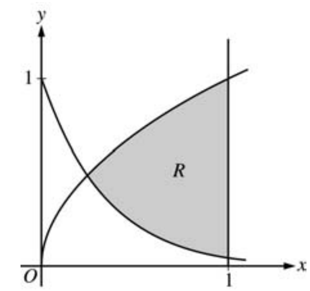
Question 1

Let R be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.



(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 1$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a rectangle whose height is 5 times the length of its base in region R. Find the volume of this solid.

Solution	Scoring
Point of intersection $e^{-3x} = \sqrt{x}$ at $(T, S) = (0.238734, 0.488604)$	1: Correct limits in an integral in (a), (b), or (c)
(a) Area $= \int_T^1 \left(\sqrt{x} - e^{-3x}\right) dx$ $= 0.442$ or 0.443	2 : { 1 : integrand 1 : answer
(b) Volume $= \pi \int_T^1 \left(\left(1 - e^{-3x}\right)^2 - \left(1 - \sqrt{x}\right)^2 \right) dx$ $= 0.453\pi$ or 1.423 or 1.424	3 : { 2 : integrand $< -1 >$ reversal $< -1 >$ error with constant $< -1 >$ omits 1 in one radius $< -2 >$ other errors 1 : answer
(c) Length $= \sqrt{x} - e^{-3x}$ Height $= 5\left(\sqrt{x} - e^{-3x}\right)$ Volume $= \int_T^1 5\left(\sqrt{x} - e^{-3x}\right)^2 dx = 1.554$	3 : { 2 : integrand $< -1 >$ incorrect but has $\sqrt{x} - e^{-3x}$ as a factor 1 : answer

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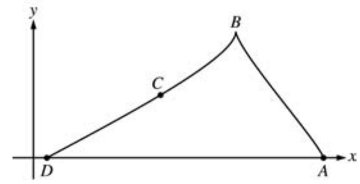
Question 2



A particle starts at point A on the positive x-axis at time $t = 0$ and travels along the curve from A to B to C to D, as shown above. The coordinates of the particle's position $\left(x(t),\, y(t)\right)$ are differentiable functions of t, where

$x'(t) = \dfrac{dx}{dt} = -9\cos\left(\dfrac{\pi t}{6}\right)\sin\left(\dfrac{\pi\sqrt{t+1}}{2}\right)$ and $y'(t) = \dfrac{dy}{dt}$ is not explicitly given.

At time $t = 9$, the particle reaches its final position at point D on the positive x-axis.

(a) At point C, is $\dfrac{dy}{dt}$ positive? At point C, is $\dfrac{dx}{dt}$ positive? Give a reason for each answer.

(b) The slope of the curve is undefined at point B. At what time t is the particle at point B?

(c) The line tangent to the curve at the point $\left(x(8),\, y(8)\right)$ has equation $y = \dfrac{5}{9}x - 2$. Find the velocity vector and the speed of the particle at this point.

(d) How far apart are points A and D, the initial and final positions, respectively, of the particle?

(a) At point C, $\dfrac{dy}{dt}$ is not positive because $y(t)$ is decreasing along the arc BD as t increases.

At point C, $\dfrac{dx}{dt}$ is not positive because $x(t)$ is decreasing along the arc BD as t increases.

2 : { 1 : $\dfrac{dy}{dt}$ not positive with reason; 1 : $\dfrac{dx}{dt}$ not positive with reason }

(b) $\dfrac{dx}{dt} = 0$; $\cos\left(\dfrac{\pi t}{6}\right) = 0$ or $\sin\left(\dfrac{\pi\sqrt{t+1}}{2}\right) = 0$

$\dfrac{\pi t}{6} = \dfrac{\pi}{2}$ or $\dfrac{\pi\sqrt{t+1}}{2} = \pi$; $t = 3$ for both.

Particle is at point B at $t = 3$.

2 : { 1 : sets $\dfrac{dx}{dt} = 0$; 1 : $t = 3$ }

(c) $x'(8) = -9\cos\left(\dfrac{4\pi}{3}\right)\sin\left(\dfrac{3\pi}{2}\right) = -\dfrac{9}{2}$

$\dfrac{y'(8)}{x'(8)} = \dfrac{dy}{dx} = \dfrac{5}{9}$

$y'(8) = \dfrac{5}{9}x'(8) = -\dfrac{5}{2}$

The velocity vector is $< -4.5, -2.5 >$.

Speed $= \sqrt{4.5^2 + 2.5^2} = 5.147$ or 5.148

3 : { 1 : $x'(8)$; 1 : $y'(8)$; 1 : speed }

(d) $x(9) - x(0) = \displaystyle\int_0^9 x'(t)\,dt$

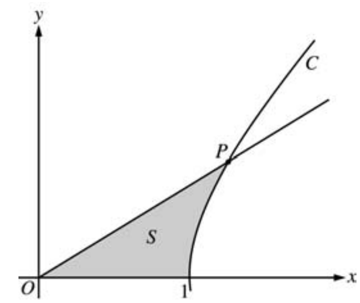
$= -39.255$

The initial and final positions are 39.255 apart.

2 : { 1 : integral; 1 : answer }

Question 3

The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1 + y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.



(a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P.

(b) Set up and evaluate an integral expression with respect to y that gives the area of S.

(c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

(d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S.

(a) At P, $\frac{5}{3}y = \sqrt{1 + y^2}$, so $y = \frac{3}{4}$.

Since $x = \frac{5}{3}y$, $x = \frac{5}{4}$.

$$\frac{dx}{dy} = \frac{y}{\sqrt{1+y^2}} = \frac{y}{x}. \text{ At } P,\ \frac{dx}{dy} = \frac{3/4}{5/4} = \frac{3}{5}.$$

2 : { 1 : coordinates of P; 1 : $\frac{dx}{dy}$ at P }

(b) Area $= \int_0^{3/4} \left(\sqrt{1+y^2} - \frac{5}{3}y\right) dy$

$= 0.346$ or 0.347

3 : { 1 : limits; 1 : integrand; 1 : answer }

(c) $x = r\cos\theta$; $y = r\sin\theta$

$x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1$

$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$

2 : { 1 : substitutes $x = r\cos\theta$ and $y = r\sin\theta$ into $x^2 - y^2 = 1$; 1 : isolates r^2 }

(d) Let β be the angle that segment OP makes with the x-axis. Then $\tan\beta = \frac{y}{x} = \frac{3/4}{5/4} = \frac{3}{5}$.

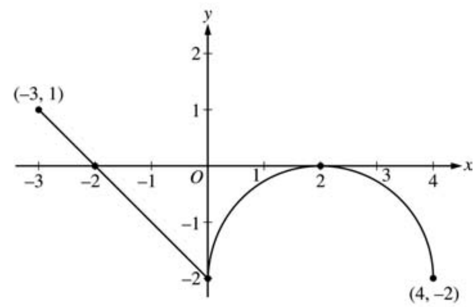
$$\text{Area} = \int_0^{\tan^{-1}(3/5)} \frac{1}{2}r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}(3/5)} \frac{1}{\cos^2\theta - \sin^2\theta}\, d\theta$$

2 : { 1 : limits; 1 : integrand and constant }

Question 4

Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown above.



Graph of f'

(a) On what intervals, if any, is f increasing? Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

(c) Find an equation for the line tangent to the graph of f at the point $(0,3)$.

(d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.

(a) The function f is increasing on $[-3,-2]$ since $f' > 0$ for $-3 \le x < -2$.

$2 : \begin{cases} 1 : \text{interval} \\ 1 : \text{reason} \end{cases}$

(b) $x = 0$ and $x = 2$

f' changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 2$

$2 : \begin{cases} 1 : x = 0 \text{ and } x = 2 \text{ only} \\ 1 : \text{justification} \end{cases}$

(c) $f'(0) = -2$

Tangent line is $y = -2x + 3$.

$1 : \text{equation}$

(d)
$$f(0) - f(-3) = \int_{-3}^{0} f'(t)\,dt = \frac{1}{2}(1)(1) - \frac{1}{2}(2)(2) = -\frac{3}{2}$$

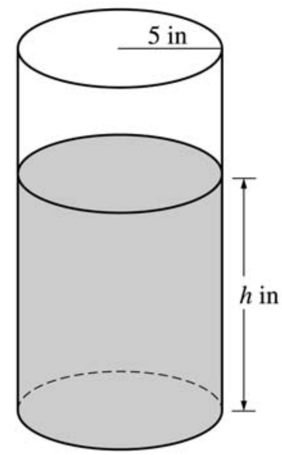
$$f(-3) = f(0) + \frac{3}{2} = \frac{9}{2}$$

$$f(4) - f(0) = \int_{0}^{4} f'(t)\,dt = -\left(8 - \frac{1}{2}(2)^2\pi\right) = -8 + 2\pi$$

$$f(4) = f(0) - 8 + 2\pi = -5 + 2\pi$$

$4 : \begin{cases} 1 : \pm\left(\frac{1}{2} - 2\right) \text{ (difference of areas of triangles)} \\ 1 : \text{answer for } f(-3) \text{ using FTC} \\ 1 : \pm\left(8 - \frac{1}{2}(2)^2\pi\right) \text{ (area of rectangle – area of semicircle)} \\ 1 : \text{answer for } f(4) \text{ using FTC} \end{cases}$

Question 5



A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

(a) $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

3 : { 1 : $\frac{dV}{dt} = -5\pi\sqrt{h}$; 1 : computes $\frac{dV}{dt}$; 1 : shows result }

(b) $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}}\,dh = -\frac{1}{5}\,dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

5 : { 1 : separates variables; 1 : antiderivatives; 1 : constant of integration; 1 : uses initial condition $h = 17$ when $t = 0$; 1 : solves for h }

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(c) $\left(-\frac{1}{10}t + \sqrt{17}\right)^2 = 0$

$$t = 10\sqrt{17}$$

1 : answer

Question 6

The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

(c) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = \cos x$.

(a) $f'(0) =$ coefficient of x term $= 0$

$f''(0) = 2$ (coefficient of x^2 term) $= 2\left(-\frac{1}{3!}\right) = -\frac{1}{3}$

f has a local maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

4 : { 1 : $f'(0)$; 1 : $f''(0)$; 1 : critical point answer; 1 : reason }

(b) $f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \cdots$

This is an alternating series whose terms decrease in absolute value with limit 0. Thus, the error is less than the first omitted term, so $\left| f(1) - \left(1 - \frac{1}{3!}\right) \right| \le \frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$.

1 : error bound $< \frac{1}{100}$

(c)

$$y' = -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots + \frac{(-1)^n 2n x^{2n-1}}{(2n+1)!} + \cdots$$

$$xy' = -\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \cdots + \frac{(-1)^n 2n x^{2n}}{(2n+1)!} + \cdots$$

$$xy' + y = 1 - \left(\frac{2}{3!} + \frac{1}{3!}\right)x^2 + \left(\frac{4}{5!} + \frac{1}{5!}\right)x^4 - \left(\frac{6}{7!} + \frac{1}{7!}\right)x^6 + \cdots + (-1)^n\left(\frac{2n}{(2n+1)!} + \frac{1}{(2n+1)!}\right)x^{2n} + \cdots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots$$

$$= \cos x$$

4 : { 1 : series for y'; 1 : series for xy'; 1 : series for $xy' + y$; 1 : identifies series as $\cos x$ }

OR

$$xy = xf(x) = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{1}{(2n+1)!} x^{2n+1} + \cdots$$

$$= \sin x$$

$$xy' + y = (xy)' = (\sin x)' = \cos x$$

OR

4 : { 1 : series for $xf(x)$; 1 : identifies series as $\sin x$; 1 : handles $xy' + y$; 1 : makes connection }