Part B (AB or BC): Graphing calculator not allowed
Question 3   9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($°C$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is $5°C$. It can be shown that $M(t) < 40$ for all values of t.

Model Solution	Scoring

(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.
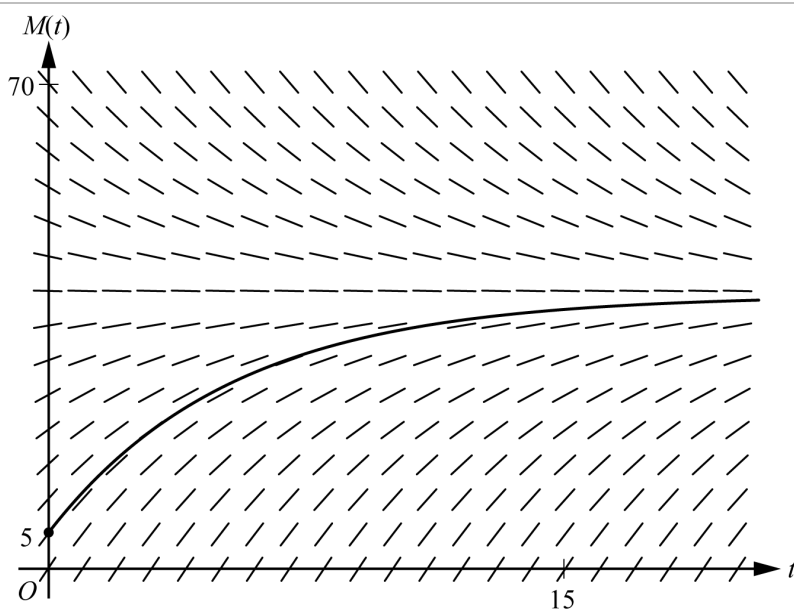
Solution curve   **1 point**

Scoring notes:

- The solution curve must pass through the point $(0, 5)$, extend reasonably close to the left and right edges of the rectangle, and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.
- The solution curve must lie entirely below the horizontal line segments at $M = 40$.

Total for part (a)   1 point

(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2),$ the temperature of the milk at time $t = 2$ minutes.

$\left.\dfrac{dM}{dt}\right|_{t=0} = \dfrac{1}{4}(40 - 5) = \dfrac{35}{4}$	$\left.\dfrac{dM}{dt}\right|_{t=0}$	**1 point**
The tangent line equation is $y = 5 + \dfrac{35}{4}(t - 0).$ $M(2) \approx 5 + \dfrac{35}{4} \cdot 2 = 22.5$ The temperature of the milk at time $t = 2$ minutes is approximately $22.5°$ Celsius.	Approximation	**1 point**

Scoring notes:

- The value of the slope may appear in a tangent line equation or approximation.

- A response of $5 + \dfrac{35}{4} \cdot 2$ is the minimal response to earn both points.

- A response of $\dfrac{1}{4}(40 - 5)$ earns the first point. If there are any subsequent errors in simplification, the response does not earn the second point.

- In order to earn the second point the response must present an approximation found by using a tangent line that:

  o passes through the point $(0, 5)$ and

  o has slope $\dfrac{35}{4}$ or a nonzero slope that is declared to be the value of $\dfrac{dM}{dt}.$

- An unsupported approximation does not earn the second point.

- The approximation need not be simplified, but the response does not earn the second point if the approximation is simplified incorrectly.

Total for part (b)   2 points

(c) Write an expression for $\dfrac{d^2M}{dt^2}$ in terms of $M.$ Use $\dfrac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2).$ Give a reason for your answer.

$\dfrac{d^2M}{dt^2} = -\dfrac{1}{4}\dfrac{dM}{dt} = -\dfrac{1}{4}\left(\dfrac{1}{4}(40 - M)\right) = -\dfrac{1}{16}(40 - M)$	$\dfrac{d^2M}{dt^2}$	**1 point**
Because $M(t) < 40,$ $\dfrac{d^2M}{dt^2} < 0,$ so the graph of M is concave down. Therefore, the tangent line approximation of $M(2)$ is an overestimate.	Overestimate with reason	**1 point**

Scoring notes:

- The first point is earned for either $\dfrac{d^2M}{dt^2} = -\dfrac{1}{4}\left(\dfrac{1}{4}(40 - M)\right)$ or $\dfrac{d^2M}{dt^2} = -\dfrac{1}{16}(40 - M)$ (or equivalent). A response that presents any subsequent simplification error does not earn the second point.

- A response that presents an expression for $\dfrac{d^2M}{dt^2}$ in terms of $\dfrac{dM}{dt}$ but fails to continue to an expression in terms of M (i.e., $\dfrac{d^2M}{dt^2} = -\dfrac{1}{4}\dfrac{dM}{dt}$) does not earn the first point but is eligible for the second point.

- If the response presents an expression for $\dfrac{d^2M}{dt^2}$ that is incorrect, the response is eligible for the second point only if the expression is a nonconstant linear function that is negative for $5 < M < 40$.

  - Special case: A response that presents $\dfrac{d^2M}{dt^2} = \dfrac{1}{16}(40 - M)$ does not earn the first point but is eligible to earn the second point for a consistent answer and reason.

- To earn the second point a response must include $\dfrac{d^2M}{dt^2} < 0$, or $\dfrac{dM}{dt}$ is decreasing, or the graph of M is concave down, as well as the conclusion that the approximation is an overestimate.

- A response that presents an argument based on $\dfrac{d^2M}{dt^2}$ or concavity at a single point does not earn the second point.

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Total for part (c)   2 points

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(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

$\dfrac{dM}{40 - M} = \dfrac{1}{4}dt$ $\displaystyle\int \dfrac{dM}{40 - M} = \int \dfrac{1}{4}\,dt$	Separates variables **1 point**
$-\ln\lvert 40 - M \rvert = \dfrac{1}{4}t + C$	Finds antiderivatives **1 point**
$-\ln\lvert 40 - 5 \rvert = 0 + C \Rightarrow C = -\ln 35$ $M(t) < 40 \Rightarrow 40 - M > 0 \Rightarrow \lvert 40 - M \rvert = 40 - M$ $-\ln(40 - M) = \dfrac{1}{4}t - \ln 35$ $\ln(40 - M) = -\dfrac{1}{4}t + \ln 35$	Constant of integration and uses initial condition **1 point**

$40 - M = 35e^{-t/4}$ $M = 40 - 35e^{-t/4}$	**Solves for** M	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 4 points.

- A response that presents an antiderivative of $-\ln(40 - M)$ without absolute value symbols is eligible for all 4 points.

- A response with no constant of integration can earn at most the first 2 points.

- A response is eligible for the third point only if it has earned the first 2 points.

    o Special Case: A response that presents $+\ln(40 - M) = \dfrac{t}{4} + C$ (or equivalent) does not earn the second point, is eligible for the third point, but not eligible for the fourth.

- An eligible response earns the third point by correctly including the constant of integration in an equation and substituting 0 for t and 5 for M.

- A response is eligible for the fourth point only if it has earned the first 3 points.

- A response earns the fourth point only for an answer of $M = 40 - 35e^{-t/4}$ or equivalent.

Total for part (d)	**4 points**
Total for question 3	**9 points**