# AP® CALCULUS BC
# 2007 SCORING GUIDELINES

## Question 1

Let $R$ be the region in the first and second quadrants bounded above by the graph of $y = \dfrac{20}{1+x^2}$ and below by the horizontal line $y = 2$.

(a) Find the area of $R$.

(b) Find the volume of the solid generated when $R$ is rotated about the $x$-axis.

(c) The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the $x$-axis are semicircles. Find the volume of this solid.

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| | |
|---|---|
| $\dfrac{20}{1+x^2} = 2$ when $x = \pm 3$ | 1 : correct limits in an integral in (a), (b), or (c) |
| (a) Area $= \displaystyle\int_{-3}^{3}\left(\frac{20}{1+x^2} - 2\right)dx = 37.961$ or $37.962$ | 2 : { 1 : integrand<br>1 : answer |
| (b) Volume $= \pi\displaystyle\int_{-3}^{3}\left(\left(\frac{20}{1+x^2}\right)^2 - 2^2\right)dx = 1871.190$ | 3 : { 2 : integrand<br>1 : answer |
| (c) Volume $= \dfrac{\pi}{2}\displaystyle\int_{-3}^{3}\left(\frac{1}{2}\left(\frac{20}{1+x^2} - 2\right)\right)^2 dx$<br>$= \dfrac{\pi}{8}\displaystyle\int_{-3}^{3}\left(\frac{20}{1+x^2} - 2\right)^2 dx = 174.268$ | 3 : { 2 : integrand<br>1 : answer |

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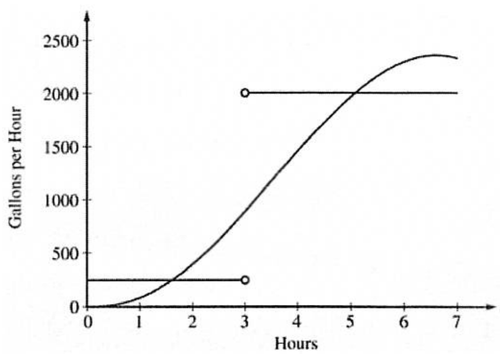
## Question 2

The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where $t$ is measured in hours. In this model, rates are given as follows:

(i) The rate at which water enters the tank is $f(t) = 100t^2 \sin\left(\sqrt{t}\right)$ gallons per hour for $0 \le t \le 7$.

(ii) The rate at which water leaves the tank is

$$g(t) = \begin{cases} 250 & \text{for } 0 \le t < 3 \\ 2000 & \text{for } 3 < t \le 7 \end{cases} \text{ gallons per hour.}$$




The graphs of $f$ and $g$, which intersect at $t = 1.617$ and $t = 5.076$, are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

(a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

(b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for each answer.

(c) For $0 \le t \le 7$, at what time $t$ is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

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(a) $\int_0^7 f(t)\,dt \approx 8264$ gallons

$$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$$

(b) The amount of water in the tank is decreasing on the intervals $0 \le t \le 1.617$ and $3 \le t \le 5.076$ because $f(t) < g(t)$ for $0 \le t < 1.617$ and $3 < t < 5.076$.

$$2: \begin{cases} 1: \text{intervals} \\ 1: \text{reason} \end{cases}$$

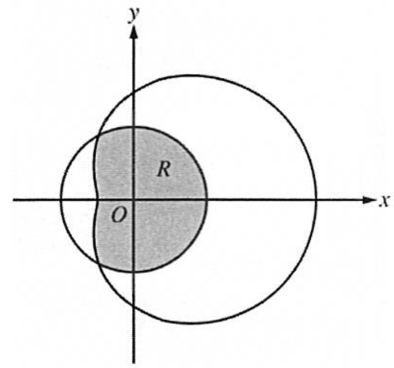
(c) Since $f(t) - g(t)$ changes sign from positive to negative only at $t = 3$, the candidates for the absolute maximum are at $t = 0$, 3, and 7.

| $t$ (hours) | gallons of water |
|---|---|
| 0 | 5000 |
| 3 | $5000 + \int_0^3 f(t)\,dt - 250(3) = 5126.591$ |
| 7 | $5126.591 + \int_3^7 f(t)\,dt - 2000(4) = 4513.807$ |

The amount of water in the tank is greatest at 3 hours. At that time, the amount of water in the tank, rounded to the nearest gallon, is 5127 gallons.

$$5: \begin{cases} 1: \text{identifies } t = 3 \text{ as a candidate} \\ 1: \text{integrand} \\ 1: \text{amount of water at } t = 3 \\ 1: \text{amount of water at } t = 7 \\ 1: \text{conclusion} \end{cases}$$

## Question 3




The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

(a) Let $R$ be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of $R$.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time $t$, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

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(a) $\text{Area} = \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3 + 2\cos\theta)^2\,d\theta$

$= 10.370$

$$4 : \begin{cases} 1 : \text{area of circular sector} \\ 2 : \text{integral for section of limaçon} \\ \quad 1 : \text{integrand} \\ \quad 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$$

(b) $\left.\frac{dr}{dt}\right|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = -1.732$

The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.

$$2 : \begin{cases} 1 : \left.\frac{dr}{dt}\right|_{\theta=\pi/3} \\ 1 : \text{interpretation} \end{cases}$$

(c) $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$

$\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} = 0.5$

The particle is moving away from the $x$-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.

$$3 : \begin{cases} 1 : \text{expression for } y \text{ in terms of } \theta \\ 1 : \left.\frac{dy}{dt}\right|_{\theta=\pi/3} \\ 1 : \text{interpretation} \end{cases}$$

## Question 4

Let $f$ be the function defined for $x > 0$, with $f(e) = 2$ and $f'$, the first derivative of $f$, given by $f'(x) = x^2 \ln x$.

(a) Write an equation for the line tangent to the graph of $f$ at the point $(e, 2)$.

(b) Is the graph of $f$ concave up or concave down on the interval $1 < x < 3$ ? Give a reason for your answer.

(c) Use antidifferentiation to find $f(x)$.

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(a) $f'(e) = e^2$

An equation for the line tangent to the graph of $f$ at the point $(e, 2)$ is $y - 2 = e^2(x - e)$.

$2 : \begin{cases} 1 : f'(e) \\ 1 : \text{equation of tangent line} \end{cases}$

(b) $f''(x) = x + 2x\ln x$.

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of $f$ is concave up on $(1, 3)$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) Since $f(x) = \int \left(x^2 \ln x\right) dx$, we consider integration by parts.

$$u = \ln x \qquad dv = x^2\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = \int\left(x^2\right)dx = \frac{1}{3}x^3$$

Therefore,

$$\begin{aligned} f(x) &= \int\left(x^2 \ln x\right) dx \\ &= \frac{1}{3}x^3 \ln x - \int\left(\frac{1}{3}x^3 \cdot \frac{1}{x}\right) dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C. \end{aligned}$$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$4 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{uses } f(e) = 2 \\ 1 : \text{answer} \end{cases}$

## Question 5

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function $W$ models the total amount of solid waste stored at the landfill. Planners estimate that $W$ will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. $W$ is measured in tons, and $t$ is measured in years from the start of 2010.

(a) Use the line tangent to the graph of $W$ at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of $W$. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

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(a) $\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons

$2 : \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$

(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \geq 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \leq t \leq \frac{1}{4}$.

The answer in part (a) is an underestimate.

$2 : \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$\int \frac{1}{W - 300}\, dW = \int \frac{1}{25}\, dt$

$\ln|W - 300| = \frac{1}{25}t + C$

$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$W(t) = 300 + 1100e^{\frac{1}{25}t},\ 0 \leq t \leq 20$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

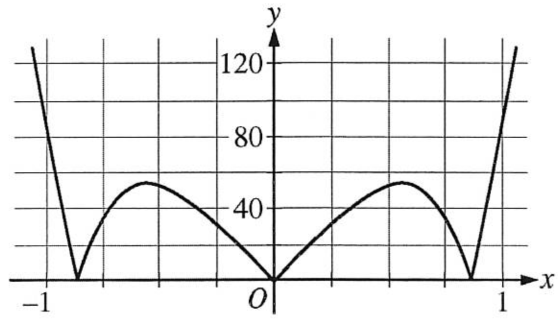
Note: $0/5$ if no separation of variables

**Question 6**

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for $f$ about $x = 0$.




Graph of $y = \left|f^{(5)}(x)\right|$

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for $f$ about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

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(a) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

$\sin\left(x^2\right) = x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \sin x \\ 2 : \text{series for } \sin\left(x^2\right) \end{cases}$

(b) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$

$f(x) = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{121x^6}{6!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \cos x \\ 2 : \text{series for } f(x) \end{cases}$

(c) $\dfrac{f^{(6)}(0)}{6!}$ is the coefficient of $x^6$ in the Taylor series for $f$ about $x = 0$. Therefore $f^{(6)}(0) = -121$.

1 : answer

(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$.

Therefore

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}.$$

$2 : \begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$