}{4}\pi(4)^2\right]$

$= \frac{1}{12}\left[32 + 4\pi\right]$

$= 3.71386$



40. As shown in the figure above, the function $f(x)$ consists of a line segment from (0,4) to (8,4) and one-quarter of a circle with a radius of 4. What is the average (mean) value of this function on the interval $[0,12]$?

$\boxed{\text{(A)}}$ 3.714 (B) 6.855 (C) 22.283 (D) 44.566

41. If f is the function defined by $f(x) = \sqrt[3]{x^2+4x}$ and g is an antiderivative of f such that $g(5) = 7$, then $g(1) \approx$

$\boxed{\text{(A)}}$ –3.882 (B) –3.557 (C) 3.557 (D) 3.882

$\int_1^5 (x^2+4x)^{1/3}dx = g(5) - g(1)$

$g(1) = g(5) - \int_1^5 (x^2+4x)^{1/3}dx$

$(7) - 10.8822 = -3.882$ Math 9

42. The amount, $A(t)$, of a certain item produced in a factory is given by

$$A(t) = 4000 + 48(t-3) - 4(t-3)^3$$

where t is the number of hours of production since the beginning of the workday at 8:00 am. At what time is the rate of production at its maximum? A'

(A) 10:00 am $\boxed{\text{(B)}}$ 11:00 am (C) 12:00 noon (D) 1:00 pm

$A' = 48 - 12(t-3)^2(1)$

$A'' = -24(t-3) = 0$

at $t = 3$

$A''' = -24$ concave down so max

8:00 am + 3 hrs = $\boxed{11:00 \text{ am}}$

43. Let g be the function defined by $g(x) = \int_3^x (5 + 4t - t^2)(2^{-t})\,dt$. Which of the following statements about g must be true?

by Fundamental Theorem of Calculus!

$g'(x) = (5+4x-x^2)(2^{-x}) = -(x^2-4x-5)(2^{-x})$

$g'(x) = -(x+1)(x-5)(2^{-x}) = -\frac{(x+1)(x-5)}{2^x}$

$-\infty \quad (-2) \quad -1 \quad (0) \quad 5 \quad (6) \quad \infty$

$\frac{(-)(-)(-)}{(+)}$ $-$ $\quad \frac{(-)(+)(-)}{(+)}$ $+$ $\quad \frac{(-)(+)(+)}{(+)}$ $-$

I. g is increasing on $(3,5)$.

II. g is increasing on $(5,7)$.

III. $g(7) < 0$

(A) $\boxed{\text{I only}}$

(B) II only

(C) III only

(D) I and III only

I true

II false

III use math 9 $\int_3^7 (5+4t-t^2)2^{-t}\,dt = .5615(+)$

False

44. A population increases according to the equation $P(t) = 6000 - 5500e^{-0.159t}$ for $t \geq 0$, t measured in years. This population will approach a limiting value as time goes on. During which year will the population reach half of this limiting value?

(A) Second

(B) Third

(C) $\boxed{\text{Fourth}}$

(D) Eighth

$\lim_{t\to\infty} (6000 - 5500e^{-.159t}) = 6000$

$6000 - 5500e^{-.159t} = 3000$

$5500e^{-.159t} = 3000$

$e^{-.159t} = \frac{3000}{5500}$, $\quad -.159t = \ln\left(\frac{3000}{5500}\right)$

$t = \frac{\ln\left(\frac{3000}{5500}\right)}{-.159} = 3.812$

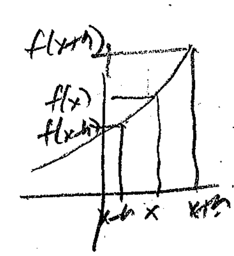
during the 4th year

45. Given function g such that $\lim_{h\to 0} \frac{g(x+h) - g(x-h)}{h} = 6 - 4x$.

Which of the following statements would be true?

$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$

so $\lim_{h\to 0} \frac{g(x+h)-g(x-h)}{h} = 2g'(x)$



I. $g'(0) = 6$

II. $g''(0) < 0$

III. $g'''(0) = 0$

(A) I and II only

(B) I and III only

(C) $\boxed{\text{II and III only}}$

(D) I, II, and III

$2g'(x) = 6 - 4x$

$g'(x) = 3 - 2x$

I) $g'(0) = 3 - 2(0) = 3$ False

II) $g''(x) = -2$ True

III) $g'''(x) = 0$ True

For each part of Section II, you may wish to look over the problems before starting to work on them. It is not expected that everyone will be able to complete all parts of all problems. All problems are given equal weight, but the parts of a particular problem are not necessarily given equal weight.

YOU SHOULD WRITE ALL WORK FOR EACH PART OF EACH PROBLEM IN THE SPACE PROVIDED FOR THAT PART. Be sure to write clearly and legibly. If you make an error, you may save time by crossing it out rather than trying to erase it. Erased or crossed-out work will not be graded. Manage your time carefully.

- Show all your work, even though a question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Your work will be graded on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.
- Justifications require that you give mathematical (noncalculator) reasons.
- Your work must be expressed in standard mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\,dx$ may not be written as fnInt(X², X, 1, 5).
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified.
- If you use decimal approximations in calculations, your work will be graded on accuracy. Unless otherwise specified, your final answers should be accurate to three places after the decimal point.
- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Section II Part A: Graphing calculator is required for these problems.

1. The number of minutes of daylight per day, $L(t)$, at 40° North latitude is modeled by the function

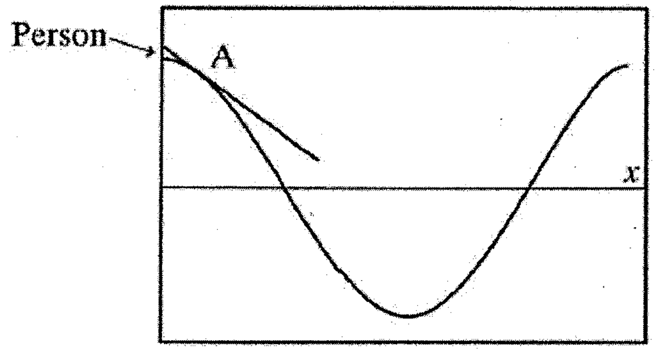
$$L(t) = 167.5\sin\left(\frac{2\pi}{366}(t-80)\right) + 731$$

where t is the number of days after the beginning of the year 2015. (Note: for Jan 1, 2015, $t = 1$; and for Dec. 31, 2015, $t = 365$)

(a) Which day (t) has the most minutes of daylight? Justify your answer using $L'(t)$.

(b) To the nearest minute what is the total number of minutes of daylight in 2015? Justify your answer.

(c) To the nearest minute what is the average number of minutes of daylight in 2015? Justify your answer.



2. As shown in the figure above a person whose eye level is 5 feet above the ground stands on the top of a hill overlooking a valley. The shape of the valley is modeled by the graph of $f(x) = 50\cos\left(\frac{x}{100}\right)$. The person's line of sight is tangent to the side of the hill at point A $\left(a, 50\cos\left(\frac{a}{100}\right)\right)$.

(a) Write an equation of the tangent line in terms of the coordinates of point A.
(b) Find the value of a.
(c) Can the person see the top of a 25-foot tall flagpole located at the lowest point of the valley? Justify your answer.

Section II Part B: No calculator is allowed for these problems.

3. For $0 \leq t \leq 4$, a particle is moving along the x-axis. The particle's position is given by $x(t) = 7t - 4t^2 + \int_0^t s^2\, ds$.

   (a) Find the position and velocity of the particle when $t = 3$.

   (b) At $t = 3$ what is the speed of the particle? Is the speed increasing or decreasing? Give a reason for your answer.

   (c) In the interval $(0,4)$ the particle changes direction once. Find the value of t when this change of direction occurs.

   (d) What is the value of t for which the particle is farthest right and at which it is farthest left? Justify your answer.

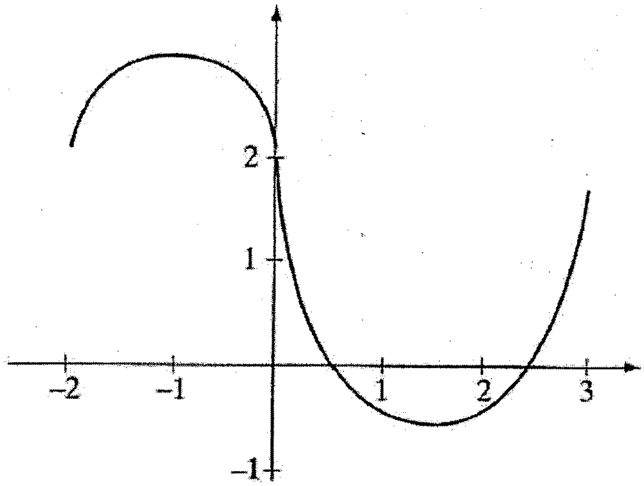
x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	2	2	9
2	5	3	4	7
3	1	5	5	k
4	2	6	5	5
5	4	8	2	3

4. The functions f and g are differentiable for all real numbers. The table above gives values of the functions and their first derivatives at selected values of x. $g'(3) = k$, where k is a constant. The function h is given by $h(x) = f(g(x)) - g(f(x))$.

   (a) Explain why there must be a value c for $1 < c < 4$ such that $h'(c) = 0$.

   (b) If $h'(3) = 0$, find the value of $g'(3)$.

   (c) Let w be the function defined by $w(x) = 7 + \int_1^{f(x)} g(t)\, dt$. Write an equation of the line tangent to w at $x = 3$.



The graph of $f(x)$

6. Let f be a continuous function defined on the closed interval $[-2,3]$. The graph of f consists of a semicircle and a semi-ellipse, as shown above. Let $G(x) = G(-2) + \int_{-2}^{x} f(t)\,dt$.

(a) On what intervals, if any, is G concave down? Justify your answer.

(b) If the equation of the line tangent to the graph of $G(x)$ at the point where $x = 0$ is $y = mx + 7$, what is the value of m and the value of $G(0)$? Justify your answer.

(c) If the average value of f on the interval $0 \le x \le 3$ is zero, find the value of $G(3)$. Show your work that leads to your answer.

Sample Exam I FRQ solutions

Part A (calculator required)

(1) $L(t) = 167.5 \sin\left(\frac{2\pi}{366}(t-80)\right) + 731$ t: days ($t=1$ = Jan 1, 2015)

(a) $L'(t) = 167.5 \cos\left(\frac{2\pi}{366}(t-80)\right) \cdot \frac{2\pi}{366}$ max when $L'(t) = 0$

$L'(t) = 0$ when $\cos\left(\frac{2\pi}{366}(t-80)\right) = 0$

when $\frac{2\pi}{366}(t-80) = \frac{\pi}{2}$ $t - 80 = \frac{\pi}{2} \cdot \frac{366}{2\pi} = 91$

$t = 171.5$

also when $\frac{2\pi}{366}(t-80) = \frac{3\pi}{2}$ $t - 80 = \frac{3\pi}{2} \cdot \frac{366}{2\pi} = 274.5$

checking min or max: $L''(t) = -167.5 \sin\left(\frac{2\pi}{366}(t-80)\right) \cdot \frac{2\pi}{366}$

$t = 171.5$, $L''(171.5) = -2.876$ concave down (maximum)

$t = 274.5$, $L''(274.5) = +.5640$ concave up (minimum)

so most minutes of daylight occurs $\boxed{\text{on the } 172^{nd} \text{ day of the year}}$.

(b) accumulated minutes found by integral of derivative of minutes which is $L(t)$ minutes of daylight per day:

$$\int_1^{365} \left(167.5 \sin\left(\frac{2\pi}{366}(t-80)\right) + 731\right) dt$$

math 9

$= 266412.4766$ so $\boxed{266{,}412 \text{ minutes}}$

(unclear why this problem uses 366 instead of 365, leap year? If so should probably be $\int_1^{366} L(t)\,dt$)

(answer key shows 266249 which you get if $L(t)$ is changed to divide by 365 instead of 366).

(c) $Av = \frac{1}{b-a}\int_a^b L(t)\,dt = \frac{1}{365-1}[266412] = \boxed{731.902 \text{ minutes/day}}$

(again, problem's key is imprecise, does time start at $t=0$ or $t=1$?)

this is the important part

:)

(2) hill: $f(x) = 50\cos\left(\frac{x}{100}\right)$

(a) slope $= m = f'(x) = -50\sin\left(\frac{x}{100}\right)\cdot\frac{1}{100} = -\frac{1}{2}\sin\left(\frac{x}{100}\right)$

at $x=a$: $m = -\frac{1}{2}\sin\left(\frac{a}{100}\right)$

so tangent line is $\boxed{\left(y - 50\cos\left(\frac{a}{100}\right)\right) = -\frac{1}{2}\sin\left(\frac{a}{100}\right)(x-a)}$

(b) tangent line goes through 2 pts, the other is eye level:

at $x=0$ $f(x) = 50\cos(0) = 50$ and the person's eyes are 5 feet higher, at $(0,55)$, substituting $x=0$, $y=55$ into solution for a:

$$\left(55 - 50\cos\left(\frac{a}{100}\right)\right) = -\frac{1}{2}\sin\left(\frac{a}{100}\right)(0-a)$$

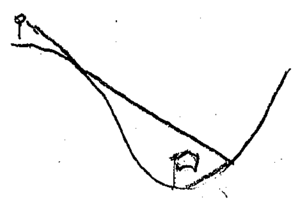
to find zeros

using calculator: $Y1 = 55 - 50\cos\left(\frac{a}{100}\right) - \frac{a}{2}\sin\left(\frac{a}{100}\right) = 0$

... first zero is at $x = 45.934793$

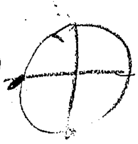
so $\boxed{a = 45.9348 \text{ ft}}$

(c)



top of flag would be 25 ft above the minimum height for $f(x) = 50\cos\left(\frac{x}{100}\right)$ lowest value of $\cos\left(\frac{x}{100}\right) = -1$ at $-50 + 25$ ft, y coordinate $= -25$

(-1,0)



this would occur when $\frac{x}{100} = \pi$, $x = 100\pi$

so top of flag is at $(100\pi, -25)$

extending tangent line:

$$y = -\frac{1}{2}\sin\left(\frac{45.9348}{100}\right)(x - 45.9348) + 50\cos\left(\frac{45.9348}{100}\right)$$

so y at $x = 100\pi$ is -14.64319 which is higher than the flagpole at $y = -25$.

Therefore, $\boxed{\text{the flagpole cannot be seen by the person.}}$

Part B (no calculator)

3) position $x(t) = 7t - 4t^2 + \int_0^t s^2\,ds$

(a) position at $t=3$ = $x(3) = 7(3) - 4(3)^2 + \int_0^3 s^2\,ds$

$= 21 - 36 + \left[\frac{1}{3}s^3\right]_0^3$

$= 21 - 36 + \left(\frac{1}{3}3^3 - 0\right) = 21 - 36 + 9 = \boxed{-6}$

velocity $v(t) = 7 - 8t + t^2$

$v(3) = 7 - 8(3) + 3^2 = 7 - 24 + 9 = \boxed{-8}$

(b) speed = |velocity| = $\boxed{8}$

$a(t) = v'(t) = -8 + 2t$

$a(3) = -8 + 2(3) = -2$

velocity is -8 and $\frac{dv}{dt}$ is $-$ so velocity is getting more negative

so $\boxed{\text{the speed is increasing}}$

(c) change in direction when

$v(t) = 0$

$7 - 8t + t^2 = 0$

$t^2 - 8t + 7 = 0$

$(t-1)(t-7) = 0$

$t = 1 \quad t = 7$

$\boxed{t=1}$ is when the particle is changing direction in the interval $(0,4)$

(d) change in direction at $t=1$

$x(0) = 7(0) - 4(0)^2 + \int_0^0 s^2\,ds = 0$

$x(1) = 7(1) - 4(1)^2 + \int_0^1 s^2\,ds = 7 - 4 + \left.\frac{1}{3}s^3\right]_0^1$

$= 3 + \frac{1}{3}(1-0) = \frac{10}{3}$

$x(4) = 7(4) - 4(4)^2 + \int_0^4 s^2\,ds = 28 - 64 + \left.\frac{1}{3}s^3\right]_0^4 = 28 - 64 + \frac{64}{3}$

$= -36 + \frac{64}{3} = \frac{-108 + 64}{3} = \frac{-44}{3}$

$\begin{array}{r} \cancel{6}4 \\ -28 \\ \hline 36 \end{array} \quad \begin{array}{r} 36 \\ 3 \\ \hline 108 \end{array} \quad \begin{array}{r} 108 \\ 64 \\ \hline 44 \end{array}$

$\boxed{\text{the particle is farthest right at } t=1 \text{ and farthest left at } t=4}$

(4) $h(x) = f(g(x)) - g(f(x))$

(a) if f is differentiable for all real numbers then f is differentiable on $(1,4)$ and also continuous on $[1,4]$ so the Mean Value Theorem applies

$h(1) = f(g(1)) - g(f(1))$
$= f(2) - g(3)$
$= 5 - 5$
$= 0$

and $h(4) = f(g(4)) - g(f(4))$
$= f(5) - g(2)$
$= 4 - 4$
$= 0$

Since $h(1) = h(4)$, Rolle's Theorem also applies: there must exist a c with $1 < c < 4$ where $f'(c) = 0$.

(b) by the Chain Rule:

$h'(x) = f'(g(x)) \cdot g'(x) - g'(f(x)) \cdot f'(x)$
$h'(3) = f'(g(3))g'(3) - g'(f(3)) \cdot f'(3)$
$0 = f'(5) \cdot g'(3) - g'(1) \cdot 5$
$0 = 8 \cdot g'(3) - 9.5$
$g'(3) = \boxed{\frac{45}{8}}$

(c) $w(x) = 7 + \int_1^{f(x)} g(t)\,dt$

so $w(3) = 7 + \int_1^{f(3)} g(t)\,dt = 7 + \int_1^1 g(t)\,dt = 7 + 0 = 7$

slope $w'(x) = 0 + g(f(x)) \cdot f'(x) \leftarrow$ (chain rule)
$w'(3) = g(f(3)) \cdot f'(3)$
$= g(1) \cdot 5$
$= 2 \cdot 5$
$= 10 = m$

So tangent line is $\boxed{(y-7) = 10(x-3)}$

(6) $G(x) = G(-2) + \int_{-2}^{x} f(t)dt$

(a) $G(x)$ is concave down when $G''(x) < 0$:

$G'(x) = 0 + f(x)$

$G''(x) = f'(x)$ so $G''(x) < 0$ when $f(x)$ is decreasing

which occurs at $\boxed{-1 < x < 1.5}$

(b) tangent line to $G(x)$ would have slope $= m = G'(x) = f(x)$

when $x = 0$ $m = G'(0) = f(0) = 2$, so $\boxed{m = 2}$

at $x = 0$, $y = mx + 7$

$y = 2x + 7$

$y = 2(0) + 7$

$y = 7$ so $\boxed{G(0) = 7}$

(c) $Av = \frac{1}{3-0} \int_0^3 f(x)dx = 0$, then $\int_0^3 f(x)dx$ must $= 0$

$G(x) = G(-2) + \int_{-2}^{x} f(t)dt$

$G(3) = G(-2) + \int_{-2}^{3} f(t)dt$

also: $G(0) = G(-2) + \int_{-2}^{0} f(t)dt$

Subtracting...

$G(3) - G(0) = \int_{-2}^{3} f(t)dt - \int_{-2}^{0} f(t)dt$

$G(3) - G(0) = \int_0^3 f(t)dt$

$G(3) - 7 = 0$

$\therefore \boxed{G(3) = 7}$