Question 3

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.

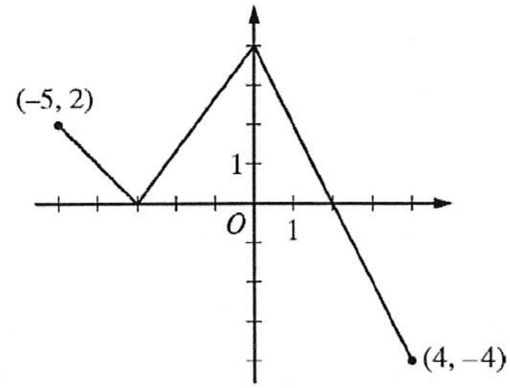
Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\, dt$.

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.



Graph of f

(a) $g(3) = \int_{-3}^{3} f(t)\, dt = 6 + 4 - 1 = 9$

1 : answer

(b) $g'(x) = f(x)$

The graph of g is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $h'(x) = \dfrac{5xg'(x) - g(x)5}{(5x)^2} = \dfrac{5xg'(x) - 5g(x)}{25x^2}$

$$h'(3) = \frac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2}$$

$$= \frac{15(-2) - 5(9)}{225} = \frac{-75}{225} = -\frac{1}{3}$$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

(d) $p'(x) = f'\left(x^2 - x\right)(2x - 1)$

$p'(-1) = f'(2)(-3) = (-2)(-3) = 6$

$3 : \begin{cases} 2 : p'(x) \\ 1 : \text{answer} \end{cases}$

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Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

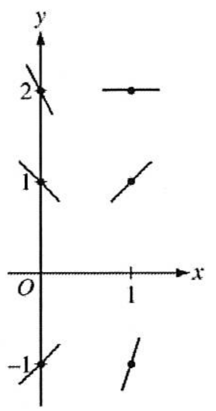
(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

(a)



$2: \begin{cases} 1: \text{slopes where } x = 0 \\ 1: \text{slopes where } x = 1 \end{cases}$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.
Therefore, all solution curves are concave up in Quadrant II.

$2: \begin{cases} 1: \frac{d^2y}{dx^2} \\ 1: \text{concave up with reason} \end{cases}$

(c) $\left.\frac{dy}{dx}\right|_{(x,\,y)=(2,\,3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

$2: \begin{cases} 1: \text{considers } \left.\frac{dy}{dx}\right|_{(x,\,y)=(2,\,3)} \\ 1: \text{conclusion with justification} \end{cases}$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$2x - y = m$

$2x - (mx + b) = m$

$(2 - m)x - (m + b) = 0$

$2 - m = 0 \Rightarrow m = 2$

$b = -m \Rightarrow b = -2$

Therefore, $m = 2$ and $b = -2$.

$3: \begin{cases} 1: \frac{d}{dx}(mx + b) = m \\ 1: 2x - y = m \\ 1: \text{answer} \end{cases}$

Question 2

(a) $\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta = 3.534292$

The area of S is 3.534.

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) $\frac{1}{\sqrt{\pi} - 0}\int_0^{\sqrt{\pi}} r(\theta) \, d\theta = 1.579933$

The average distance from the origin to a point on the curve $r = r(\theta)$ for $0 \le \theta \le \sqrt{\pi}$ is 1.580 (or 1.579).

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(c) $\tan\theta = \frac{y}{x} = m \Rightarrow \theta = \tan^{-1} m$

$$\frac{1}{2}\int_0^{\tan^{-1} m} (r(\theta))^2 \, d\theta = \frac{1}{2}\left(\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta\right)$$

$3: \begin{cases} 1: \text{equates polar areas} \\ 1: \text{inverse trigonometric function applied to } m \text{ as limit of integration} \\ 1: \text{equation} \end{cases}$

(d) As $k \to \infty$, the circle $r = k\cos\theta$ grows to enclose all points to the right of the y-axis.

$$\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\pi/2} (r(\theta))^2 \, d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2 \, d\theta = 3.324$$

$2: \begin{cases} 1: \text{limits of integration} \\ 1: \text{answer with integral} \end{cases}$

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Question 6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

(a) $f(1) = 1,\ f'(1) = -\frac{1}{2},\ f''(1) = \frac{1}{2^2},\ f'''(1) = -\frac{2}{2^3}$

$$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \cdots + \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \cdots$$

$4 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \\ 1 : \text{general term} \end{cases}$

(b) $R = 2$. The series converges on the interval $(-1, 3)$.

When $x = -1$, the series is $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the harmonic series diverges, this series diverges.

When $x = 3$, the series is $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is $-1 < x \leq 3$.

$2 : \begin{cases} 1 : \text{identifies both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

(c) $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$

1 : approximation

(d) The series for $f(1.2)$ alternates with terms that decrease in magnitude to 0.

$$|f(1.2) - T_2(1.2)| \leq \left| \frac{-1}{2^3 \cdot 3}(0.2)^3 \right| = \frac{1}{3000} \leq 0.001$$

$2 : \begin{cases} 1 : \text{error form} \\ 1 : \text{analysis} \end{cases}$