AP® CALCULUS BC
2012 SCORING GUIDELINES

Question 1

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

(a) $W'(12) \approx \dfrac{W(15) - W(9)}{15 - 9} = \dfrac{67.9 - 61.8}{6}$
$= 1.017$ (or 1.016)

The water temperature is increasing at a rate of approximately 1.017 °F per minute at time $t = 12$ minutes.

2 : { 1 : estimate; 1 : interpretation with units }

(b) $\int_0^{20} W'(t)\,dt = W(20) - W(0) = 71.0 - 55.0 = 16$

The water has warmed by 16 °F over the interval from $t = 0$ to $t = 20$ minutes.

2 : { 1 : value; 1 : interpretation with units }

(c) $\frac{1}{20}\int_0^{20} W(t)\,dt \approx \frac{1}{20}(4 \cdot W(0) + 5 \cdot W(4) + 6 \cdot W(9) + 5 \cdot W(15))$
$= \frac{1}{20}(4 \cdot 55.0 + 5 \cdot 57.1 + 6 \cdot 61.8 + 5 \cdot 67.9)$
$= \frac{1}{20} \cdot 1215.8 = 60.79$

This approximation is an underestimate, because a left Riemann sum is used and the function W is strictly increasing.

3 : { 1 : left Riemann sum; 1 : approximation; 1 : underestimate with reason }

(d) $W(25) = 71.0 + \int_{20}^{25} W'(t)\,dt$
$= 71.0 + 2.043155 = 73.043$

2 : { 1 : integral; 1 : answer }

Question 2

For $t \geq 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 2$, the particle is at position $(1, 5)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+2}}{e^t}$ and $\frac{dy}{dt} = \sin^2 t$.

(a) Is the horizontal movement of the particle to the left or to the right at time $t = 2$? Explain your answer. Find the slope of the path of the particle at time $t = 2$.

(b) Find the x-coordinate of the particle's position at time $t = 4$.

(c) Find the speed of the particle at time $t = 4$. Find the acceleration vector of the particle at time $t = 4$.

(d) Find the distance traveled by the particle from time $t = 2$ to $t = 4$.

(a) $\left.\frac{dx}{dt}\right|_{t=2} = \frac{2}{e^2}$

Because $\left.\frac{dx}{dt}\right|_{t=2} > 0$, the particle is moving to the right at time $t = 2$.

$\left.\frac{dy}{dx}\right|_{t=2} = \frac{\left.dy/dt\right|_{t=2}}{\left.dx/dt\right|_{t=2}} = 3.055$ (or 3.054)

3 : { 1 : moving to the right with reason; 1 : considers $\frac{dy/dt}{dx/dt}$; 1 : slope at $t = 2$ }

(b) $x(4) = 1 + \int_2^4 \frac{\sqrt{t+2}}{e^t}\,dt = 1.253$ (or 1.252)

2 : { 1 : integral; 1 : answer }

(c) Speed $= \sqrt{(x'(4))^2 + (y'(4))^2} = 0.575$ (or 0.574)

Acceleration $= \langle x''(4), y''(4) \rangle$
$= \langle -0.041, 0.989 \rangle$

2 : { 1 : speed; 1 : acceleration }

(d) Distance $= \int_2^4 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$
$= 0.651$ (or 0.650)

2 : { 1 : integral; 1 : answer }

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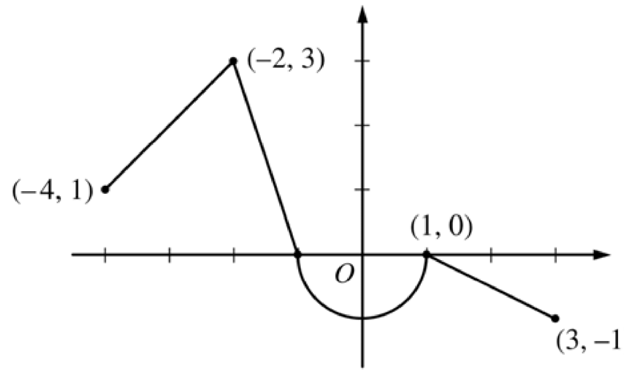
Question 3



Graph of f

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt.$

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

Solution	Scoring
(a) $g(2) = \int_1^2 f(t)\,dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$ $g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt$ $= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$	2 : { 1 : $g(2)$ 1 : $g(-2)$
(b) $g'(x) = f(x) \Rightarrow g'(-3) = f(-3) = 2$ $g''(x) = f'(x) \Rightarrow g''(-3) = f'(-3) = 1$	2 : { 1 : $g'(-3)$ 1 : $g''(-3)$
(c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$. $g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, g has a relative maximum at $x = -1$. $g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.	3 : { 1 : considers $g'(x) = 0$ 1 : $x = -1$ and $x = 1$ 1 : answers with justifications
(d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.	2 : { 1 : answer 1 : explanation

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Question 4

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

(a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

(c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

(d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

(a) $f(1) = 15,\quad f'(1) = 8$

An equation for the tangent line is
$y = 15 + 8(x - 1)$.

$f(1.4) \approx 15 + 8(1.4 - 1) = 18.2$

$2 : \begin{cases} 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$

(b) $\int_1^{1.4} f'(x)\,dx \approx (0.2)(10) + (0.2)(13) = 4.6$

$f(1.4) = f(1) + \int_1^{1.4} f'(x)\,dx$

$f(1.4) \approx 15 + 4.6 = 19.6$

$3 : \begin{cases} 1 : \text{midpoint Riemann sum} \\ 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

(c) $f(1.2) \approx f(1) + (0.2)(8) = 16.6$

$f(1.4) \approx 16.6 + (0.2)(12) = 19.0$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

(d) $T_2(x) = 15 + 8(x - 1) + \dfrac{20}{2!}(x - 1)^2$
$= 15 + 8(x - 1) + 10(x - 1)^2$

$f(1.4) \approx 15 + 8(1.4 - 1) + 10(1.4 - 1)^2 = 19.8$

$2 : \begin{cases} 1 : \text{Taylor polynomial} \\ 1 : \text{approximation} \end{cases}$

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Question 5

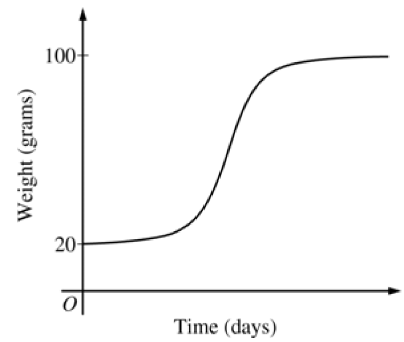
The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

(a) $\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(60) = 12$

$\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(30) = 6$

Because $\left.\frac{dB}{dt}\right|_{B=40} > \left.\frac{dB}{dt}\right|_{B=70}$, the bird is gaining weight faster when it weighs 40 grams.

$2 : \begin{cases} 1 : \text{uses } \frac{dB}{dt} \\ 1 : \text{answer with reason} \end{cases}$

(b) $\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt} = -\frac{1}{5}\cdot\frac{1}{5}(100 - B) = -\frac{1}{25}(100 - B)$

Therefore, the graph of B is concave down for $20 \le B < 100$. A portion of the given graph is concave up.

$2 : \begin{cases} 1 : \frac{d^2B}{dt^2} \text{ in terms of } B \\ 1 : \text{explanation} \end{cases}$

(c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$\int \frac{1}{100 - B}\, dB = \int \frac{1}{5}\, dt$

$-\ln|100 - B| = \frac{1}{5}t + C$

Because $20 \le B < 100$, $|100 - B| = 100 - B$.

$-\ln(100 - 20) = \frac{1}{5}(0) + C \Rightarrow -\ln(80) = C$

$100 - B = 80e^{-t/5}$

$B(t) = 100 - 80e^{-t/5}, \quad t \ge 0$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } B \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

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Question 6

The function g has derivatives of all orders, and the Maclaurin series for g is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots$.

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\left|\frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}}\right| = \left(\frac{2n+3}{2n+5}\right) \cdot x^2$

$\lim_{n \to \infty}\left(\frac{2n+3}{2n+5}\right) \cdot x^2 = x^2$

$x^2 < 1 \Rightarrow -1 < x < 1$

The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$

This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \le x \le 1$.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies interior of interval of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

(b) $\left|g\left(\frac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

2 :
- 1 : uses the third term as an error bound
- 1 : error bound

(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n\left(\frac{2n+1}{2n+3}\right)x^{2n} + \cdots$

2 :
- 1 : first three terms
- 1 : general term